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THE DOUGLAS-RACHFORD ALGORITHM WITH NEW ERROR SEQUENCES FOR AN INCLUSION PROBLEM

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Abstract. In recent years, the Douglas-Rachford algorithm has received much attention due to its various applications in image recovery, signal processing, and machine learning. In this paper, we introduce the Mann iteration of Douglas-Rachford algorithm with a new error sequence in Hilbert spaces, and establish its weak convergence under some mild conditions. Furthermore, we propose the Halpern iteration of Douglas-Rachford algorithm with two different error sequences, and prove their strong convergence under some proper conditions. Finally, a feasibility problem is also considered.

Keywords. Douglas-Rachford algorithm; Image recovery; Reflected resolvent; Strong convergence; Weak convergence.

1. Introduction

In the paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. An important problem in the theory of nonlinear analysis is to find a solution of the following inclusion problem:

$$0 \in (A+B)x, \tag{1.1}$$

where A and B are two monotone operators in Hilbert space H. Problem (1.1) includes many optimization problems arising from various applied areas, such as signal processing, image recovery, statistical regression, and machine learning; see, e.g., [21, 23] and the references therein.

There are many effective methods to solve problem (1.1); see, e.g. [6, 7, 8, 10, 13, 18, 20]. The most popular one among these methods is the Douglas-Rachford algorithm (DRA for short), which was first introduced in [11] to numerically solve certain types of heat equation. Lions and Mercier [15] further extended the algorithm to the case of inclusion problems, not necessarily for linear and possibly set-valued maximally monotone operators.

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Let A and B be two maximal monotone operators. For some $\gamma > 0$, the sequence $\{z_k\}_{k=0}^{\infty}$ is said to obey the *Douglas-Rachford algorithm* for γ, A, and B if

$$z_{k+1} = J_{\gamma A}((2J_{\gamma B} - I)(z_k)) + (I - J_{\gamma B})(z_k).$$

In 1987, Lawrence and Spingarn [16] first interpreted DRA as a proximal point method. In 1992, Eckstein and Bertsekas [12] studied the DRA with summable errors with over/under the relaxation: for $\{z_k\}_{k=0}^{\infty}, \{u_k\}_{k=0}^{\infty}, \{v_k\}_{k=1}^{\infty} \subseteq R^n$, $\{\alpha_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty} \subseteq [0,\infty]$, and $\{\rho_k\}_{k=0}^{\infty} \subseteq (0,2)$ provided that following conditions were satisfied,

(T1) $\|u_k - J_{\gamma B}(z_k)\| \leq \beta_k$, $\forall k \geq 0$,
(T2) $\|v_{k+1} - J_{\gamma A}(2u_k - z_k)\| \leq \alpha_k$, $\forall k \geq 0$,
(T3) $\sum_{k=0}^{\infty} \alpha_k < \infty$, $\sum_{k=0}^{\infty} \beta_k < \infty$, $0 < \liminf_{k\geq 0} \rho_k \leq \limsup_{k\geq 0} \rho_k < 2$,

where $J_{\gamma A} := (I + \gamma A)^{-1}$ is the *resolvent* of A. The generalized DRA:

$$z_{k+1} = z_k + \rho_k(v_{k+1} - u_k), \quad \forall k \geq 0$$

converges weakly to some element of $Z_{\lambda}^* = \{u + \lambda b | b \in Bu, -b \in Au\}$ provided that the solution set of problem (1.1) is nonempty.

In 2004, Combettes [6] further proposed a relaxed extension of DRA: for any initial $x_0 \in H$,

$$x_{n+1} = x_n + \nu_n(J_{\gamma A}(2(J_{\gamma B}x_n + b_n) - x_n) + a_n - (J_{\gamma B}x_n + b_n)),$$

where $\{\nu_n\}$ is a sequence in $(0,2)$, $\{a_n\}$, and $\{b_n\}$ are summable errors sequences in H. Moreover, they proved its weak convergence under some appropriate conditions in H. Further improvements of the convergence analysis of the method with summable errors were established in [20].

In 2011, Svaiter [22] studied the following DRA with summable errors. Let $\lambda > 0$, (x_0, b_0) such that $b_0 \in B(x_0)$, and for $k = 1, 2, \ldots$

(a) find (y_k, a_k) such that $a_k \in A(y_k)$, $\|y_k + \lambda a_k - (x_{k-1} - \lambda b_{k-1})\| \leq \alpha_k$;

(b) find (x_k, y_k) such that $b_k \in B(x_k)$, $\|x_k + \lambda b_k - (y_k + \lambda b_{k-1})\| \leq \beta_k$, where $\{\alpha_k\}$ and $\{\beta_k\}$ are the sequences of positive error tolerance such that $\sum_{k=1}^{\infty} \alpha_k < \infty$ and $\sum_{k=1}^{\infty} \beta_k < \infty$. Additionally, they proved the algorithm's weak convergence.

In 2015, Bot et al. [5] proposed an inertial DRA for inclusion problem (1.1). Their algorithm is generated by

$$\begin{cases} y_n = J_{\gamma B}[x_n + \alpha_n(x_n - x_{n-1})], \\ z_n = J_{\gamma A}[2y_n - x_n - \alpha_n(x_n - x_{n-1})], \\ x_{n+1} = x_n + \alpha_n(x_n - x_{n-1}) + \lambda_n(z_n - y_n), \end{cases}$$

where $\gamma > 0$, x_0, x_1 are arbitrarily chosen in H, and $\{\alpha_n\}$ is a nondecreasing sequence with $\alpha_1 = 0$ and $0 \leq \alpha_n \leq \alpha < 1$ for every $n \geq 1$. In addition, they also investigated the weak and strong convergence of $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ under some appropriate conditions.

In 2017, Bauschke and Moursi [2] proved the shadow sequence of DRA converges weakly to a best approximation problem. The shadow sequence was generated by $\{J_A T^n x\}_{n\in N}$, where $T := \frac{1}{2}(I + R_B R_A) = I - J_A + J_B R_A$, and $R_A = 2J_A - I$ is *reflected resolvent*. Specifically, their proof relied on a new convergence principle for Fejér monotone sequences.

In 2019, Wang and Wang [24] presented the α-DRA in a finite dimensional Euclidean space:

$$\begin{cases} y_n = J_B x_n, \\ z_n = J_A(\alpha y_n - x_n), \\ x_{n+1} = x_n + (z_n - y_n), \end{cases}$$

where $\alpha \in (1,2)$. Furthermore, the convergence of $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ was proved under some mild conditions.

In this paper, we introduce the Mann iterative DRA with a new error sequence in Hilbert space, and establish its weak convergence under some appropriate conditions. Furthermore, we propose the Halpern iteration of DRA with two different error sequences, and prove their strong convergences under some proper conditions. We also apply the conclusions to a feasibility problem. To solve the problem, we propose two DRAs with projection operators and give their weak convergence and strong convergences.

2. Preliminaries

Let $A : H \to 2^H$ be a set valued operator. Recall that A is said to be *monotone* if $\langle x - y, Ax - Ay \rangle \geq 0$ for all $x, y \in H$; and that A is *maximally monotone* if it is monotone and its graph cannot be extended without destroying monotonicity. An operator $T : H \to H$ is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$.

Let C be a nonempty, closed, and convex subset of H. Recall that the *projection* from H onto C, denoted by P_C, assigns $x \in H$ to the unique point $P_C x \in C$ with the property:

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

It is well known that $P_C x$ is characterized by the inequality:

$$P_C x \in C, \quad \langle x - P_C x, z - P_C x \rangle \leq 0, \quad \forall z \in C. \tag{2.1}$$

Definition 2.1. Let $T : H \to H$ be an operator. T is *firmly nonexpansive* if one of the following conditions is satisfied

(a) $2T - I$ is nonexpansive;
(b) $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle$ for all $x, y \in H$;
(c) $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2$ for all $x, y \in H$.

Next we collect several lemmas that are be used in the following part.

Lemma 2.2 ([14]). *Let C be a nonempty, closed, and convex subset of H, and $T : C \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)x_n \to 0$, then $(I - T)x = 0$, i.e., $x \in Fix(T)$.*

Lemma 2.3 ([17]). *Let $\{\alpha_n\}$ be a sequence in $[0, +\infty)$, let $\{\beta_n\}$ be a summable sequence in $[0, +\infty)$, and let $\{\varepsilon_n\}_{n \in N}$ be a summable sequence in $[0, +\infty)$ such that $(\forall n \in N)$ $\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + \varepsilon_n$. Then $\{\alpha_n\}$ converges.*

Lemma 2.4 ([6]). *Let C be a nonempty, closed, and convex subset of H, and let $\{x_n\}_{n \in N}$ be a sequence in H, which is quasi-Fejer monotone with respect to C, i.e., there exists a summable sequence $\{\varepsilon_n\}_{n \in N}$ in $[0, +\infty)$ such that*

$$(\forall x \in C)(\forall n \in N) \ \|x_{n+1} - x\| \leq \|x_n - x\| + \varepsilon_n.$$

Then

(i) $\{x_n\}_{n\in N}$ *is bounded;*
(ii) $\{x_n\}_{n\in N}$ *converges weakly to a point in* C *if and only if* $w(x_n)_{n\in N} \subset C$, *where* $w(x_n)_{n\in N}$ *denotes the set of weak cluster points of* $\{x_n\}_{n\in N}$.

We can find more properties of quasi-Fejer monotonicity in [9], and the proof of (ii) can be obtained in Proposition 3.2 (i) and Theorem 3.8 in [9].

Lemma 2.5 ([3])**.** *Let* m *be an integer such that* $m \geq 2$, *set* $I = \{1,2,...m\}$, *and let* $\{A_i\}_{i\in I}$ *be maximally monotone operators from* H *to* 2^H. *For every* $i \in I$, *let* $\{x_{i,n}, u_{i,n}\}_{n\in N}$ *be a sequence in* $graA_i$ *and let* $\{x_i, u_i\} \in H \times H$. *Suppose that*

$$\sum_{i\in I} u_{i,n} \to 0 \text{ and } (\forall i \in I) \begin{cases} x_{i,n} \rightharpoonup x_i, \\ u_{i,n} \rightharpoonup u_i, \\ mx_{i,n} - \sum_{j\in I} x_{j,n} \to 0. \end{cases}$$

Then there exists $x \in zer\sum_{i\in I} A_i$ *such that the following hold:*

(i) $x = x_1 = x_2 = ... = x_m$;
(ii) $\sum_{i\in I} u_i = 0$;
(iii) $(\forall i \in I)\ (x, u_i) \in graA_i$;
(iv) $\sum_{i\in I} \langle x_{i,n}, u_{i,n}\rangle \to \langle x, \sum_{i\in I} u_i\rangle = 0$.

Lemma 2.6 ([3])**.** *Let* $\{x_n\}_{n\in N}$ *be a sequence in* H. *Then* $\{x_n\}_{n\in N}$ *converges weakly if and only if it is bounded and possesses at most one weak sequential cluster point.*

Lemma 2.7 ([25])**.** *Assume that* $\{s_k\}_{k=0}^{\infty}$ *is a sequence of nonnegative real numbers such that* $s_{k+1} \leq (1-\lambda_k)s_k + \lambda_k b_k + c_k$, *where* $\{\lambda_k\}, \{b_k\}$, *and* $\{c_k\}$ *satisfy the conditions:*

(i) $\lim_{k\to\infty} \lambda_k = 0$, $\sum_{k=0}^{\infty} \lambda_k = \infty$;
(ii) *either* $\limsup_{k\to\infty} b_k \leq 0$ *or* $\sum_{k=0}^{\infty} |\lambda_k b_k| < \infty$;
(iii) $c_k \geq 0$ *for all* k *and* $\sum_{k=0}^{\infty} c_k < \infty$.

Then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.8. *Assume* $x, y \in H$, *and* $\alpha \in R$. *Then*

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$;
(ii) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$.

Lemma 2.9 ([3])**.** *Let* $A : H \to 2^H$, *let* $B : H \to 2^H$, *and let* $\gamma > 0$. *Suppose that* A *and* B *are monotone. Then* $zer(A+B) = J_{\gamma B}(\mathrm{Fix}(\mathrm{R_{\gamma A}R_{\gamma B}})) = \mathrm{J_{\gamma B}(FixT)}$, *where* $R_{\gamma A} = 2J_{\gamma A} - I$, *and* $T = J_{\gamma A}(2J_{\gamma B} - I) + I - J_{\gamma B}$.

Lemma 2.10. *Let* $T = J_{\gamma A}(2J_{\gamma B} - I) + I - J_{\gamma B}$, *where* $\gamma > 0$. *Then* T *is firmly nonexpansive.*

3. The Weak Convergence

In this section, we propose the Mann iteration of DRA with a new error sequence. To this end, we assume that problem (1.1) is consistent, namely, its solution set is nonempty. We prove DRA's weak convergence under some proper conditions. First, we give a lemma, which will be used in proofs of the weak and strong convergence of DRA.

Lemma 3.1. *Let $\eta \in (0,1/2)$, $x,e \in H$, and $\tilde{x} = T(x+e)$, where $T = J_{\gamma A}(2J_{\gamma B} - I) + I - J_{\gamma B}$. If $\|e\| \leq \eta\|x-\tilde{x}\|$, then $\|\tilde{x}-z\|^2 \leq (1+(2\eta)^2)\|x-z\|^2 - \frac{1}{2}\|\tilde{x}-x\|^2$, $\forall z \in FixT$.*

Proof. Since $z \in FixT$ and T is firmly nonexpansive,

$$\begin{aligned}\|\tilde{x}-z\|^2 \leq& \|x+e-z\|^2 - \|T(x+e)-x-e\|^2\\ =&\|x-z\|^2 + \|e\|^2 + 2\langle x-z,e\rangle - \|\tilde{x}-x-e\|^2\\ =&\|x-z\|^2 - \|\tilde{x}-x\|^2 + 2\langle \tilde{x}-z,e\rangle.\end{aligned} \tag{3.1}$$

Using inequality $2\langle a,b\rangle \leq 2\eta^2\|a\|^2 + \|b\|^2/2\eta^2$, we have

$$2\langle \tilde{x}-z,e\rangle \leq 2\eta^2\|\tilde{x}-z\|^2 + \|e\|^2/2\eta^2.$$

Subsisting the last inequality into (3.1) and noting $\|e\| \leq \eta\|\tilde{x}-x\|$, we see that

$$\begin{aligned}\|\tilde{x}-z\|^2 \leq& \|x-z\|^2 - \|\tilde{x}-x\|^2 + 2\eta^2\|\tilde{x}-z\|^2 + \|e\|^2/2\eta^2\\ \leq& \|x-z\|^2 + 2\eta^2\|\tilde{x}-z\|^2 - \frac{1}{2}\|\tilde{x}-x\|^2.\end{aligned}$$

It follows that

$$\|\tilde{x}-z\|^2 \leq (1+\frac{2\eta^2}{1-2\eta^2})\|x-z\|^2 - \frac{1}{2(1-2\eta^2)}\|\tilde{x}-x\|^2.$$

Consequently, we obtain the desired result from the fact $\eta \in (0,1/2)$. □

Theorem 3.2. *Let A and B be maximally monotone operators from H to 2^H such that $zer(A+B) \neq \emptyset$, and let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that*

$$\sum_{n=0}^{\infty} \alpha_n(\frac{3}{2} - \alpha_n) = \infty. \tag{3.2}$$

Let $e_n \in H$, $\eta_n > 0$, and $T = J_{\gamma A}(2J_{\gamma B} - I) + I - J_{\gamma B}$,

$$\|e_n\| \leq \eta_n\|T(x_n+e_n) - x_n\|, \text{ and } \sum_{n=0}^{\infty} \eta_n < \infty. \tag{3.3}$$

For any $x_0 \in H$, $\gamma \in R_{++}$,

$$\begin{cases} y_n = J_{\gamma B}(x_n+e_n)\\ z_n = J_{\gamma A}(2y_n - x_n - e_n)\\ x_{n+1} = \alpha_n(z_n - y_n + e_n) + x_n.\end{cases}$$

Then there exists $x \in \Omega = FixT$ such that the following hold:

(i) *$\{x_n\}$ converges weakly to x;*
(ii) *$\{y_n - z_n\}_{n\in N}$ converges strongly to 0;*
(iii) *$\{y_n\}$ converges weakly to $J_{\gamma B}x$, which is a solution to problem (1.1);*
(iv) *$\{z_n\}$ converges weakly to $J_{\gamma B}x$, which is a solution to problem (1.1).*

Proof. By the definition of T, x_{n+1} can be rewritten as $x_{n+1} = \alpha_n T(x_n + e_n) + (1-\alpha_n)x_n$. For any $z \in FixT = FixR_{\gamma A}R_{\gamma B}$, Lemma 2.8 (ii) and Lemma 3.1 yield that

$$\begin{aligned}
\|x_{n+1}-z\|^2 =& \alpha_n\|T(x_n+e_n)-z\|^2 + (1-\alpha_n)\|x_n-z\|^2 \\
& -\alpha_n(1-\alpha_n)\|T(x_n+e_n)-x_n\|^2 \\
\leq& \alpha_n[(1+(2\eta_n)^2)\|x_n-z\|^2 - \frac{1}{2}\|T(x_n+e_n)-x_n\|^2] \\
& +(1-\alpha_n)\|x_n-z\|^2 - \alpha_n(1-\alpha_n)\|T(x_n+e_n)-x_n\|^2 \\
=& (1+4\eta_n^2\alpha_n)\|x_n-z\|^2 - \alpha_n(\frac{3}{2}-\alpha_n)\|T(x_n+e_n)-x_n\|^2.
\end{aligned} \tag{3.4}$$

Since $\alpha_n \in (0,1)$, we have

$$\|x_{n+1}-z\|^2 \leq (1+4\eta_n^2)\|x_n-z\|^2. \tag{3.5}$$

From (3.3), we have $\lim\limits_{n\to\infty}\eta_n = 0$. Without loss generality, we assume that $\eta_n \in (0,1/2)$, which together with (3.3) yields $\sum_{n=0}^{\infty} 4\eta_n^2 < \infty$. According to Lemma 2.3, $\lim_{n\to\infty}\|x_n - z\|$ exists. Hence, $\{x_n\}$ is bounded. Combining the boundedness of $\{x_n\}$ with inequality (3.4), we have

$$\begin{aligned}
\alpha_n(\frac{3}{2}-\alpha_n)\|T(x_n+e_n)-x_n\|^2 \leq& \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + 4\eta_n^2\alpha_n\|x_n-z\|^2 \\
\leq& \|x_n-z\|^2 - \|x_{n+1}-z\|^2 + 4\eta_n^2 M,
\end{aligned}$$

where $M > 0$ is a sufficient large number. Therefore,

$$\sum_{n=0}^{\infty}\alpha_n(\frac{3}{2}-\alpha_n)\|T(x_n+e_n)-x_n\|^2 < \infty.$$

From (3.2), we can immediately obtain

$$\liminf_{n\to\infty}\|T(x_n+e_n)-x_n\| = 0. \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}
&\|T(x_{n+1}+e_{n+1})-x_{n+1}\| \\
&\leq \|x_{n+1}-x_n+e_{n+1}-e_n\| + (1-\alpha_n)\|T(x_n+e_n)-x_n\| \\
&\leq \alpha_n\|T(x_n+e_n)-x_n\| + (1-\alpha_n)\|T(x_n+e_n)-x_n\| + \|e_{n+1}-e_n\| \\
&\leq \|T(x_n+e_n)-x_n\| + \|e_{n+1}\| + \|e_n\| \\
&\leq \|T(x_n+e_n)-x_n\| + \eta_{n+1}\|T(x_{n+1}+e_{n+1})-x_{n+1}\| + \eta_n\|T(x_n+e_n)-x_n\| \\
&\leq \|T(x_n+e_n)-x_n\| + M_1(\eta_{n+1}+\eta_n),
\end{aligned}$$

where the last inequality bases on Lemma 3.1 and the boundedness of $\{x_n\}$, and M_1 satisfies $\max\{\|T(x_{n+1}+e_{n+1})-x_{n+1}\|, \|T(x_n+e_n)-x_n\|\} \leq M_1$. By (3.3), we have $\sum_{n=0}^{\infty} M_1(\eta_n + \eta_{n+1}) < \infty$. By Lemma 2.3, the limit of sequence $\{\|T(x_n+e_n)-x_n\|\}_{n\in N}$ exists, which together with (3.6) finds

$$\|T(x_n+e_n)-x_n\| \to 0. \tag{3.7}$$

as $n \to \infty$. Observe that

$$\begin{aligned}\|Tx_n - x_n\| =&\|Tx_n - T(x_n+e_n) + T(x_n+e_n) - x_n\|\\ \leq&\|Tx_n - T(x_n+e_n)\| + \|T(x_n+e_n) - x_n\|\\ \leq&\|e_n\| + \|T(x_n+e_n) - x_n\|\\ \leq&\eta_n\|T(x_n+e_n) - x_n\| + \|T(x_n+e_n) - x_n\| \to 0. \end{aligned} \tag{3.8}$$

If, in addition, $x_{n_k} \rightharpoonup x^*$, then it follows from Lemma 2.2 that $Tx^* = x^*$. Thus $w(x_n) \subset FixT$. By (3.5), we have $\|x_{n+1} - z\|^2 \leq (1+2\eta_n)^2\|x_n - z\|^2$ for all $z \in FixT$. Since $\{x_n\}$ is bounded, there exists $M_2 > 0$ such that

$$\|x_{n+1} - z\| \leq (1+2\eta_n)\|x_n - z\| \leq \|x_n - z\| + 2M_2\eta_n,$$

where M_2 satisfies $\|x_n - z\| \leq M_2$. According to (3.3), $\sum_{n=0}^{\infty} 2M_2\eta_n < \infty$. It follows from Lemma 2.4 (ii) that $\{x_n\}$ converges weakly to x, which is a point of $FixT$.

(ii) By (3.7), we can deduce that

$$\begin{aligned}\|y_n - z_n\| =&\|J_{\gamma A}(2J_{\gamma B} - I)(x_n+e_n) - J_{\gamma B}(x_n+e_n) + x_n + e_n - x_n - e_n\|\\ \leq&\|T(x_n+e_n) - x_n\| + \|e_n\|\\ \leq&\|T(x_n+e_n) - x_n\| + \eta_n\|T(x_n+e_n) - x_n\| \to 0. \end{aligned} \tag{3.9}$$

(iii) Setting $v_n = x_n - y_n + e_n$ and $w_n = 2y_n - x_n - z_n - e_n$, we have

$$\begin{cases}(z_n, w_n) \in gra\gamma A,\\ (y_n, v_n) \in gra\gamma B,\\ v_n + w_n = y_n - z_n.\end{cases} \tag{3.10}$$

Since $J_{\gamma B}$ is nonexpansive, we have

$$\begin{aligned}\|y_n - y_0\| \leq&\|x_n - x_0\| + \|e_n\| + \|e_0\|\\ \leq&\|x_n - x_0\| + \eta_n\|T(x_n+e_n) - x_n\| + \|e_0\|.\end{aligned}$$

According to the boundedness of $\{x_n\}$ and (3.7), we have that $\{y_n\}$ is bounded. Now let y be a weak sequential cluster point of $\{y_n\}$, say $y_{n_k} \rightharpoonup y$. From the definition of v_n and w_n, we find from (i) and (ii) that

$$y_{n_k} \rightharpoonup y,\ z_{n_k} \rightharpoonup y,\ v_{n_k} \rightharpoonup x - y, \text{ and } w_{n_k} \rightharpoonup y - x.$$

(3.10), (ii), and Lemma 2.5 yield

$$y \in zer(\gamma A + \gamma B) = zer(A+B),\ (y, y-x) \in gra\gamma A, \text{ and } (y, x-y) \in gra\gamma B.$$

Hence, $y = J_{\gamma B}x$ and $y \in domA$. Thus $J_{\gamma B}x$ is the unique weak sequential cluster point of $\{y_n\}$. From Lemma 2.6, we conclude that $y_n \rightharpoonup J_{\gamma B}x$. Moreover, $J_{\gamma B}x$ is a solution of problem (1.1) by Lemma 2.9.

(iv) According to (ii) and (iii), we obtain the conclusion immediately. □

Remark 3.3. Theorem 3.2 shows the weak convergence of DRA with a new error sequence, which generalizes Corollary 5.2 of [6] and Theorem 2.1 of [20].

4. The Strong Convergence

We start with some auxiliary results.

Lemma 4.1 ([19]). *Given $\beta > 0$, let $\{s_n\}$ be a nonnegative real sequence satisfying $s_{n+1} \leq (1-\lambda_n)(1+\varepsilon_n)s_n + \lambda_n\beta$, where $\{\lambda_n\} \subset (0,1)$ and $\{\varepsilon_n\} \in l_1$ are real sequences. Then $\{s_n\}$ is bounded. More precisely, $s_n \leq \max\{\beta, s_0\} \exp(\sum_{n=0}^{\infty} \varepsilon_n) < \infty$.*

Lemma 4.2 ([19]). *Given $\beta > 0$, let $\{s_n\}$ be a nonnegative real sequence satisfying $s_{n+1} \leq (1-\lambda_n)(1+\varepsilon_n)s_n + \lambda_n\beta$, where $\{\lambda_n\} \subset (0,1)$ and $\{\varepsilon_n\} \subset [0,\infty)$ are real sequences. If $2\varepsilon_n(1-\lambda_n) \leq \lambda_n$, then $\{s_n\}$ is bounded. More precisely, $s_n \leq \max\{2\beta, s_0\} < \infty$.*

In the following, we give the Halpern iteration of DRA: for $x_0, u \in H$, and $\gamma \in R_{++}$,

$$\begin{cases} y_n = J_{\gamma B}(x_n + e_n) \\ z_n = J_{\gamma A}(2y_n - x_n - e_n) \\ x_{n+1} = \lambda_n u + (1-\lambda_n)(z_n - y_n + x_n + e_n). \end{cases} \tag{4.1}$$

We are now in a position to state two strong convergence results of algorithm (4.1) under two different error sequences.

Theorem 4.3. *Let A and B be maximally monotone operators from H to 2^H such that $zer(A+B) \neq \emptyset$, and let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that*

$$\lim_{n\to\infty} \lambda_n = 0, \ \sum_{n=0}^{\infty} \lambda_n = \infty. \tag{4.2}$$

Let $e_n \in H$, $\eta_n > 0$, $T = J_{\gamma A}(2J_{\gamma B} - I) + I - J_{\gamma B}$, and

$$\|e_n\| \leq \eta_n \|T(x_n + e_n) - x_n\|, \text{ and let } \sum_{n=0}^{\infty} \eta_n^2 < \infty. \tag{4.3}$$

Then, for algorithm (4.1), there exists $z \in \Omega = FixT$ such that the following hold:

(i) *$\{x_n\}$ converges strongly to z;*
(ii) *$\{y_n - z_n\}_{n\in N}$ converges strongly to 0;*
(iii) *$\{y_n\}$ converges strongly to $J_{\gamma B}z$, which is a solution of problem (1.1);*
(iv) *$\{z_n\}$ converges strongly to $J_{\gamma B}z$, which is a solution of problem (1.1).*

Proof. By the definition of T, x_{n+1} can be rewritten as $x_{n+1} = \lambda_n u + (1-\lambda_n)T(x_n + e_n)$. Let $z = P_\Omega u$. For convenient, we denote $u_n = T(x_n + e_n)$, and

$$x_{n+1} = \lambda_n u + (1-\lambda_n)u_n. \tag{4.4}$$

By (4.3), we may assume without loss of generality that $\eta_n \in (0, 1/2)$. Thus Lemma 3.1 implies

$$\|u_n - z\|^2 \leq (1+\varepsilon_n)\|x_n - z\|^2 - \frac{1}{2}\|u_n - x_n\|^2, \tag{4.5}$$

where $\varepsilon_n := (2\eta_n)^2$ satisfies $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. It then follows from algorithm (4.4) that

$$\|x_{n+1} - z\|^2 \leq (1-\lambda_n)\|u_n - z\|^2 + \lambda_n\|u - z\|^2,$$

which together with (4.5) yields

$$\|x_{n+1}-z\|^2 \leq (1-\lambda_n)(1+\varepsilon_n)\|x_n-z\|^2+\lambda_n\|u-z\|^2.$$

Applying Lemma 4.1 to the last inequality, we conclude that $\{x_n\}$ is bounded. It follows from Lemma 2.8 that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|(1-\lambda_n)(u_n-z)+\lambda_n(u-z)\|^2\\ &\leq (1-\lambda_n)\|u_n-z\|^2+2\lambda_n\langle u-z, x_{n+1}-z\rangle.\end{aligned}$$

Combining the last inequality with (4.5) yields

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq&(1-\lambda_n)\|x_n-z\|^2-\frac{1-\lambda_n}{2}\|u_n-x_n\|^2\\ &+2\lambda_n\langle u-z,x_{n+1}-z\rangle+M\varepsilon_n\\ =&(1-\lambda_n)\|x_n-z\|^2+\lambda_n[-\frac{1-\lambda_n}{2\lambda_n}\|u_n-x_n\|^2\\ &+2\langle u-z,x_{n+1}-z\rangle]+M\varepsilon_n,\end{aligned} \tag{4.6}$$

where M is a sufficient largely number. Set $s_n := \|x_n-z\|^2$ and

$$b_n := -\frac{1-\lambda_n}{2\lambda_n}\|u_n-x_n\|^2+2\langle u-z,x_{n+1}-z\rangle.$$

Then we can rewrite the above inequality as

$$s_{n+1} \leq (1-\lambda_n)s_n+\lambda_n b_n+M\varepsilon_n. \tag{4.7}$$

Next we prove $-\delta \leq \limsup_{n\to\infty} b_n < +\infty$ for some $\delta > 0$, which indicates that $\limsup_{n\to\infty} b_n$ is finite. On the one hand, since $\{x_n\}$ is bounded, we have

$$\sup b_n \leq \sup 2\langle u-z,x_{n+1}-z\rangle \leq 2\sup\|u-z\|\|x_{n+1}-z\| < +\infty.$$

On the other hand, we show $\limsup_{n\to\infty} b_n > -\delta$. To this aim, we proceed by contradiction. Assume that $\limsup_{n\to\infty} b_n < -\delta$, which implies that there exists $n_0 \in N$ such that $b_n \leq -\delta$ for all $n \geq n_0$. It follows from that

$$\begin{aligned}s_{n+1} &\leq (1-\lambda_n)s_n+\lambda_n b_n+M\varepsilon_n\\ &\leq (1-\lambda_n)s_n-\lambda_n\delta+M\varepsilon_n\\ &\leq s_n-\lambda_n\delta+M\varepsilon_n.\end{aligned}$$

Hence $s_{n+1} \leq s_{n_0}-\delta\sum_{i=n_0}^{n}\lambda_n+M\sum_{i=n_0}^{n}\varepsilon_n$, which implies that $\limsup_{n\to\infty} s_{n+1} < -\infty$. As a matter of fact, $\{s_n\}$ is a nonnegative real sequence, which is a contradiction. Therefore, $\limsup\limits_{n\to\infty} b_n$ is finite.

Finally, we show that $\{x_n\}$ converges to $z = P_\Omega u$. By the above proof, we first take a subsequence $\{n_k\}$ such that

$$\limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty}[-\frac{1-\lambda_{n_k}}{2\lambda_{n_k}}\|u_{n_k}-x_{n_k}\|^2+2\langle u-z,x_{n_k+1}-z\rangle]. \tag{4.8}$$

Since $\{2\langle u-z,x_{n_k+1}-z\rangle\}$ is a bounded sequence for every n, we may assume that without loss of generality there exists the limit $\lim_{k\to\infty}\langle u-z,x_{n_k+1}-z\rangle$. Combine the limit with (4.8), we obtain the following limit exists $\lim_{k\to\infty}\frac{1-\lambda_{n_k}}{\lambda_{n_k}}\|u_{n_k}-x_{n_k}\|^2$. On the other hand, by (4.2), we have

$\lim_{n\to\infty} \frac{\lambda_n}{1-\lambda_n} = 0$. It follows that the limit of sequence $\{\|u_{n_k} - x_{n_k}\|\}$ exists, and $\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0$. It is similar with inequality (3.8) that $\|Tx_{n_k} - x_{n_k}\| \le (1+\eta_{n_k})\|u_{n_k} - x_{n_k}\| \to 0$, as $n \to \infty$, which shows that $\lim_{k\to\infty} \|Tx_{n_k} - x_{n_k}\| = 0$. It follows from Lemma 2.2 that any weak cluster point of $\{x_{n_k}\}$ belongs to Ω.

Next the definition of $\{x_{n_k+1}\}$ yields that

$$\begin{aligned} \|x_{n_k+1} - x_{n_k}\| &= \|\lambda_{n_k}(u - x_{n_k}) + (1-\lambda_{n_k})(u_{n_k} - x_{n_k})\| \\ &\le \lambda_{n_k}\|u - x_{n_k}\| + (1-\lambda_{n_k})\|u_{n_k} - x_{n_k}\| \to 0, \end{aligned} \tag{4.9}$$

so any weak cluster point of $\{x_{n_k+1}\}$ also belongs to Ω. Without loss of generality, we assume that $\{x_{n_k+1}\}$ converges weakly to x^*. Then $x^* \in \Omega$. Now by (4.8), (2.1), and $z = P_\Omega u$, we obtain that

$$\limsup_{n\to\infty} b_n \le \lim_{k\to\infty} 2\langle u - z, x_{n_k+1} - z\rangle = 2\langle u - z, x^* - z\rangle \le 0.$$

Consequently, we apply Lemma 2.7 to (4.7) to obtain that $\|x_n - z\| \to 0$.

(ii) By the proof of (i), we have $\|x_n - z\| \to 0$ with $z = P_\Omega u$. It follows that

$$\begin{aligned} \|T(x_n + e_n) - x_n\| &\le \|T(x_n + e_n) - z\| + \|x_n - z\| \\ &\le 2\|x_n - z\| + \|e_n\| \\ &\le 2\|x_n - z\| + \eta_n\|T(x_n + e_n) - x_n\|. \end{aligned}$$

Furthermore, we see that $\|T(x_n + e_n) - x_n\| \le \frac{2}{1-\eta_n}\|x_n - z\| \to 0$ as $n \to \infty$. Hence $\|T(x_n + e_n) - x_n\| \to 0$ as $n \to \infty$. This together with (3.9) implies $\|y_n - z_n\| \le (1+\eta_n)\|T(x_n + e_n) - x_n\| \to 0$. as $n \to \infty$.

(iii) According to the result $\|x_n - z\| \to 0$, we have

$$\begin{aligned} \|y_n - J_{\gamma B} z\| &\le \|x_n + e_n - z\| \\ &\le \|x_n - z\| + \|e_n\| \\ &\le \|x_n - z\| + \eta_n\|T(x_n + e_n) - x_n\| \to 0. \end{aligned}$$

So $\|y_n - J_{\gamma B} z\| \to 0$. Furthermore, $\{y_n\}$ converges strongly to a solution of problem (1.1) by Lemma 2.9.

(iv) Combining (ii) and (iii), we have the desired result immediately. □

Theorem 4.4. *If the condition* (4.3) *of Theorem 4.3 is substituted the following condition*

$$\|e_n\| \le \eta_n\|T(x_n + e_n) - x_n\|, \text{ and } \lim_{n\to\infty} \eta_n^2/\lambda_n = 0, \tag{4.10}$$

then the conclusions of Theorem 4.3 are still valid.

Proof. Let $z \in P_\Omega u$. Similar to Theorem 4.3, we have

$$\|x_{n+1} - z\|^2 \le (1-\lambda_n)(1+\varepsilon_n)\|x_n - z\|^2 + \lambda_n\|u - z\|^2,$$

where $\varepsilon_n := (2\eta_n)^2$ satisfies $\varepsilon_n/\lambda_n \to 0$. Assume without loss of generality that $2\varepsilon_n(1-\lambda_n) \le \lambda_n$. Applying Lemma 4.2, we conclude that $\{x_n\}$ is bounded. According to (4.6), we have

$$\begin{aligned} \|x_{n+1} - z\|^2 \le &(1-\lambda_n)\|x_n - z\|^2 + \lambda_n[-\frac{1-\lambda_n}{2\lambda_n}\|u_n - x_n\|^2 \\ &+ 2\langle u - z, x_{n+1} - z\rangle + M\frac{\varepsilon_n}{\lambda_n}], \end{aligned} \tag{4.11}$$

where M is a sufficient largely number. Set $s_n := \|x_n - z\|^2$ and

$$b_n := -\frac{1-\lambda_n}{2\lambda_n}\|u_n - x_n\|^2 + 2\langle u - z, x_{n+1} - z\rangle + M\frac{\varepsilon_n}{\lambda_n}.$$

Then we can rewrite inequality (4.11) as

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n b_n. \tag{4.12}$$

Next we prove $-\delta \le \limsup_{n\to\infty} b_n < +\infty$ for some $\delta > 0$, which indicates that $\limsup_{n\to\infty} b_n$ is finite. On the one hand, since $\{x_n\}$ is bounded, we have

$$\sup b_n \le \sup 2\langle u - z, x_{n+1} - z\rangle + M\frac{\varepsilon_n}{\lambda_n} \le 2\sup\|u - z\|\|x_{n+1} - z\| + M < +\infty.$$

On the other hand, we show that $\limsup_{n\to\infty} b_n > -\delta$. To this aim, we proceed by contradiction. Assume that $\limsup_{n\to\infty} b_n < -\delta$, which implies that there exists $n_0 \in N$ such that $b_n \le -\delta$ for all $n \ge n_0$. It follows from that

$$s_{n+1} \le (1-\lambda_n)s_n + \lambda_n b_n \le (1-\lambda_n)s_n - \lambda_n\delta \le s_n - \lambda_n\delta.$$

Hence $s_{n+1} \le s_{n_0} - \delta\sum_{i=n_0}^{n}\lambda_n$, and we also have $\limsup_{n\to\infty} s_{n+1} < s_{n_0} - \delta\sum_{i=n_0}^{\infty}\lambda_n = -\infty$. As a matter of fact, $\{s_n\}$ is a nonnegative real sequence, which is a contradiction. Thus $\limsup_{n\to\infty} b_n$ is finite.

Finally, we prove that $\{x_n\}$ converges strongly to $z = P_\Omega u$. From the above, we first take a subsequence $\{n_k\}$ such that

$$\limsup_{n\to\infty} b_n = \lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty}[-\frac{1-\lambda_{n_k}}{2\lambda_{n_k}}\|u_{n_k} - x_{n_k}\|^2 + 2\langle u - z, x_{n_k+1} - z\rangle + M\frac{\varepsilon_{n_k}}{\lambda_{n_k}}]. \tag{4.13}$$

Since $\{2\langle u - z, x_{n_k+1} - z\rangle\}$ is a bounded sequence for every n, we may assume that without loss of generality there exists the limit $\lim_{k\to\infty}\langle u - z, x_{n_k+1} - z\rangle$. Combining (4.10) with (4.13), we obtain the following limit exists $\lim_{k\to\infty}\frac{1-\lambda_{n_k}}{\lambda_{n_k}}\|u_{n_k} - x_{n_k}\|^2$.

On the other hand, by (4.2), we have $\lim_{n\to\infty}\frac{\lambda_n}{1-\lambda_n} = 0$. It follows that the limit of sequence $\{\|u_{n_k} - x_{n_k}\|\}$ exists, and $\lim_{k\to\infty}\|u_{n_k} - x_{n_k}\| = 0$. It is similar with inequality (3.8) that $\|Tx_{n_k} - x_{n_k}\| \le (1+\eta_{n_k})\|u_{n_k} - x_{n_k}\| \to 0$ as $n \to \infty$. The last inequality also based on $\lim_{n\to\infty}\eta_n = 0$, which can be obtained from (4.2) and (4.10). Thus $\lim_{k\to\infty}\|Tx_{n_k} - x_{n_k}\| = 0$. It follows from Lemma 2.2 that any weak cluster point of $\{x_{n_k}\}$ belongs to Ω.

Similar to (4.9), we can obtain that any weak cluster point of $\{x_{n_k+1}\}$ also belongs to Ω. Without loss of generality, we assume that $\{x_{n_k+1}\}$ converges weakly to x^*. Then $x^* \in \Omega$. Now by (4.13), (2.1), and $z = P_\Omega u$, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} b_n &\le \lim_{k\to\infty}(2\langle u - z, x_{n_k+1} - z\rangle + M\frac{\varepsilon_{n_k}}{\lambda_{n_k}})\\ &= 2\langle u - z, x^* - z\rangle \le 0.\end{aligned}$$

Consequently, we can apply Lemma 2.7 to (4.12) to obtain $\|x_n - z\| \to 0$.

(ii)-(iv) The proofs are similar to Theorem 4.3 (ii)-(iv), so they are omitted. □

Remark 4.5. Theorem 4.3 and Theorem 4.4 present strong convergences of Halpern DRA under two different new error sequences, which generalize the results of [5] and [24].

5. Applications

Consider the following feasibility problem,

$$\text{find } x \in C \cap D, \tag{5.1}$$

where C and D are nonempty closed convex subsets of Hilbert space H. Assume that $C \cap D$ is nonempty. To solve feasibility problem (5.1), Bauschke and Noll [4] studied the classical DRA:

$$x_{n+1} \in T(x_n),\ T := \frac{1}{2}(R'_D R'_C + I), n \in N, \tag{5.2}$$

$R'_C = 2P_C - I$. In addition, they proved (5.2) converges to a fixed point of T.

Aragón Artacho, Campoy, and Tam [1] further investigated the following DRA in a Euclidean space:

$$x_{k+1} = T_{C,D}(x_k), \text{with } T_{C,D} = \frac{I + R'_D R'_C}{2}. \tag{5.3}$$

They discussed the convergence of algorithm (5.3) under different conditions.

In this section, we propose Mann and Halpern iteration of DRA with projection operators to solve problem (5.1). We further show the weak convergence of Mann iteration, and the strong convergence of Halpern iteration under two different error sequences, respectively. We first give a lemma about the solution set of problem (5.1).

Lemma 5.1. *Let C and D be nonempty closed and convex subsets of Hilbert space. Then $zer(N_C + N_D) = C \cap D = P_D(FixT')$, where N_C and N_D are normal cone to C and D, and $T' = P_C(2P_D - I) + I - P_D$.*

Proof. For any $x \in FixT'$, we have that $FixT'$ is equivalent to $P_C(2P_D x - x) = P_D x$. Hence, $P_D x \in C \cap D$, and we can immediately obtain $P_D(FixT') \subset C \cap D$.

Next, we prove $C \cap D \subset P_D(FixT')$. For any $x \in C \cap D$, we have $P_C(2P_D x - x) = P_D x = x$, which implies that $x \in FixT'$ and $x = P_D x \in P_D(FixT')$. So we can obtain $C \cap D \subset P_D(FixT')$. Thus $P_D(FixT') = C \cap D$. According to Lemma 2.9, the conclusion can be easily obtained. □

Letting $A = N_C$ and $B = N_D$ in Theorem 3.2, Theorem 4.3, and Theorem 4.4, we obtain from Lemma 5.1 Theorem 5.2, Theorem 5.4, and Theorem 5.5.

Theorem 5.2. *Let C and D be nonempty, closed, and convex subsets of Hilbert space H. Let $\{\alpha_n\}$ be a sequence in $[0,1]$ such that $\sum_{n=0}^{\infty} \alpha_n(\frac{3}{2} - \alpha_n) = \infty$. Let $e_n \in H$, $\eta_n > 0$, $T' = P_C(2P_D - I) + I - P_D$, $\|e_n\| \le \eta_n \|T'(x_n + e_n) - x_n\|$, and $\sum\limits_{n=0}^{\infty} \eta_n < \infty$. For $x_0 \in H$,*

$$\begin{cases} y_n = P_D(x_n + e_n), \\ z_n = P_C(2y_n - x_n - e_n), \\ x_{n+1} = \alpha_n(z_n - y_n + e_n) + x_n. \end{cases}$$

Then there exists $x \in FixT'$ such that the following hold:

(i) *$\{x_n\}$ converges weakly to x;*
(ii) *$\{y_n - z_n\}_{n \in N}$ converges strongly to 0;*
(iii) *$\{y_n\}$ converges weakly to $P_D x$, which is a solution of problem (5.1);*
(iv) *$\{z_n\}$ converges weakly to $P_D x$, which is a solution of problem (5.1).*

Remark 5.3. Theorem 5.2 presentes the weak convergence of DRA with a new error sequence, which generalizes the result of [1] and [4].

Theorem 5.4. *Let C and D be nonempty, closed, and convex subsets of Hilbert space H, and let $\{\lambda_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty}\lambda_n = 0$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$. Let $e_n \in H$, $\eta_n > 0$, $T' = P_C(2P_D - I) + I - P_D$, and*

$$\|e_n\| \leq \eta_n \|T'(x_n + e_n) - x_n\|, \text{ and } \sum_{n=0}^{\infty}\eta_n^2 < \infty. \tag{5.4}$$

For $x_0, u \in H$, $\gamma \in R_{++}$,

$$\begin{cases} y_n = P_D(x_n + e_n), \\ z_n = P_C(2y_n - x_n - e_n), \\ x_{n+1} = \lambda_n u + (1-\lambda_n)(z_n - y_n + x_n + e_n). \end{cases}$$

Then there exists $z \in \Omega = FixT'$ such that the following hold:

(i) *$\{x_n\}$ converges strongly to z;*
(ii) *$\{y_n - z_n\}_{n\in N}$ converges strongly to 0;*
(iii) *$\{y_n\}$ converges strongly to $P_D z$, which is a solution of problem (5.1);*
(iv) *$\{z_n\}$ converges strongly to $P_D z$, which is a solution of problem (5.1).*

Theorem 5.5. *If the condition (5.4) of Theorem 5.4 is substituted by the following condition $\|e_n\| \leq \eta_n \|T'(x_n + e_n) - x_n\|$ and $\lim_{n\to\infty}\eta_n^2/\lambda_n = 0$, then the conclusions of Theorem 5.4 are still valid.*

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