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# MK VISCOSITY APPROXIMATION FOR FIXED POINTS AND EQUILIBRIUM PROBLEMS IN HILBERT SPACES

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**Abstract.** In this paper, an MK viscosity iteration is introduced and investigated for solving an equilibrium problem and a fixed point problem with a nonexpansive operator. A theorem of strong convergence is established in the setting of Hilbert spaces. There is no any compact restriction imposed on the operator and the subset involved.

**Keywords.** Approximate solution; Equilibrium problem; Fixed point; Viscosity method.

## 1. Introduction

Let $H$ be a Hilbert space with inner $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\mathscr{B}$ be a bifunction from $S\times S$ to $\mathbb{R}$ with $\mathscr{B}(x,x)=0$, $\forall x\in S$, where $S$ is a convex, closed, and nonempty subset of space $H$. By an equilibrium problem, as understood by Blum and Oettli [1] is the problem, which consists of finding

$$\bar{x}\in S \text{ such that } \mathscr{B}(\bar{x},y)\geq 0,\quad \forall y\in S.$$

The equilibrium problem is quite general, and its solution set is presented by $Sol(\mathscr{B},S)$ in this paper. For example, let $\mathscr{B}(x,x)=\langle Ay,x-y\rangle$ for all $x,y\in S$. Then, $z\in Sol(\mathscr{B},S)$ if and only if $\langle Az,x-z\rangle\geq 0$ for all $x\in S$, that is, point $z$ solve the classical variational inequality. The equilibrium problem also includes celebrated saddle problems as special cases. In the real world, numerous problems in computer science, physics, and economics reduce to find a solution of the equilibrium problem. Recently, various solution methods have been presented to solve the equilibrium problem numerically; see, e.g., [10, 20, 25] and the references therein.

Let $\mathscr{N}$ be mapping on $H$. A point $x$ in $H$ is said to be a fixed point of $\mathscr{N}$ iff $\mathscr{N}x=x$. The fixed point set of the mapping is denoted by $Fix(\mathscr{N})$. Finding fixed points of nonlinear operators is an interesting field. It has many theoretical applications, such as differential equations and real applications, such as machine learning; see, e.g., [7, 13, 27] and the references therein.

Recall that $\mathscr{N}$ is contractive if

$$\|\mathscr{N}x-\mathscr{N}y\|\leq c\|x-y\|,\quad \forall x,y\in H,$$

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where $c$ is a constant in $(0,1)$. It is known that every contractive mapping has a unique fixed point and the simple Picard works for the contractive mappings.

Recall that $\mathscr{N}$ is Meir-Keeler contractive (MK contractive for short) if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for each $x, y \in X$ with $\varepsilon \leq \|x-y\| < \varepsilon + \delta$, $\|f(x) - f(y)\| < \varepsilon$.

In 1969, Meir and Keeler [16] proved that every MK contractive mapping has a unique fixed point in metric spaces.

Recall that $\mathscr{N}$ is nonexpansive if

$$\|\mathscr{N}x - \mathscr{N}y\| \leq \|x-y\|, \quad \forall x, y \in H.$$

Recall that $\mathscr{N}$ is firmly nonexpansive if

$$\|\mathscr{N}x - \mathscr{N}y\|^2 \leq \langle x-y, \mathscr{N}x - \mathscr{N}y\rangle, \quad \forall x, y \in H.$$

The class of (firmly) nonexpansive mappings is important from the viewpoint of mathematical programming computation. Indeed, many optimization problems can be solved via its resolvent operators, which are firmly nonexpansive; see, e.g., [2,3,14,21] and the references therein. One important example is nearest point projection, $Proj_S$, which reads $Proj_S(y) := \arg\min\{\|x - y\|, x \in S\}$ for any $y \in H$.

$$\|Proj_S y - Proj_S x\|^2 \leq \langle y-x, Proj_S y - Proj_S(x)\rangle, \quad \forall x, y \in H.$$

For fixed points of nonlinear operators, Mann iteration [15] is efficient in finding approximate fixed points of nonexpansive operators in Euclidean spaces. Let $\{\alpha_n\}$ be a real number sequence in the interval $(0,1)$. The Mann iteration reads as follows

$$x_0 \in H, \quad x_{n+1} = (1-\alpha_n)\mathscr{N}x_n + \alpha_n x_n, \quad n \geq 0.$$

It deserves mentioning that the Mann iteration is only weakly convergent in the framework of infinite dimensional spaces; see, e.g., [8] and the references therein. To force the strong convergence of the Mann iteration, various modified methods were introduced and stuided in Hilbert spaces and Banach spaces recently; see, e.g., [4, 11, 12, 18]. Here, we mention the celebrated Halpern iteration [9]. It was introuced by Halpern and reads as follows

$$x_0 \in H, \quad x_{n+1} = (1-\alpha_n)\mathscr{N}x_n + \alpha_n u, \quad \forall n \geq 0,$$

where $\mathscr{N}$ is a nonexpansive mapping on $S$, $u$ is a fixed anchor in $S$, and $\{\alpha_n\}$ is a real sequence in $(0,1)$. To force the convergence, It is known that the conditions (c1) $\alpha_n \to 0$ as $n \to \infty$ and (c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary if the Halpern iteration converges strongly (due to the convex combination of a nonexpansive maping with the fixed anchor is a contractive mapping). From the structure, we hope that $\alpha_n \to 0$ as fast as possible. In view of restriction (c2), Halpern iteration may not be a fast iteration.

In 2000, Moudafi [17] introduced a viscosity approximation iteration, which is known as the Moudafi's viscosity and reads as follows

$$x_0 \in S, \quad x_{n+1} = \alpha_n \mathscr{C}x_n + (1-\alpha_n)\mathscr{N}x_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$, $\mathscr{N}$ is a nonexpansive mapping with fixed points, and $\mathscr{C}$ is a contractive mapping. Moudafi proved that $\{x_n\}$ converges strongly a fixed point, $x$, of mapping $\mathscr{N}$ under some assumptions on $\{\alpha_n\}$, and the fixed point also is a unique solution to the variational inequality: $\langle \mathscr{C}x - x, x-y\rangle \geq 0$ for all $y \in Fix(\mathscr{N})$. Recently, many authors investigated the fixed point problems of various nonlinear operators based on the Moudafi's viscosity; see, e.g., [22,28] and the references therein.

Now, let us turn back to the equilibrium problem. To study approximate solutions of the equilibrium problem, one usually imposes the following restrictions on the bifunction $\mathscr{B}$.

(R1) $\mathscr{B}(y,y)=0$, $\forall y\in S$;

(R2) $\mathscr{B}(y,x)+\mathscr{B}(x,y)\le 0$, $\forall x,y\in S$ (monotone);

(R3) $\limsup_{t\to 1}\mathscr{B}((1-t)z+x,y)\le \mathscr{B}(x,y)$;

(R4) for each $x\in S$, $y\mapsto \mathscr{B}(x,y)$ is convex and lower semicontinuous.

In 2007, S. Takahashi and W. Takahashi [26] introduced a moudafi's viscosity methods for fixed points of nonexpansive mappings and an equilibrium problem in Hilbert spaces. Their viscosity methods reads as follows

$$\begin{cases} \mathscr{B}(y_n,y)+\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\ge 0, & \forall y\in S,\\ x_{n+1}=\alpha_n\mathscr{C}x_n+(1-\alpha_n)\mathscr{N}y_n, & \forall n\ge 0,\end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $(0,1)$ and $\{r_n\}$ is a nonnegative real sequence. They proved that the sequence $\{x_n\}$ generated by their method converges strongly to a common solution of the two problem, and it also is a unique solution to another variational inequality involving $\mathscr{C}$. Subsequently, numerous new methods were introduced and investigated; see, e.g., [5,10,20,23, 25] and the references.

In this paper, we consider the fixed points of nonexpansive mappings and the solutions of the equilibrium problem via a MK viscosity method. Under some mild conditions on the control sequences, we obtain a strongly convergent theorem without compact assumptions on any operators and subsets.

## 2. Tools

In this section, we list some lemmas, which are needed for the main convergence theorem.

**Lemma 2.1.** *[30] Let $H$ be a Hilbert space and let $S$ be a nonempty, convex, and closed subset of $H$. Let $\mathscr{N}$ be a nonexpansive mapping on $S$. Then $Fix(\mathscr{N})$ is convex and closed.*

**Lemma 2.2.** *[1] Let $H$ be a Hilbert space and let $S$ be a nonempty, convex, and closed subset of $H$. Let $\mathscr{B}$ be a bifunction of $S\times S$ to $\mathbb{R}$ satisfying (R1), (R2), (R3), and (R4). Let $x\in H$ and $r>0$. Then, there exists $z$ such that*

$$\langle y-z,z-x\rangle+r\mathscr{B}(z,y)\ge 0,\quad \forall y\in S.$$

**Lemma 2.3.** *[6] Let $H$ be a Hilbert space and let $S$ be a nonempty, convex, and closed subset of $H$. Let $\mathscr{B}$ be a bifunction of $S\times S$ to $\mathbb{R}$ satisfying (R1), (R2), (R3), and (R4). Let $x\in H$ and $r>0$. Define the resolvent $Res_r^{\mathscr{B}}$ from $H$ to $S$ by*

$$Res_r^{\mathscr{B}}x=\{z\in S:\langle y-z,z-x\rangle+r\mathscr{B}(z,y)\ge 0,\quad \forall y\in S\}.$$

*Then $Res_r^{\mathscr{B}}$ is single-valued and firmly nonexpansive, i.e.,*

$$\|Res_r^{\mathscr{B}}x-Res_r^{\mathscr{B}}x'\|^2\le\langle x-x',Res_r^{\mathscr{B}}x-Res_r^{\mathscr{B}}x'\rangle,\quad \forall x,x'\in H.$$

*Additionly, $Fix(Res_r^{\mathscr{B}})=Sol(\mathscr{B},S)$ is a convex and close set.*

**Lemma 2.4.** *[30] Let $H$ be a Hilbert space, and let $\mathscr{C}$ be a MK contraction. Then, for each $\varepsilon>0$, there exists $c_\varepsilon$ in $(0,1)$ such that $\|x-y\|\ge\varepsilon$ implies $\|f(x)-f(y)\|\le c_\varepsilon\|x-y\|$, $\forall x,y\in H$.*

**Lemma 2.5.** *[24] Let $\{y_n\}$ and $\{x_n\}$ be two bounded vector sequences in a Hilbert space $H$. Let $\varepsilon_n$ be a real sequence in $(0,1)$ with $0 < \liminf_{n\to\infty} \varepsilon_n \leq \limsup_{n\to\infty} \varepsilon_n < 1$. Put $x_{n+1} = (1-\varepsilon_n)x_n + \varepsilon_n y_n$. If $\limsup_{n\to\infty}(\|\varepsilon_n - \varepsilon_{n+1}\| - \|x_n - x_{n+1}\|) = 0$, then $\lim_{n\to\infty} \|x_n - \varepsilon_n\| = 0$.*

**Lemma 2.6.** *[19] Let $H$ be a Hilbert space. Let $\{x_n\}$ be a sequence in $X$ converging converges weakly to a point $y$. Then $\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - y'\|$ for any $y' \neq y$ in $H$.*

**Lemma 2.7.** *[29] Let $\{a_n\}$ be a nonnegative real sequence such that $a_{n+1} \leq (1-t_n)a_n + b_n$, for all $n$, where $\{t_n\}$ and $\{b_n\}$ are sequences of real numbers with $\{t_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} t_n = \infty$, $\lim_{n\to\infty} t_n = 0$, and $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.*

## 3. Main Results

**Theorem 3.1.** *Let $H$ be a Hilbert space with inner $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $\mathscr{B}$ be a bifunction from $S \times S$ to $\mathbb{R}$ with restrictions (R1), (R2), (R3), and (R4), where $S$ is a convex, closed, and nonempty subset of space $H$. Let $\mathscr{N}$ be a nonexpansive from $S$ to $H$, and let $\mathscr{C}$ be a fixed MK contractive mapping from $S$ to $H$. Let $\{r_n\}$ be a nonnegative regular sequence. Let $\{x_n\}$ be the iterative sequence generated in the following process with $x_0 \in H$ and*

$$\begin{cases} \langle y - y_n, y_n - x_n \rangle + r_n \mathscr{B}(y_n, y) \geq 0, \quad \forall y \in S, \\ x_{n+1} = \alpha_n \mathscr{N} y_n + \beta_n \mathscr{C} y_n + \gamma_n x_n, \quad \forall n \geq 0, \end{cases}$$

*where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real number sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$. Assumed that $r_n \geq r$, where $r$ is some positive real number, $\sum_{n=0}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, and $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$. If $\Omega := Fix(\mathscr{N}) \cap Sol(\mathscr{B}, S)$ is not empty, then $\{x_n\}$ converges strongly to a point $\bar{x} \in \Omega$, and $Proj_{\Omega}\mathscr{C}(\bar{x}) = \bar{x}$.*

*Proof.* From Lemma 2.2, we have that $\{y_n\}$ is well defined. From the assumption and Lemmas 2.1 and 2.3, we see that $\Omega$ is closed, convex, and nonempty set. Hence, the nearest point projection on it is well defined. In view of Lemma 2.3, we have $y_n = Res_{r_n}^{\mathscr{B}} x_n$, where $Res_{r_n}^{\mathscr{B}}$ is the resolvent of bifunction $\mathscr{B}$. Fix a common solution in $\Omega$, say $x' \in \Omega$. Thus $x' = Res_{r_n}^{\mathscr{B}} x'$ and $x' = \mathscr{N} x'$ It follows from the nonexpansivity of the resolvent that

$$\|x' - y_n\| = \|Res_{r_n}^{\mathscr{B}} x' - Res_{r_n}^{\mathscr{B}} x_n\| \leq \|x' - x_n\|.$$

Next, we show both $\{x_n\}$ and $\{y_n\}$ are bounded. Let $\varepsilon >$ be any positive constant. If $\|x' - x_n\| < \varepsilon$, then $\{x_n\}$ is bounded. This is obvious. If $\|x' - x_n\| \geq \varepsilon$, From Lemma 2.4, we have that there holds $\|\mathscr{C}x - \mathscr{C}y\| \leq c_\varepsilon \|x - y\|$, $\forall x, y \in S$, where $c_\varepsilon$ is a real constant in $(0,1)$. Thus

$$\begin{aligned} &\|x' - x_{n+1}\| \\ &= \|x' - \alpha_n \mathscr{N} y_n - \beta_n \mathscr{C} y_n - \gamma_n x_n\| \\ &\leq \alpha_n \|\mathscr{N} x' - \mathscr{N} y_n\| + \beta_n \|x' - \mathscr{C} y_n\| + \gamma_n \|x' - x_n\| \\ &\leq \alpha_n \|x' - y_n\| + \beta_n \|x' - \mathscr{C} x'\| + \beta_n \|\mathscr{C} x' - \mathscr{C} y_n\| + \gamma_n \|x' - x_n\| \\ &\leq (1-\beta_n)\|x' - x_n\| + \beta_n \|x' - \mathscr{C} x'\| + \beta_n c \|x' - y_n\| \\ &\leq (1 - \beta_n(1-c))\|x' - x_n\| + \beta_n \|x' - \mathscr{C} x'\|, \end{aligned}$$

which implies that

$$\|x' - x_{n+1}\| \leq (1 - \beta_n(1-c))\|x' - x_n\| + \beta_n(1-c)\frac{\|x' - \mathscr{C} x'\|}{1-c},$$

that is, $\|x'-x_{n+1}\| \le \max\{\|x'-x_n\|, \frac{\|x'-\mathscr{C}x'\|}{1-c}\}$. Mathematical induction indicates that

$$\|x'-x_{n+1}\| \le \max\{\|x'-x_0\|, \frac{\|x'-\mathscr{C}x'\|}{1-c}\}.$$

This proves that $\{x_n\}$ is bounded. Hence, $\{x_n\}$ is always bounded for the two cases. From the way that $\{y_n\}$ is generated, we see that

$$\langle y-y_{n+1}, y_{n+1}-x_{n+1}\rangle + r_{n+1}\mathscr{B}(y_{n+1},y) \ge 0, \quad \forall y \in S \tag{3.1}$$

and

$$\langle y-y_n, y_n-x_n\rangle + r_n\mathscr{B}(y_n,y) \ge 0, \quad \forall y \in S. \tag{3.2}$$

Putting $y_n$ into (3.1) and $y_{n+1}$ into (3.2), we have

$$\langle y_n-y_{n+1}, y_{n+1}-x_{n+1}\rangle + r_{n+1}\mathscr{B}(y_{n+1},y_n) \ge 0 \tag{3.3}$$

and

$$\langle y_{n+1}-y_n, y_n-x_n\rangle + r_n\mathscr{B}(y_n,y_{n+1}) \ge 0. \tag{3.4}$$

Using Lemma 2.3, we conclude from (3.3) and (3.4) that

$$\langle y_n-y_{n+1}, \frac{y_{n+1}-x_{n+1}}{r_{n+1}} - \frac{y_n-x_n}{r_n}\rangle \ge 0,$$

that is, $\langle y_{n+1}-y_n, y_{n+1}-x_n-\frac{r_n}{r_{n+1}}(y_{n+1}-x_{n+1})\rangle \ge \|y_n-y_{n+1}\|^2$, which is equivalent to

$$\begin{aligned}\|y_n-y_{n+1}\|^2 &\le \langle y_{n+1}-y_n, (1-\frac{r_n}{r_{n+1}})(y_{n+1}-x_{n+1})\rangle + \langle y_{n+1}-y_n, x_{n+1}-x_n\rangle \\ &\le |1-\frac{r_n}{r_{n+1}}|\|y_{n+1}-x_{n+1}\|\|y_{n+1}-y_n\| + \|x_n-x_{n+1}\|\|y_{n+1}-y_n\|.\end{aligned}$$

It follows that

$$\|y_n-y_{n+1}\| \le |1-\frac{r_n}{r_{n+1}}|\|y_{n+1}-x_{n+1}\| + \|x_n-x_{n+1}\|.$$

Without loss of generality, one may assume that there exists a positive constant $r$ such that $0<r<r_n$, $\forall n \ge 0$. Thus

$$\begin{aligned}\|y_n-y_{n+1}\| &\le \frac{|r_{n+1}-r_n|}{r_{n+1}}\|y_{n+1}-x_{n+1}\| + \|x_n-x_{n+1}\| \\ &\le \frac{|r_{n+1}-r_n|}{r}\|y_{n+1}-x_{n+1}\| + \|x_n-x_{n+1}\| \\ &\le \|x_n-x_{n+1}\| + \frac{M}{r}|r_{n+1}-r_n|,\end{aligned}$$

where $M$ is an appropriate constant such $M \ge \sup_{n\ge 0}\{\|y_{n+1}-x_{n+1}\|\}$.

To use Lemma 2.5, one sets $\varepsilon_n = \frac{x_{n+1}-\gamma_n x_n}{1-\gamma_n}$. Hence,

$$\begin{aligned}&\varepsilon_n-\varepsilon_{n+1} \\ &= \frac{\alpha_n\mathscr{N}y_n+\beta_n\mathscr{C}y_n}{1-\gamma_n} - \frac{\alpha_{n+1}\mathscr{N}y_{n+1}+\beta_{n+1}\mathscr{C}y_{n+1}}{1-\gamma_{n+1}} \\ &= \frac{(1-\beta_n-\gamma_n)\mathscr{N}y_n+\beta_n\mathscr{C}y_n}{1-\gamma_n} - \frac{(1-\beta_{n+1}-\gamma_{n+1})\mathscr{N}y_{n+1}+\beta_{n+1}\mathscr{C}y_{n+1}}{1-\gamma_{n+1}} \\ &= \frac{(1-\gamma_n)\mathscr{N}y_n+\beta_n(\mathscr{C}y_n-\mathscr{N}y_n)}{1-\gamma_n} - \frac{(1-\gamma_{n+1})\mathscr{N}y_{n+1}+\beta_{n+1}(\mathscr{C}y_{n+1}-\mathscr{N}y_{n+1})}{1-\gamma_{n+1}}.\end{aligned}$$

Hence,

$$\varepsilon_n - \varepsilon_{n+1} = \mathscr{N} y_n + \frac{\beta_n}{1-\gamma_n}(\mathscr{C} y_n - \mathscr{N} y_n) - \mathscr{N} y_{n+1} - \frac{\beta_{n+1}}{1-\gamma_{n+1}}(\mathscr{C} y_{n+1} - \mathscr{N} y_{n+1}).$$

We estimate as follows

$$\begin{aligned}
&\|\varepsilon_n - \varepsilon_{n+1}\| \\
&\le \frac{\beta_n}{1-\gamma_n}\|\mathscr{C} y_n - \mathscr{N} y_n\| + \frac{\beta_{n+1}}{1-\gamma_{n+1}}\|\mathscr{C} y_{n+1} - \mathscr{N} y_{n+1}\| + \|\mathscr{N} y_n - \mathscr{N} y_{n+1}\| \\
&\le \frac{\beta_n}{1-\gamma_n}\|\mathscr{C} y_n - \mathscr{N} y_n\| + \frac{\beta_{n+1}}{1-\gamma_{n+1}}\|\mathscr{C} y_{n+1} - \mathscr{N} y_{n+1}\| + \|y_n - y_{n+1}\| \\
&\le \frac{\beta_n}{1-\gamma_n}\|\mathscr{C} y_n - \mathscr{N} y_n\| + \frac{\beta_{n+1}}{1-\gamma_{n+1}}\|\mathscr{C} y_{n+1} - \mathscr{N} y_{n+1}\| + \|x_n - x_{n+1}\| + \frac{M}{r}|r_{n+1} - r_n|.
\end{aligned}$$

This indicates that

$$\begin{aligned}
&\|\varepsilon_{n+1} - \varepsilon_n\| - \|x_{n+1} - x_n\| \\
&\le \frac{\beta_n}{1-\gamma_n}\|\mathscr{C} y_n - \mathscr{N} y_n\| + \frac{\beta_{n+1}}{1-\gamma_{n+1}}\|\mathscr{C} y_{n+1} - \mathscr{N} y_{n+1}\| + \frac{M}{r}|r_{n+1} - r_n|.
\end{aligned}$$

From the assumptions on the parameters, we find that

$$\limsup_{n\to\infty}(\|\varepsilon_n - \varepsilon_{n+1}\| - \|x_n - x_{n+1}\|) = 0.$$

An application of Lemma 2.5 indicates that, as $n \to \infty$, $\|\varepsilon_n - x_n\| \to 0$. Observe that $x_{n+1} - x_n = (1-\gamma_n)(\varepsilon_n - x_n)$, which further indicates that, as $n \to \infty$, $\|x_{n+1} - x_n\| \to 0$ due to $0 < \liminf_{n\to\infty}\gamma_n \le \limsup_{n\to\infty}\gamma_n < 1$. We also have $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$. Observe that

$$\begin{aligned}
\|x_n - \mathscr{N} y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - \mathscr{N} y_n\| \\
&\le \|x_n - x_{n+1}\| + \beta_n\|\mathscr{C} y_n - \mathscr{N} y_n\| + \gamma_n\|x_n - \mathscr{N} y_n\|,
\end{aligned}$$

which together with the assumptions on $\{\beta\}$ and $\{\gamma_n\}$ and the boundedness of $\{x_n\}$ and $\{y_n\}$ that $\|x_n - \mathscr{N} y_n\| \to 0$ as $n \to \infty$. Since the resolvent is firmly nonexpansive, then

$$\begin{aligned}
\|\mu - y_n\|^2 &= \|Res^{\mathscr{B}}_{r_n}\mu - Res^{\mathscr{B}}_{r_n}x_n\|^2 \\
&\le \langle \mu - x_n, Res^{\mathscr{B}}_{r_n}\mu - Res^{\mathscr{B}}_{r_n}x_n\rangle \\
&= \langle \mu - x_n, \mu - y_n\rangle \\
&= \frac{1}{2}(\|\mu - x_n\|^2 + \|\mu - y_n\|^2 - \|x_n - y_n\|^2).
\end{aligned}$$

This shows $\|\mu - y_n\|^2 \le \|\mu - x_n\|^2 - \|x_n - y_n\|^2$. In view of the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
\|\mu - x_{n+1}\|^2 &= \|\alpha_n(\mu - \mathscr{N} y_n) + \beta_n(\mu - \mathscr{C} y_n) + \gamma_n(\mu - x_n)\|^2 \\
&\le \alpha_n\|\mathscr{N}\mu - \mathscr{N} y_n\|^2 + \beta_n\|\mu - \mathscr{C} y_n\|^2 + \gamma_n\|\mu - x_n\|^2 \\
&\le \alpha_n\|\mu - y_n\|^2 + \beta_n\|\mu - \mathscr{C} y_n\|^2 + \gamma_n\|\mu - x_n\|^2 \\
&\le \alpha_n\|\mu - x_n\|^2 - \alpha_n\|x_n - y_n\|^2 + \beta_n\|\mu - \mathscr{C} y_n\|^2 + \gamma_n\|\mu - x_n\|^2 \\
&\le \|\mu - x_n\|^2 - \alpha_n\|x_n - y_n\|^2 + \beta_n\|\mu - \mathscr{C} y_n\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}\alpha_n\|x_n-y_n\|^2 &\le \beta_n\|\mu-\mathscr{C}y_n\|^2+\|\mu-x_n\|^2-\|\mu-x_{n+1}\|^2\\ &\le \beta_n\|\mu-\mathscr{C}y_n\|^2+(\|\mu-x_n\|+\|\mu-x_{n+1}\|)\|x_n-x_{n+1}\|.\end{aligned}$$

From the assumptions on $\{\alpha_n\}$ and $\{\beta_n\}$, we have $\lim_{n\to\infty}\|x_n-y_n\|=0$.

Next, we prove $\limsup_{n\to\infty}\langle x_n-\bar{x},\mathscr{C}\bar{x}-\bar{x}\rangle\le 0$. To prove this inequality, we choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle x_n-\bar{x},\mathscr{C}\bar{x}-\bar{x}\rangle=\lim_{n\to\infty}\langle x_{n_j}-\bar{x},\mathscr{C}\bar{x}-\bar{x}\rangle.$$

Since $\{x_{n_j}\}$ is a bounded vector sequence, one asserts that there exists a weakly converging subsequence $\{x_{j_i}\}$ of $\{x_j\}$. Without loss of generality, one just assume that $\{x_{n_j}\}\rightharpoonup x^*\in S$. Further, one proves $x^*\in\Omega$. Indeed, from the way that $\{y_n\}$ is generated, one has $\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle+\mathscr{B}(y_n,y)\ge 0$, $\forall y\in S$. By (R2), we have $\frac{1}{r_n}\langle y-y_n,y_n-x_n\rangle\ge\mathscr{B}(y,y_n)$, $\forall y\in S$. Obviously,

$$\langle y-y_{n_j},\frac{y_{n_j}-x_{n_j}}{r_{n_j}}\rangle\ge\mathscr{B}(y,y_{n_j}),\quad \forall y\in S.$$

Note that $\lim_{n\to\infty}\|\frac{y_{n_j}-x_{n_j}}{r_{n_j}}\|=0$ and $\{y_{n_j}\}\rightharpoonup x^*$. Hence,

$$\mathscr{B}(y,x^*)\le 0,\quad \forall y\in S.$$

Let $y^z_{x^*}=(1-z)x^*+zy$, where $z$ is a constant in $0\le z<1$. Since both $x^*$ and $y$ are in $S$, we have $y^z_{x^*}\in S$. It follows that $\mathscr{B}(y^z_{x^*},x^*)\le 0$, $\forall y\in S$. By restrictions (R1) and (R4), we have

$$\mathscr{B}(y^z_{x^*},y^z_{x^*})=\mathscr{B}(y^z_{x^*},(1-z)x^*+zy)=(1-z)\mathscr{B}(y^z_{x^*},x^*)+z\mathscr{B}(y^z_{x^*},y).$$

This shows that $\mathscr{B}((1-z)x^*+zy,y)\ge 0$. By (R3), we have $\mathscr{B}(x^*,y)\ge 0$, $\forall y\in S$, that is, $x^*\in Sol(\mathscr{B},S)$. On the other hand,

$$\begin{aligned}\|x_n-\mathscr{N}x_n\| &\le \|x_n-\mathscr{N}y_n\|+\|\mathscr{N}y_n-\mathscr{N}x_n\|\\ &\le \|x_n-\mathscr{N}y_n\|+\|y_n-x_n\|.\end{aligned}$$

This finds $\|x_n-\mathscr{N}x_n\|\to 0$ as $n\to\infty$. Assume $x^*\notin Fix(\mathscr{N})$. From Opial's condition (Lemma 2.6), we have

$$\begin{aligned}\|x_{n_i}-x^*\| &< \|x_{n_i}-\mathscr{N}x^*\|\\ &\le \|x_{n_i}-\mathscr{N}x_{n_i}\|+\|\mathscr{N}x_{n_i}-\mathscr{N}x^*\|\\ &\le \|x_{n_i}-x^*\|.\end{aligned}$$

This reach a contradiction, which presents $x^*\in Fix(\mathscr{N})$. This finishes the proof that

$$\limsup_{n\to\infty}\langle x_n-\bar{x},\mathscr{C}\bar{x}-\bar{x}\rangle\le 0.$$

Finally, we show that $\|x_n-\bar{x}\|\to\infty$ as $n\to\infty$. Assume that the sequence $\{x_n\}$ does not converge to $\bar{x}$ strongly. Thus there exists $\varepsilon>0$ and a subsequence $\{x_{n_i}\}$ such that $\|x_{n_i}-\bar{x}\|\ge\varepsilon$

for all $i$. From Lemma 2.4, we observe that

$$\begin{aligned}
&\|x_{n+1}-\bar{x}\|^2 \\
&= \langle \alpha_n(\mathscr{N}y_n-\bar{x})+\beta_n(\mathscr{C}y_n-\bar{x})+\gamma_n(x_n-\bar{x}), x_{n+1}-\bar{x}\rangle \\
&= \alpha_n\langle \mathscr{N}y_n-\bar{x}, x_{n+1}-\bar{x}\rangle+\beta_n\langle \mathscr{C}y_n-\mathscr{C}\bar{x}, x_{n+1}-\bar{x}\rangle+\beta_n\langle \mathscr{C}\bar{x}-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\quad +\gamma_n\langle x_n-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\le \alpha_n\|\mathscr{N}y_n-\mathscr{N}\bar{x}\|\|x_{n+1}-\bar{x}\|+\beta_n\|\mathscr{C}y_n-\mathscr{C}\bar{x}\|\|x_{n+1}-\bar{x}\|+\beta_n\langle \mathscr{C}\bar{x}-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\quad +\gamma_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\
&\le \alpha_n\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\beta_n c_\varepsilon\|y_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\beta_n\langle \mathscr{C}\bar{x}-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\quad +\gamma_n\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\| \\
&\le (1-\beta_n(1-c_\varepsilon))\|x_n-\bar{x}\|\|x_{n+1}-\bar{x}\|+\beta_n\langle \mathscr{C}\bar{x}-\bar{x}, x_{n+1}-\bar{x}\rangle \\
&\le \frac{1-\beta_n(1-c_\varepsilon)}{2}(\|x_n-\bar{x}\|^2+\|x_{n+1}-\bar{x}\|^2)+\beta_n\langle \mathscr{C}\bar{x}-\bar{x}, x_{n+1}-\bar{x}\rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1}-\bar{x}\|^2 \le (1-\beta_n(1-c_\varepsilon))\|x_n-\bar{x}\|^2+2\beta_n\langle \mathscr{C}\bar{x}-\bar{x}, x_{n+1}-\bar{x}\rangle.$$

By using Lemma 2.7, one obtains $x_n \to \bar{x}$ as $n \to \infty$. This finishes the proof of this theorem. □

From Theorem 3.1, we have the following result on the Halpern-based method immediately.

**Corollary 3.1.** *Let $H$ be a Hilbert space with inner $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\mathscr{B}$ be a bifunction from $S\times S$ to $\mathbb{R}$ with restrictions (R1), (R2), (R3), and (R4), where $S$ is a convex, closed, and nonempty subset of space $H$. Let $\mathscr{N}$ be a nonexpansive from $S$ to $H$. Let $\{r_n\}$ be a nonnegative regular sequence. Let $\{x_n\}$ be the iterative sequence generated in the following process with $x_0 \in H$ and $x_{n+1} = \alpha_n\mathscr{N}y_n+\beta_n u+\gamma_n x_n$, where $\{y_n\}$ is defined by $\langle y-y_n, y_n-x_n\rangle + r_n\mathscr{B}(y_n,y)\ge 0$, $\forall y\in S$, where $u$ is a fixed vector in $S$, $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real number sequences in $(0,1)$ with $\alpha_n+\beta_n+\gamma_n=1$. Assumed that $r_n\ge r$, where $r$ is some positive real number, $\sum_{n=0}^\infty \beta_n=\infty$, $\lim_{n\to\infty}\beta_n=0$, $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$, and $\lim_{n\to\infty}|r_{n+1}-r_n|=0$. If $\Omega := Fix(\mathscr{N})\cap Sol(\mathscr{B},S)$ is not empty, then $\{x_n\}$ converges strongly to a point $\bar{x}\in\Omega$, and $Proj_\Omega u=\bar{x}$.*

From Theorem 3.1, we also have the following result on fixed points of nonself mappings method immediately.

**Corollary 3.2.** *Let $H$ be a Hilbert space with inner $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $\mathscr{N}$ be a nonexpansive from $S$ to $H$, and let $\mathscr{C}$ be a fixed MK contractive mapping from $S$ to $H$, where $S$ is a convex, closed, and nonempty subset of space $H$. Let $\{x_n\}$ be the iterative sequence generated in the following process with $x_0\in H$ and $x_{n+1}=\alpha_n\mathscr{N}Proj_S x_n+\beta_n\mathscr{C}Proj_S x_n+\gamma_n x_n$, $\forall n\ge 0$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are real number sequences in $(0,1)$ with $\alpha_n+\beta_n+\gamma_n=1$. Assumed that $\sum_{n=0}^\infty\beta_n=\infty$, $\lim_{n\to\infty}\beta_n=0$, and $0<\liminf_{n\to\infty}\gamma_n\le\limsup_{n\to\infty}\gamma_n<1$. If $Fix(\mathscr{N})$ is not empty, then $\{x_n\}$ converges strongly to a point $\bar{x}\in Fix(\mathscr{N})$, and $Proj_{Fix(\mathscr{N})}\mathscr{C}(\bar{x})=\bar{x}$.*

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