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APPROXIMATION OF SOLUTIONS TO THE SFPMOS PROBLEM

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Abstract. In this paper, we study the split feasibility problem with multiple output sets in Hilbert spaces. For solving this problem, we propose an iterative method and construct two selection strategies of stepsizes. Under appropriate conditions, we prove the strong convergence of the proposed iterative method.

Keywords. Demiclosedness principle; Metric projection; Split feasibility problem; stepsize.

1. INTRODUCTION

Given two Hilbert spaces H and H_1, the split feasibility problem (SFP) [5] can be mathematically expressed as the problem of finding a point $x^\dagger \in H$ such that

$$x^\dagger \in C \bigcap A^{-1}(Q). \tag{1.1}$$

Here C and Q are respectively nonempty, closed and convex subsets in H and H_1, and $A^{-1}(Q) = \{x \in H : Ax \in Q\}$ with $A : H \to H_1$ a given bounded linear mapping. This problem has attracted increasing attention due to its wide applications in applied disciplines, such as signal processing and image reconstruction [3, 4, 7]. Historically, there are many iterative methods to solve (1.1), among which the most popular method is the CQ method proposed by Byrne [2], which generates the sequence $\{x_n\}$ through a recursive process:

$$x_{n+1} = P_C\big[x_n - \tau A^*(I - P_Q)Ax_n\big], \tag{1.2}$$

where A^* is the conjugate of A, I stands for the identity mapping, $\tau > 0$ is a properly chosen stepsize, and P_C and P_Q are the metric projections onto C and Q, respectively. It was shown that if τ is chosen in $(0, \frac{2}{\|A\|^2})$, then (1.2) converges weakly to a solution of (1.1) whenever such a solution exists. For more progress on this issue, we refer the reader to [6, 14, 15, **?**, 16, 17, 18, 19] and the references therein.

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In the literature, there are various generalizations of SFP, one of which is the split feasibility problem with multiple output sets (SFPMOS) [11]. For real Hilbert spaces H and H_i, $i = 1,2,\dots,N$, the SFPMOS consists of finding an element $x^\dagger \in H$ such that

$$x^\dagger \in C \cap \Big(\bigcap_{i=1}^{N} A_i^{-1}(Q_i) \Big), \tag{1.3}$$

where $C \subset H$ and $Q_i \subset H_i$ are closed and convex subsets, and each $A_i : H \to H_i$ is a bounded linear mapping for $i = 1,2,\dots,N$. To solve problem (1.3), Reich, Truong and Mai [11] proposed two novel iterative methods. For any initial guesses x_0, let $\{x_n\}$ be a sequence generated by:

$$x_{n+1} = P_C\left[x_n - \tau \sum_{i=1}^{N} A_i^* \big(I - P_{Q_i}\big) A_i x_n \right], \tag{1.4}$$

where $\tau > 0$ is a properly chosen stepsize. It was shown that if τ is chosen such that

$$0 < \tau < \frac{2}{N \max_{1\le i\le N} \|A_i\|^2}, \tag{1.5}$$

then method (1.4) converges weakly to a solution of problem (1.3). In order to reach a strongly convergent method, they modified the above method as the following form: for any initial guesses y_0, let $\{y_n\}$ be a sequence generated by:

$$y_{n+1} = \gamma_n f(y_n) + (1-\gamma_n) P_C\left[y_n - \tau \sum_{i=1}^{N} A_i^* \big(I - P_{Q_i}\big) A_i y_n \right], \tag{1.6}$$

where $\{\gamma_n\} \subset (0,1)$, and f is a contraction. It was shown that method (1.6) converges strongly to a solution of problem (1.3) under some certain conditions. For more progress on this issue, we refer the reader to [11, 13] and the references therein.

In this paper, we continue to study the SFPMOS and construct a new class of strongly convergent iterative methods. The paper is organized as follows. In Section 2, we gather some preliminary knowledge and some related lemmas. In Section 3, we show that the problem under consideration is equivalent to a fixed point equation. In Section 4, we propose another iteration method and prove that the method has strong convergence. In Section 5, we construct the variable step size strategy and prove its strong convergence.

2. PRELIMINARIES

In this section, we assume that T is a mapping from H into itself and $C \subset H$ is a nonempty closed convex subset in H. We first recall the definition of several important classes of nonlinear mappings.

Definition 2.1. T is called *firmly nonexpansive* if, for each $x, y \in H$,

$$\|Tx - Ty\|^2 \le \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2.$$

T is called *nonexpansive* if, for each $x, y \in H$,

$$\|Tx - Ty\| \le \|x - y\|.$$

Definition 2.2. T is said to satisfy the demiclosedness principle if $I-T$ is *demiclosed at 0*, that is, for any sequence $\{x_n\}\subset H$ and $x^\dagger\in H$,

$$\left.\begin{array}{c} x_n \rightharpoonup x^\dagger \\ x_n - Tx_n \to 0 \end{array}\right] \Longrightarrow Tx^\dagger = x^\dagger.$$

Here "$\to$" stands for strong convergence and "$\rightharpoonup$" weak convergence.

Lemma 2.3. [1, 9] *Every nonexpansive mapping and firmly nonexpansive mapping have the demiclosedness principle.*

More information on firmly nonexpansive and nonexpansive mappings can be found, for example, in [8, 12], and the references therein. A typical example of firmly nonexpansive mappings is the metric projection.

Definition 2.4. For any $x\in H$, the metric projection $P_C : H \to C$ is defined by

$$P_C x = \arg\min_{y\in C} \|x-y\|, x\in H.$$

Lemma 2.5. *The metric projection P_C is firmly nonexpansive. Moreover, for any $x\in H$, it follows that*

$$\|x-P_C x\|^2 \le \langle x-P_C x, x-z\rangle,\ \forall z\in C.$$

The following lemma plays an important role in our convergence analysis.

Lemma 2.6. *Suppose that $\{x_n\}$ is a sequence in H satisfying the following conditions:*

(1) $\omega_w(x_n)\subseteq C$;
(2) $\{\|x_n-x_0\|\}$ *is convergent;*
(3) $\|x_n-x_0\|\le\|x_0-P_C(x_0)\|, \forall n\ge 0.$

Then $\{x_n\}$ converges strongly to $P_C(x_0)$.

Proof. Fix any weak cluster point x of $\{x_n\}$. Hence there exists a subsequence $\{x_{n_k}\}$ that converges weakly to $x\in C$. It then follows from the property of metric projections and the lower semi-continuity of the norm that

$$\begin{aligned}\|x_0-P_C(x_0)\| &\le \|x_0-x\| \\ &\le \lim_{k\to\infty}\|x_0-x_{n_k}\| \\ &\le \|x_0-P_C(x_0)\|,\end{aligned}$$

where the last inequality follows from conditon (3). Hence,

$$\lim_{k\to\infty}\|x_0-x_{n_k}\| = \|x_0-P_C(x_0)\| = \|x_0-x\|.$$

This implies that $\{x_n\}$ converges weakly to $P_C(x_0)$ and $\|x_0-x_n\|\to\|x_0-P_C(x_0)\|$ as $n\to\infty$. By property of the inner product, we conclude that

$$\|x_n-P_C(x_0)\|^2 = \|x_n-x_0\|^2 + 2\langle x_n-x_0, x_0-P_C(x_0)\rangle + \|x_0-P_C(x_0)\|^2.$$

Hence, $\{x_n\}$ converges strongly to $P_C(x_0)$ as desired. □

3. Equivalent Fixed Point Problems

In this section, problem (1.3) is first transformed into a fixed point problem. The the SFPMOS is called consistent, which means that its solution set denoted by $\mathscr{S}$ is nonempty. For the convenience, we set below $Q_0 = C$ and $A_0 = I$ the identity mapping onto H.

Theorem 3.1. *For $r > 0$, let*

$$T := I - r\sum_{i=0}^{N} A_i^*(A_i - P_{Q_i}(A_i)).$$

If the SFPMOS is consistent, then $\mathscr{S} = \mathrm{Fix}(T)$.

Proof. We just verify $\mathrm{Fix}(T) \subseteq \mathscr{S}$ since the converse is obvious. Fix $x^\dagger \in \mathrm{Fix}(T)$ and choose any $z \in \mathscr{S}$. It then follows from Lemma 2.5 that

$$\begin{aligned}
&\|A_i x^\dagger - P_{Q_i}(A_i x^\dagger)\|^2 \\
&\le \langle (A_i x^\dagger - P_{Q_i}(A_i x^\dagger)), A_i x^\dagger - A_i z\rangle \\
&= \langle A_i^*(A_i x^\dagger - P_{Q_i}(A_i x^\dagger)), x^\dagger - z\rangle,
\end{aligned}$$

for each $i = 0, 1, \ldots, N$. Adding up these inequalities, we have

$$\begin{aligned}
&\sum_{i=0}^{N} \|A_i x^\dagger - P_{Q_i}(A_i x^\dagger)\|^2 \\
&\le \sum_{i=0}^{N} \langle A_i^*(A_i x^\dagger - P_{Q_i}(A_i x^\dagger)), x^\dagger - z\rangle \\
&\le \left\langle \sum_{i=0}^{N} A_i^*(A_i x^\dagger - P_{Q_i}(A_i x^\dagger)), x^\dagger - z \right\rangle \\
&\le \frac{1}{r}\left\langle (x^\dagger - Tx^\dagger), x^\dagger - z \right\rangle = 0.
\end{aligned}$$

This yields $\mathrm{Fix}(T) \subseteq \mathscr{S}$ and thus $\mathscr{S} = \mathrm{Fix}(T)$. □

4. Iterative Method with Constant Stepsizes

Motivated by Theorem 3.1, we propose our first method for solving the problem.

Algorithm 4.1. Choose an arbitrary initial guess $x_0 \in H$. Given the current iteration x_n, update the next iteration x_{n+1} by the formula:

$$\begin{cases}
z_n = x_n - r_n\left[\sum_{i=0}^{N} A_i^*(A_i x_n - P_{Q_i}(A_i x_n))\right] \\
\mathscr{E}_n = \{z \in H : \langle z_n - z, x_n - z_n\rangle \ge 0\} \\
\mathscr{F}_n = \{z \in H : \langle x_n - z, (x_0 - x_n)\rangle \ge 0\} \\
x_{n+1} = P_{\mathscr{E}_n \cap \mathscr{F}_n}(x_0),
\end{cases}$$

where $\{r_n\}$ is an appropriately chosen step size.

Now let us state the convergence of $\{x_n\}$ generated by the above method.

Theorem 4.2. *For each $n \geq 0$, let the step size r_n satisfy*

$$0 < r \leq r_n \leq \frac{1}{\sum_{i=0}^{N} \|A_i\|^2}. \tag{4.1}$$

If the SFPMOS is consistent, then the sequence $\{x_n\}$ generated by Algorithm 4.1 is well defined and converges strongly to $P_{\mathscr{S}}(x_0)$.

Proof. We first show that $\{x_n\}$ generated by Algorithm 4.1 is well defined. To this end, it suffices to show that $\mathscr{E}_n \cap \mathscr{F}_n$ is nonempty because the set $\mathscr{E}_n \cap \mathscr{F}_n$ is clearly closed and convex. Fix any $z \in \mathscr{S}$. By Lemma 2.5, we have

$$\begin{aligned}
\langle x_n - z, x_n - z_n \rangle &= r_n \left\langle x_n - z, \sum_{i=0}^{N} A_i^*(A_i x_n - P_{Q_i}(A_i x_n)) \right\rangle \\
&= r_n \sum_{i=0}^{N} \langle x_n - z, A_i^*(A_i x_n - P_{Q_i}(A_i x_n)) \rangle \\
&= r_n \sum_{i=0}^{N} \langle A_i x_n - A_i z, (A_i x_n - P_{Q_i}(A_i x_n)) \rangle \\
&\geq r_n \sum_{i=0}^{N} \|A_i x_n - P_{Q_i}(A_i x_n)\|^2.
\end{aligned} \tag{4.2}$$

By Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|x_n - z_n\|^2 &= r_n^2 \left\| \sum_{i=0}^{N} A_i^*(A_i x_n - P_{Q_i}(A_i x_n)) \right\|^2 \\
&\leq r_n^2 \left(\sum_{i=0}^{N} \|A_i^*(A_i x_n - P_{Q_i}(A_i x_n))\| \right)^2 \\
&\leq r_n^2 \left(\sum_{i=0}^{N} \|A_i^*\| \|(A_i x_n - P_{Q_i}(A_i x_n))\| \right)^2 \\
&= r_n^2 \left(\sum_{i=0}^{N} \|A_i\| \|A_i x_n - P_{Q_i}(A_i x_n)\| \right)^2 \\
&\leq r_n^2 \left(\sum_{i=0}^{N} \|A_i\|^2 \right) \left(\sum_{i=0}^{N} \|A_i x_n - P_{Q_i}(A_i x_n)\|^2 \right).
\end{aligned}$$

Combining the last two inequalities, we have

$$\begin{aligned}
\langle z_n - z, x_n - z_n \rangle &= \langle z_n - x_n, x_n - z_n \rangle + \langle x_n - z, x_n - z_n \rangle \\
&= -\|z_n - x_n\|^2 + \langle x_n - z, x_n - z_n \rangle \\
&\geq r_n \left(1 - r_n \left(\sum_{i=0}^{N} \|A_i\|^2 \right) \right) \sum_{i=0}^{N} \|A_i x_n - P_{Q_i}(A_i x_n)\|^2.
\end{aligned}$$

In view of (4.1), we deduce that

$$\langle z_n - z, x_n - z_n \rangle \geq 0,$$

which yields $\mathscr{S} \subseteq \mathscr{E}_n$ for all $n \geq 0$. We now show $\mathscr{S} \subseteq \mathscr{F}_n$ by induction. It is easy to check that $\mathscr{S} \subseteq Q_0$. Now assume that $\mathscr{S} \subseteq Q_k$ for some $k \geq 1$, which implies $\mathscr{S} \subseteq \mathscr{E}_k \cap \mathscr{F}_k$. Since x_{k+1} lies in $\mathscr{E}_k \cap \mathscr{F}_k$, this gives $\langle x_{k+1} - z, (x_0 - x_{k+1}) \rangle \geq 0, \forall z \in \mathscr{S}$, which implyies $\mathscr{S} \subseteq \mathscr{F}_{k+1}$. Hence, $\mathscr{S} \subseteq \mathscr{F}_n$ for all $n \geq 0$. Altogether, $\mathscr{S} \subseteq \mathscr{E}_n \cap \mathscr{F}_n$ and the set $\mathscr{E}_n \cap \mathscr{F}_n$ is thus nonempty. Therefore, the sequence $\{x_n\}$ defined above is well defined.

Next, we show $\lim_n \|z_n - x_n\| = 0$. Indeed, as $P_{\mathscr{S}}(x_0) \in \mathscr{S} \subseteq \mathscr{E}_n \cap \mathscr{F}_n$, we have

$$\|x_0 - x_{n+1}\| = \|x_0 - P_{\mathscr{E}_n \cap \mathscr{F}_n}(x_0)\| \leq \|x_0 - P_{\mathscr{S}}(x_0)\|$$

for all $n \geq 0$. On the other hand, since x_{n+1} is in $\mathscr{F}_n$, we arrive at

$$\|x_0 - x_n\| = \|x_0 - P_{\mathscr{F}_n}(x_0)\| \leq \|x_0 - x_{n+1}\|.$$

Hence the sequence $\{\|x_0 - x_n\|\}$ is convergent. Note that $\frac{1}{2}(x_n + x_{n+1})$ is also in $\mathscr{F}_n$, we arrive at

$$\|x_0 - x_n\| = \|x_0 - P_{\mathscr{F}_n}(x_0)\| \leq \left\| x_0 - \frac{1}{2}(x_n + x_{n+1}) \right\|.$$

Since $z_n = P_{\mathscr{E}_n}(x_n)$, we have that

$$\begin{aligned}
\|x_n - z_n\|^2 &\leq \|x_n - x_0 + x_0 - x_{n+1}\|^2 \\
&= 2\|x_n - x_0\|^2 + 2\|x_0 - x_{n+1}\|^2 - 4\left\| x_0 - \frac{1}{2}(x_n + x_{n+1}) \right\|^2 \\
&\leq 2\|x_n - x_0\|^2 + 2\|x_0 - x_{n+1}\|^2 - 4\|x_0 - x_n\|^2 \\
&= 2\|x_n - x_0\|^2 - \|x_0 - 2x_{n+1}\|^2.
\end{aligned}$$

Letting $n \to \infty$, we obtain the desired result.

Finally, we prove $\lim_n x_n = P_{\mathscr{S}}(x_0)$. It suffices to verify $\omega_w(x_n) \subseteq \mathscr{S}$. As a matter of fact, we deduce from (4.2) that

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=0}^{N} \|A_i x_n - P_{Q_i}(A_i x_n)\|^2 &\leq \lim_{n\to\infty} \frac{1}{r_n} \langle x_n - z, x_n - z_n \rangle \\
&\leq \lim_{n\to\infty} \frac{1}{r} \|x_n - z\| \|x_n - z_n\| \\
&= 0.
\end{aligned}$$

Fix any $x \in \omega_w(x_n)$ and take a subsequence $\{x_{n_k}\}$ that converges weakly to x. Since $A_i x_{n_k} \rightharpoonup A_i x$, the demiclosedness principle implies

$$\|A_i x - P_{Q_i}(A_i x)\| = 0, i = 0, 1, \ldots, N.$$

This indicates $\omega_w(x_n) \subseteq \mathscr{S}$ and the proof is complete. □

5. ITERATIVE METHOD WITH VARIABLE STEPSIZES

Obviously, for the realization of the previous method, one has to calculate the value of $\|A_i\|$ since its upper bound depends on the value of $\|A_i\|$. However, this is usually not easy in practice. In order to avoid this situation, we thus construct a variable step size so that its calculation is independent of $\|A_i\|$.

Algorithm 5.1. Choose an arbitrary initial guess $x_0 \in H$. Given the current iteration x_n, if

$$\left\| \sum_{i=0}^{N} A_i^*(A_i x_n - P_{Q_i}(A_i x_n)) \right\| = 0, \tag{5.1}$$

then stop; otherwise update the next iteration x_{n+1} by the formula:

$$\begin{cases} z_n = x_n - r_n[\sum_{i=0}^N A_i^*(A_i x_n - P_{Q_i}(A_i x_n))] \\ \mathscr{E}_n = \{z \in H : \langle z_n - z, x_n - z_n \rangle \geq 0\} \\ \mathscr{F}_n = \{z \in H : \langle x_n - z, (x_0 - x_n) \rangle \geq 0\} \\ x_{n+1} = P_{\mathscr{E}_n \cap \mathscr{F}_n}(x_0), \end{cases}$$

where the step size r_n is chosen as

$$r_n = \frac{\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2}{\|\sum_{i=0}^N A_i^*(A_i x_n - P_{Q_i}(A_i x_n))\|^2}. \tag{5.2}$$

It is easy to check that the current iteration x_n is a solution of the SFPMOS if it satisfies conditon (5.1). Without loss of generality, we assume that the above method produces an infinite iterative sequence.

Theorem 5.2. *If the SFPMOS is consistent, then $\{x_n\}$ produced by Algorithm 5.1 is well defined and converges strongly to $P_{\mathscr{S}}(x_0)$.*

Proof. To show the first assertion, it is enough to verify $\mathscr{S} \subseteq \mathscr{E}_n \cap \mathscr{F}_n$ for each $n \geq 0$. To see this, fix any $z \in \mathscr{S}$. It then follows from Lemma 2.5 that

$$\begin{aligned} &\langle z_n - z, x_n - z_n \rangle \\ &= \langle z_n - x_n, x_n - z_n \rangle + \langle x_n - z, x_n - z_n \rangle \\ &\geq r_n \left(\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2 \right) - \|x_n - z_n\|^2 \\ &= r_n \left(\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2 \right) - r_n^2 \left\| \sum_{i=0}^N A_i^*(A_i x_n - P_{Q_i}(A_i x_n)) \right\|^2. \end{aligned}$$

By formula (5.2), we have $\langle z_n - z, x_n - z_n \rangle \geq 0$, that is, $z \in \mathscr{E}_n$. Since z is chosen arbitrarily, we conclude that $\mathscr{S} \subseteq \mathscr{E}_n$ for all $n \geq 0$. By induction, we can show $\mathscr{S} \subseteq \mathscr{F}_n$ for all $n \geq 0$. Altogether, we obtain $\mathscr{S} \subseteq \mathscr{E}_n \cap \mathscr{F}_n$ for each $n \geq 0$ as desired.

We next show the norm convergence of $\{x_n\}$. Similarly, we can prove that the sequences $\{\|x_0 - x_n\|\}$ is convergent and $\|x_0 - x_n\| \leq \|x_0 - P_{\mathscr{S}}(x_0)\|$ for all $n \geq 0$. It remains to show $\omega_w(x_n) \subseteq \mathscr{S}$ holds true. As a matter of fact, it follows from (4.2) that

$$\begin{aligned} \lim_{n\to\infty} r_n \sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2 &\leq \lim_{n\to\infty} \langle x_n - z, x_n - z_n \rangle \\ &\leq \lim_{n\to\infty} \|x_n - z\| \|x_n - z_n\| = 0. \end{aligned}$$

However, from the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
r_n &= \frac{\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2}{\|\sum_{i=0}^N A_i^*(A_i x_n - P_{Q_i}(A_i x_n))\|^2} \\
&\geq \frac{\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2}{(\sum_{i=0}^N \|A_i^*\| \|(A_i x_n - P_{Q_i}(A_i x_n))\|)^2} \\
&= \frac{\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2}{(\sum_{i=0}^N \|A_i\| \|A_i x_n - P_{Q_i}(A_i x_n)\|)^2} \\
&\geq \frac{\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2}{(\sum_{i=0}^N \|A_i\|^2)(\sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\|^2)} \\
&\geq \frac{1}{\sum_{i=0}^N \|A_i\|^2}.
\end{aligned}
$$

Consequently, this yields $\lim_n \sum_{i=0}^N \|A_i x_n - P_{Q_i}(A_i x_n)\| = 0$. Thus we can prove by a similar method that $\omega_w(x_n) \subseteq \mathscr{S}$. □

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