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HYBRID FUNCTIONS APPROACH FOR FREDHOLM INTEGRAL EQUATION OF THE FIRST KIND

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Abstract. This paper introduces a new computational direct method using the hybrid block-pulse functions and the Euler polynomials for solving the Fredholm integral equation of the first kind. The properties of hybrid functions are utilized for converting the integral equations to a linear system of algebraic ones. The key feature of our method is the low cost in setting up the equations without the aid of projections. Finally, some computational instances are presented to demonstrate the high accuracy and wide applicability of our method.

Keywords. Block-pulse functions; Direct method; Euler polynomials; Fredholm integral equation; First kind.

1. INTRODUCTION

Mathematical tools are essential in modeling and solving various problems [1, 14, 15, 26, 31, 34]. One of the equations used in various fields of science and engineering is the integral equation; see, e.g., [3–5, 10, 32, 33]. Fredholm's integral equation of the first kind occurs in several engineering fields and physical subfields, for example, in plasma diagnostics, physical electronics, nuclear physics, and optimal imaginary [16]. Also, the exact solutions of integral equations play a key role in adequately comprehending the qualitative features of phenomena and processes in different fields of natural sciences. Hence, studying and solving such problems is very practical.

This paper deals with the numerical solutions of the Fredholm integral equation as follows:

$$\int_0^1 k(t,s)f(s)\mathrm{d}s = g(t), \qquad 0 \le t \le 1, \tag{1.1}$$

where $g(t)$ and $k(t,s)$ are known functions belonging to $[0,1]$ and $[0,1]\times[0,1]$, and $f(t)$ is the unknown function.

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In many science problems, the Fredholm integral equation of the first kind appears as an ill-posed problem. It means that: (a) there is no solution to the Fredholm integral equation of the first kind, (b) its solution is not unique, and (c) it is not continuously dependent on the problem data [11, 17]. Recently, many numerical methods were utilized for estimating the solution of Eq. (1.1). Babolian and Delves [6] illustrated the augmented Galerkin method for the Fredholm integral equations of the first kind. Lewis [19] worked on a numerical method for solving the first kind of integral equations. Haar wavelets were used to solve the Fredholm integral equation of the first kind in [20]. Rabbani et al. [27] studied the Fredholm's integral equation by using computational projection methods. Maleknejad and Sohrabi [21] applied the Legendre wavelets to achieve a proximate solution for the Fredholm integral equation of the first kind. In [29], the authors used Legendre multi-wavelets to solve Eq. (1.1). Using the Chebyshev wavelet method, Adibi and Assari [2] solved the Fredholm integral equation of the first kind. Maleknejad and Saeedipoor [23] solved the Fredholm integral equation of the first kind, based on hybrid functions. Also, Bahmanpour et al. [7] solved the Fredholm integral equations of the first kind by the Müntz wavelets.

In this work, we introduce a computational method based on the hybrid block-pulse functions and the Euler polynomials for finding the solution of Fredholm integral equations of the first kind. In recent years, such hybrid functions have been increasingly used to solve several types of integral equations [12, 28], integro-differential equations [22, 24], and control problems [25]. As a significant property of the mentioned hybrid functions, we can recall their efficiency and wide range of applications. The block-pulse functions and the Euler polynomials give us a more highly accurate numerical solution with other existing basis functions. Another benefit of the hybrid function is its helpfulness in dealing with non-sufficiently smooth solutions related to C^1 and C^2 classes. This paper consists of the following sections. We provide some preliminaries of the hybrid block-pulse functions and Euler polynomials in Section 2. Section 3 briefly introduces some concepts, such as the function of a variable approximation, the function of two variable approximations, and the cross-product integration. Section 4 explains our numerical method for solving Eq. (1.1) and illustrates the approximation errors. Section 5 provides some numerical instances to demonstrate the validity and applicability of our scheme. Finally, in Section 6, a summarized conclusion is given.

2. Hybrid Block-pulse Functions and Euler Polynomials

In this section, the Euler polynomials are introduced, and some of their properties are expressed, and then hybrid functions based on these polynomials and the block-pulse functions are established.

The Euler polynomials of order m are defined by (see [9])

$$E_m(x) = \sum_{k=0}^{m} \binom{m}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{m-k},$$

where $E_k,\ k = 0, 1, \ldots, m$ are Euler numbers. The Euler numbers E_k are defined by the following generating functions:

$$\frac{2e^t}{e^{2t}+1} = \sum_{k=0}^{\infty} E_k \frac{t^k}{k!}.$$

The first few Euler numbers are

$$E_0 = 1, \qquad E_1 = 0, \qquad E_2 = -1, \qquad E_4 = 5,$$

with $E_{2k+1} = 0$, $k = 1, 2, \ldots$. The first few Euler polynomials are

$$E_0(x) = 1, \quad E_1(x) = x - \frac{1}{2}, \quad E_2(x) = x^2 - x,$$

$$E_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{4}, \quad E_4(x) = x^4 - 2x^3 + x.$$

These polynomials satisfy the following formula [30]

$$E_k = 2^k E_k(\frac{1}{2}), \quad k \geq 0,$$

$$E_n(x+1) + E_n(x) = 2x^n, \quad n \geq 0,$$

$$E_n'(x) = nE_{n-1}(x), \quad n \geq 1,$$

and

$$\int_y^x E_n(t)dt = \frac{E_{n+1}(x) - E_{n+1}(y)}{n+1}.$$

A set of the block-pulse functions $\psi_j(t)$, $j = 1, 2, \ldots, N$, is defined on the interval $[0, 1)$ as follows:

$$\psi_j(t) = \begin{cases} 1, & \frac{j-1}{N} \leq t \leq \frac{j}{N}, \\ 0, & otherwise, \end{cases}$$

where N is a positive integer and $\psi_j(t)$ is the j-th block-pulse function. The most consequential specialty of block-pulse functions is their orthogonality, disjointness, and completeness [13].

The hybrid functions $e_{nm}(t)$, $n = 1, 2, \ldots, N$, $m = 0, 1, 2, \ldots, M$ are expressed on the interval $[0, t_f)$ as follows:

$$e_{nm}(t) = \begin{cases} E_m(Nt - (n-1)t_f), & \frac{n-1}{N} t_f \leq t < \frac{n}{N} t_f, \\ 0, & otherwise, \end{cases}$$

where n and m are the order of the block-pulse functions and the Euler polynomials, respectively.

3. Function Approximation

Let $V = L^2[0, 1]$, $\{e_{10}(t), e_{11}(t), \ldots, e_{NM}(t)\} \subset V$ be the set of hybrid block-pulse functions and Euler polynomials,

$$U = span\{e_{10}(t), e_{11}(t), \ldots, e_{1M}(t), e_{20}(t), \ldots, e_{2M}(t), \ldots, e_{N0}(t), \ldots, e_{NM}(t)\},$$

and f be an arbitrary element of V. Since U is a finite-dimensional vector space, f has the unique best approximation out of U, such as $f_0 \in U$, which is

$$\forall u \in U, \parallel f - f_0 \parallel \leq \parallel f - u \parallel.$$

Since $f_0 \in U$, there exists the unique coefficients $c_{10}, c_{11}, \ldots, c_{NM}$ such that

$$f \simeq f_0 = P_M^N(f) = \sum_{m=0}^{M} \sum_{n=1}^{N} c_{nm} e_{nm}(t) = \mathbf{C}^T \mathbf{E}(t), \tag{3.1}$$

where

$$\mathbf{C} = [c_{10}(t), c_{11}(t), \ldots, c_{1M}(t), c_{20}(t), c_{21}(t), \ldots, c_{2M}(t), \ldots, c_{N0}(t), \ldots, c_{NM}(t)]^T$$

and

$$\mathbf{E}(t)=[e_{10}(t),e_{11}(t),\ldots,e_{1M}(t),e_{20}(t),e_{21}(t),\ldots,e_{2M}(t),\ldots,e_{N0}(t),\ldots,e_{NM}(t)]^T. \tag{3.2}$$

Now, we specify the matrix $\mathbf{D}$. Using Eq. (3.1), we have

$$f_{pq}=<\sum_{m=0}^{M}\sum_{n=1}^{N}c_{nm}e_{nm}(t),e_{pq}(t)>=\sum_{m=0}^{M}\sum_{n=1}^{N}c_{nm}d_{nm}^{pq},$$

$$p=1,2,\ldots,N,\ q=0,1,2,\ldots,M,$$

where $f_{pq}=<f,e_{pq}(t)>$, $d_{nm}^{pq}=<e_{nm}(t),e_{pq}(t)>$, and $<.,.>$ denotes the inner product. Accordingly, $f_{pq}=\mathbf{C}^T[d_{10}^{pq},d_{11}^{pq},\ldots,d_{1M}^{pq},d_{20}^{pq},d_{21}^{pq},\ldots,d_{2M}^{pq},\ldots,d_{N0}^{pq},d_{N1}^{pq},\ldots,d_{NM}^{pq}]^T$, $p=1,2,\ldots,N$ and $q=0,1,2,\ldots,M$. Hence, $\varphi=\mathbf{D}^T\mathbf{C}$ with

$$\varphi=[f_{10},f_{11},\ldots,f_{1M},f_{20},f_{21},\ldots,f_{2M},\ldots,f_{N0},f_{N1},\ldots,f_{NM}]^T,$$

and $\mathbf{D}=[d_{nm}^{pq}]$, where $\mathbf{D}$ is a matrix of order $N(M+1)\times N(M+1)$ and is resulted by

$$\mathbf{D}=\int_0^1\mathbf{E}(t)\mathbf{E}^T(t)\mathrm{d}t. \tag{3.3}$$

Using Eq. (3.2) in each interval $n=1,2,\ldots,N$, we are capable to obtain matrix $\mathbf{D}$. For instance, for $N=3$ and $M=3$, $\mathbf{D}$ is

$$\mathbf{D}=\begin{pmatrix}
\frac{1}{3}&0&0&0&0&0&0&0&0&0&0&0\\
0&\frac{1}{36}&0&-\frac{1}{360}&0&0&0&0&0&0&0&0\\
0&0&\frac{1}{540}&0&0&0&0&0&0&0&0&0\\
0&-\frac{1}{360}&0&\frac{1}{2520}&0&0&0&0&0&0&0&0\\
0&0&0&0&\frac{1}{3}&0&0&0&0&0&0&0\\
0&0&0&0&0&\frac{1}{36}&0&-\frac{1}{360}&0&0&0&0\\
0&0&0&0&0&0&\frac{1}{540}&0&0&0&0&0\\
0&0&0&0&0&-\frac{1}{360}&0&\frac{1}{2520}&0&0&0&0\\
0&0&0&0&0&0&0&0&\frac{1}{3}&0&0&0\\
0&0&0&0&0&0&0&0&0&\frac{1}{36}&0&-\frac{1}{360}\\
0&0&0&0&0&0&0&0&0&0&\frac{1}{540}&0\\
0&0&0&0&0&0&0&0&0&-\frac{1}{360}&0&\frac{1}{2520}
\end{pmatrix}.$$

It is observed that matrix $\mathbf{D}$ is a sparse matrix. In addition, if we select large values of M and N, non-zero elements of $\mathbf{D}$ will tend to zero. Since the best estimation is unique, $\mathbf{D}$ is invertible [18], and we can also obtain the approximation of the two variables function. If $k(t,s)$ is a function of two variables defined over the interval $t\in[0,1]$ and $s\in[0,1]$, then $k(t,s)$ can be extended $k(t,s)=\mathbf{E}^T(t)\hat{K}\mathbf{E}(s)$. We put row vector $K_{(A)}$ as

$$K_{(A)}=[K_{(A)}^{10}(t),K_{(A)}^{11}(t),\ldots,K_{(A)}^{1M}(t),\ldots,K_{(A)}^{N1}(t),K_{(A)}^{N2}(t),\ldots,K_{(A)}^{NM}(t)],$$

where $K_{(A)}^{pq}(t)=\int_0^1k(t,s)e_{pq}(s)\mathrm{d}s$. Then we put row vector $K_{(B)}$ as

$$K_{(B)}=K_{(A)}.\mathbf{D}^{-1}=[K_{(B)}^{10}(t),\ldots,K_{(B)}^{1M}(t),\ldots,K_{(B)}^{N1}(t),K_{(B)}^{N2}(t),\ldots,K_{(B)}^{NM}(t)].$$

Also, we have row vector $K_{(C)}^{pq}$, $p=1,2,\ldots,N$, $q=0,1,2,\ldots,M$, as

$$K_{(C)}^{pq}=[K_{(C)}^{10}(t),K_{(C)}^{11}(t),\ldots,K_{(C)}^{1M}(t),\ldots,K_{(C)}^{N1}(t),K_{(C)}^{N2}(t),\ldots,K_{(C)}^{NM}(t)], \tag{3.4}$$

where $K_{nm}^{pq} = \int_0^1 K_{(B)}^{pq} e_{nm}(t)\mathrm{d}t$. By using Eq. (3.4), we have matrix $\bar{K}$ as

$$\bar{K} = [K_{(C)}^{10}(t), K_{(C)}^{11}(t), \ldots, K_{(C)}^{1M}(t), \ldots, K_{(C)}^{N1}(t), K_{(C)}^{N2}(t), \ldots, K_{(C)}^{NM}(t)]^T. \tag{3.5}$$

So by using Eq. (3.5), we have $\hat{K} = \bar{K}.\mathbf{D}^{-1}$. For example, we show the graph of the function $k(t,s) = \sin(ts)$ with its approximations for $M = 2, N = 2$ and $M = 2, N = 3$ in Figures 1-3.

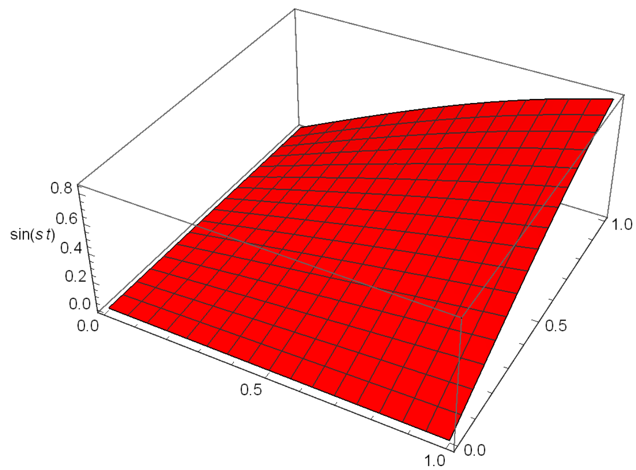

FIGURE 1. $g(t,s) = \sin(ts)$

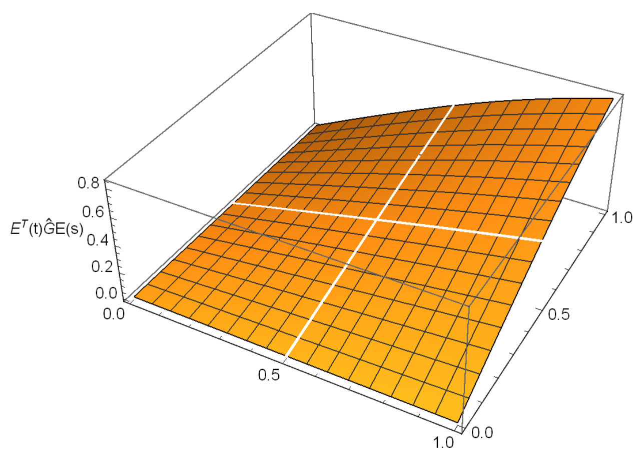

FIGURE 2. $g(t,s) \approx E^T(t)\hat{G}E(s)$ with $N = 2, M = 2$

4. DESCRIPTION OF THE PROPOSED METHOD AND ITS CONVERGENCE

This section illustrates our scheme for solving Eq. (1.1) via hybrid functions, and the convergence rate is investigated. Therefore, the functions in Eq. (1.1) are estimated, as discussed in the previous section, in the following order:

$$f(t) = \mathbf{F}^T\mathbf{E}(t), \tag{4.1}$$

$$g(t) = \mathbf{G}^T\mathbf{E}(t), \tag{4.2}$$

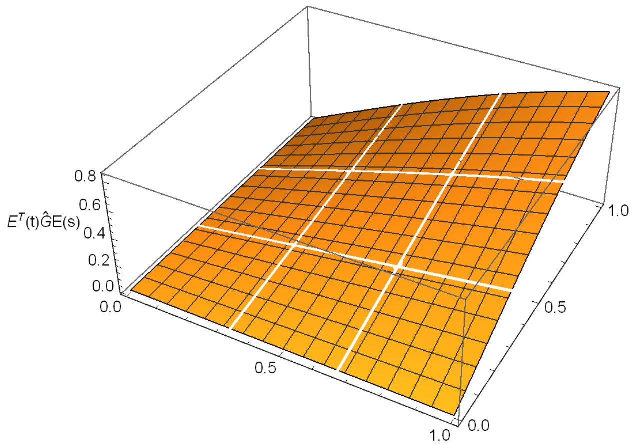

FIGURE 3. $g(t,s) \approx E^T(t)\hat{G}E(s)$ with $N=3$, $M=2$

and

$$k(t,s) = \mathbf{E}^T(t)\hat{\mathbf{K}}\mathbf{E}(s). \tag{4.3}$$

The matrix $\hat{\mathbf{K}}$ is known by the dimension $N(M+1) \times N(M+1)$. Also, the vector $\mathbf{G}$ with dimension $N(M+1)$ is known. According to Eq. (4.1), $\mathbf{F}$ with dimension $N(M+1)$ is an unknown vector. Substituting Eqs. (4.1)–(4.3) to Eq. (1.1), it results

$$\int_0^1 \mathbf{E}^T(t)\hat{\mathbf{K}}\mathbf{E}(s)\mathbf{E}^T(s)\mathbf{F}\mathrm{d}s = \mathbf{E}^T(t)\mathbf{G},$$

or

$$\mathbf{E}^T(t)\hat{\mathbf{K}}\left(\int_0^1 \mathbf{E}(s)\mathbf{E}^T(s)\mathrm{d}s\right)\mathbf{F} = \mathbf{E}^T(t)\mathbf{G}. \tag{4.4}$$

Also, using Eqs. (3.3) and (4.4), we obtain

$$\mathbf{E}^T(t)\hat{\mathbf{K}}\mathbf{D}\mathbf{F} = \mathbf{E}^T(t)\mathbf{G}.$$

Hence, the following linear system of equations is resulted: $\hat{\mathbf{K}}\mathbf{D}\mathbf{F} = \mathbf{G}$. Solving the linear system, it is possible to achieve the unknown vector $\mathbf{F}$.

Theorem 4.1. *Let H be a Hilbert space, and let W be a closed subspace of H such that $\dim W < \infty$ and $\{w_1, w_2, w_3, \dots, w_n\}$ is any basis for W. Let g be an arbitrary element of H, and let g_0 be the unique best approximation to g out of W. $\| g - g_0 \|_2 = G_g$, where*

$$G_g = \left(\frac{F(g, w_1, \dots, w_n)}{F(w_1, \dots, w_n)}\right)^{1/2},$$

and F is defined in [18].

Recall that the Sobolev norm of integer order $\mu \geq 0$ in the interval (a,b) is resulted by

$$\| f \|_{H^\mu(a,b)} = \Big(\sum_{k=0}^{\mu} \int_a^b |f^{(k)}(x)|^2 \mathrm{d}x\Big)^{1/2} = \Big(\sum_{k=0}^{\mu} \|f^{(k)}\|^2_{L^2(a,b)}\Big)^{1/2},$$

where $f^{(k)}$ denotes the (distributional) derivative of f of order k. To estimate the error of the hybrid Euler truncated series, $f - P_M^N(f)$, we recall from [8] that the truncation error, $f - P_M(f)$, where $P_M(f) = \sum_{m=0}^{M} c_m e_m(t)$, is the truncated Euler series of f and can be approximated as follows. For all $f \in H^{\mu}(0,1)$, $\mu \geq 0$, we have

$$\parallel f - P_M(f) \parallel_{L^2(0,1)} \leq cM^{-\mu} |f|_{H^{\mu,M}(0,1)},$$

where c is dependent on μ, and

$$|f|_{H^{\mu,M}(0,1)} = \Big(\sum_{k=\min(\mu,M+1)}^{\mu} \|f^{(k)}\|^2_{L^2(0,1)} \Big)^{1/2}.$$

Also, in the cases related to the truncation error of the derivatives, the following proximate expands Eq. (3.1) to higher order Sobolev norms:

$$\parallel f - P_M(f) \parallel_{H^r(0,1)} \leq cM^{2r-1/2-\mu} |f|_{H^{\mu,M}(0,1)}, \tag{4.5}$$

for $f \in H^{\mu}(0,1)$ with $\mu \geq 0$ and for any r such that $1 \leq r \leq \mu$. It is appropriate to introduce the following seminorm defined for $f \in H^{\mu}(0,1), 0 \leq r \leq \mu$, $M \geq 0$ and $N \geq 1$, as

$$|f|_{H^{r;\mu;M;N}(0,1)} = \Big(\sum_{k=\min(\mu,M+1)}^{\mu} N^{2r-2k} \|f^{(k)}\|^2_{L^2(0,1)} \Big)^{1/2}.$$

Recall that, whenever $M \geq \mu - 1$, there exists

$$|f|_{H^{r;\mu;M;N}(0,1)} = N^{r-\mu} \|f^{(\mu)}\|_{L^2(0,1)}. \tag{4.6}$$

Remark 4.2. The case $N = 1, |.|_{H^{r;\mu;M;N}}$ coincides with $|.|_{H^{\mu;M}}$ was presented and utilized in [8].

For defining the principal results, the following lemma is needed.

Lemma 4.3. *For $n = 1, 2, \ldots, N$, assume that $f_n : \left(\frac{n-1}{N}, \frac{n}{N}\right) \longrightarrow \mathbb{R}$ is a function in $H^{\mu}\left(\frac{n-1}{N}, \frac{n}{N}\right)$. If the function $F_n f_n : (0,1) \longrightarrow \mathbb{R}$ is in a way that $(F_n f_n)(x) = f_n\Big(\frac{1}{N}(x+n-1)\Big)$ for all $x \in (0,1)$, then, for $0 \leq l \leq \mu$, $\|(F_n f_n)^{(l)}\|_{L^2(0,1)} = N^{1/2-l} \|f_n^{(l)}\|_{L^2(\frac{n-1}{N},\frac{n}{N})}$.*

Proof. For $0 \leq l \leq \mu$, one has

$$\begin{aligned} \|(F_n f_n)^{(l)}\|^2_{L^2(0,1)} &= \int_0^1 |(F_n f_n)^{(l)}(x)|^2 \mathrm{d}x \\ &= \int_0^1 |f_n^{(l)}\Big(\frac{1}{N}(x+n-1)\Big)|^2 \mathrm{d}x \\ &= \int_{\frac{n-1}{N}}^{\frac{n}{N}} N^{-2l} |f_n^{(l)}(t)|^2 N \mathrm{d}t = N^{1-2l} \|f_n^{(l)}\|^2_{L^2(\frac{n-1}{N},\frac{n}{N})}, \end{aligned}$$

where, for the third section of the above equation, the change of the variable rule by defining $t = \frac{1}{N}(x+n-1)$ is applied. □

Finally, for error approximation, we have the following main theorem.

Theorem 4.4. *Let $f \in H^{\mu}(0,1)$, with $\mu \geq 0$. Then*

$$\parallel f - P_M^N(f) \parallel_{L^2(0,1)} \leq cM^{-\mu} |f|_{H^{0;\mu,M;N}(0,1)}, \tag{4.7}$$

and for the case $1 \leq r \leq \mu$,

$$\parallel f - P_M^N(f) \parallel_{H^r(0,1)} \leq cM^{2r-1/2-\mu} |f|_{H^{r;\mu,M;N}(0,1)}. \tag{4.8}$$

Proof. For $n = 1,2,3,\ldots,N$, we consider the function $f_n : \left(\frac{n-1}{N}, \frac{n}{N}\right) \longrightarrow \mathbb{R}$ such that $f_n(t) = f(t)$, for all $t \in \left(\frac{n-1}{N}, \frac{n}{N}\right)$, and we have

$$\begin{aligned}\| f - P_M^N(f) \|_{L^2(0,1)} &= \sum_{n=1}^{N} \| f_n - \sum_{m=0}^{M} c_{nm} e_{nm}(t) \|^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \\ &= N^{-1} \sum_{n=1}^{N} \| F_n f_n - P_M(F_n f_n) \|_{L^2(0,1)},\end{aligned}$$

where Lemma 4.3 is sued for the second equality. From Eq. (4.6), one has

$$\begin{aligned}\| f - P_M^N(f) \|_{L^2(0,1)} &\le cN^{-1} M^{-2\mu} \sum_{n=1}^{N} |F_n f_n|^2_{H^{\mu,M}(0,1)} \\ &= cN^{-1} M^{-2\mu} \sum_{n=1}^{N} \sum_{k=\min(\mu,M+1)}^{\mu} \|(F_n f_n)^{(k)}\|^2_{L^2(0,1)} \\ &= cM^{-2\mu} \sum_{k=\min(\mu,M+1)}^{\mu} N^{-2k} \|f_n^{(k)}\|^2_{L^2(0,1)}.\end{aligned}$$

Thus we obtain (4.7). Also, for $1 \le r \le \mu$, we conclude by Eq. (4.5) that

$$\begin{aligned}\| f - P_M^N(f) \|_{H^r(0,1)} &= \sum_{n=1}^{N} \| f_n - \sum_{m=0}^{M} c_{nm} e_{nm}(t) \|^2_{H^r(\frac{n-1}{N},\frac{n}{N})} \\ &= \sum_{n=1}^{N} \sum_{p=0}^{r} \| (f_n)^p - (\sum_{m=0}^{M} c_{nm} e_{nm}(t))^p \|^2_{L^2(\frac{n-1}{N},\frac{n}{N})} \\ &= \sum_{n=1}^{N} \sum_{p=0}^{r} N^{2p-1} \| (F_n f_n)^p - (P_M(F_n f_n))^p \|^2_{L^2(0,1)} \\ &\le \sum_{n=1}^{N} N^{2r-1} \| F_n f_n - P_M(F_n f_n) \|^2_{H^r(0,1)} \\ &\le \sum_{n=1}^{N} cN^{2r-1} M^{4r-1-2\mu} (|F_n f_n|_{H^{\mu;M}(0,1)})^2 \\ &= cN^{2r-1} M^{4r-1-2\mu} \sum_{n=1}^{N} \sum_{k=\min(\mu,M+1)}^{\mu} \|(F_n f_n)^{(k)}\|^2_{L^2(0,1)} \\ &= cM^{4r-1-2\mu} \sum_{k=\min(\mu,M+1)}^{\mu} N^{2r-2k} \|f_n^{(k)}\|^2_{L^2(0,1)}.\end{aligned}$$

This yields Eq. (4.8). □

Remark 4.5. By setting $M \ge \mu - 1$ in Eqs. (4.7) and (4.8), and using Eq. (4.6), we have

$$\| f - P_M^N(f) \|_{L^2(0,1)} \le cM^{-\mu} N^{-\mu} \|f_n^{(\mu)}\|^2_{L^2(0,1)}, \tag{4.9}$$

and

$$\| f - P_M^N(f) \|_{H^r(0,1)} \le cM^{2r-1/2-\mu} N^{r-\mu} \|f_n^{(\mu)}\|^2_{L^2(0,1)}, \qquad r \ge 1. \tag{4.10}$$

For the case that f is infinitely smooth, relations (4.9) and (4.10) demonstrate that the rate of convergency of $P_M^N(f)$ to f is faster than $\frac{1}{N}$ to the power of $M+1-r$ and any power of $\frac{1}{M}$, which is superior to that of the classical spectral methods.

5. Numerical Experiments

In this section, we introduce some examples to estimate a solution of Fredholm integral equations of the first kind by using the numerical method illustrated in the previous sections. To show the precision and the utility of the presented scheme, we separately compare the results of the proposed scheme with the results from other ones in such a way that $f(t)$ and $f_{N(M+1)}(t)$ are the exact and the approximated solutions, respectively. The Mathematica 7 software is used for the numerical simulations.

Example 5.1. Consider the following first kind integral equation:

$$\int_0^1 \sin(ts)f(s)\mathrm{d}s = \frac{\sin t - t\cos t}{t^2}, \quad 0 \le t \le 1,$$

where $f(t) = t$ is the exact solution. The absolute errors of the presented scheme and other methods in [2,23] are tabulated in Table 1. This table confirms that the accuracy of our method is better than those cited methods. Also, Table 1 explains that suitable estimated results are achieved with a few hybrid functions.

Table 1. Absolute errors for Example 5.1

t	Proposed method with $N=1, M=1$	The method in [23] with $N=2, M=3$	The method in [2] with $k=2, m=2$
1/10	$5.05707E-14$	$5.37E-11$	$5.32E-5$
2/10	$4.14668E-14$	$1.81E-11$	$1.37E-4$
3/10	$3.23908E-14$	$3.61E-11$	$2.21E-4$
4/10	$2.33008E-14$	$1.57E-13$	$3.04E-4$
5/10	$1.87558E-14$	$1.70E-11$	$5.98E-4$
6/10	$5.12090E-15$	$8.14E-12$	$3.80E-4$
7/10	$3.96905E-15$	$2.16E-13$	$5.51E-4$
8/10	$1.30451E-14$	$8.99E-13$	$7.21E-4$
9/10	$2.21489E-14$	$1.06E-12$	$8.92E-4$

Example 5.2. Consider the following Fredholm integral equation:

$$\int_0^1 e^{ts}f(s)\mathrm{d}s = \frac{e^{t+1}-1}{t+1}, \quad 0 \le t \le 1,$$

$f(t) = e^t$ is the exact solution to this problem. Table 2 demonstrates the numerical results for this instance. This table shows the comparative results between the proposed scheme and the ones in [21,23].

Example 5.3. Consider the following equation:

$$\int_0^1 \sqrt{t^2+s^2}f(s)\mathrm{d}s = w(t), \quad 0 \le t \le 1,$$

TABLE 2. Absolute errors illustrated in Example 5.2

t	Proposed method with $N=2, M=2$	The method in [23] with $N=2, M=3$	The method in [21] with $k=2, M=3$
1/10	$2.14641E-4$	$2.04E-4$	$5.01E-4$
2/10	$2.11801E-4$	$2.29E-4$	$3.33E-4$
3/10	$4.61740E-4$	$4.60E-4$	$4.61E-4$
4/10	$7.50456E-4$	$7.85E-4$	$5.95E-4$
5/10	$2.34321E-3$	$6.17E-3$	$2.24E-3$
6/10	$5.55213E-4$	$5.88E-4$	$8.39E-4$
7/10	$9.01169E-4$	$9.01E-4$	$7.83E-4$
8/10	$1.97461E-4$	$2.11E-4$	$4.89E-4$
9/10	$5.46689E-4$	$5.37E-4$	$8.57E-4$

where

$$w(t)=\frac{1}{48}\left(16\left(t^2\right)^{3/2}-2\sqrt{1+t^2}\left(2+5t^2\right)+3t^4\left(Log\left[t^2\right]-2Log\left[1+\sqrt{1+t^2}\right]\right)\right).$$

The exact solution to this problem is $f(t)=t(t-1)$. The absolute errors of the introduced method for $N=1$ and $M=2$ are presented in Table 3. In this table, the results of the proposed scheme are compared with the ones in [23].

TABLE 3. Comparison between proposed method and method [23] for Example 5.3

t	Proposed method with $N=1, M=2$	The method in [23] with $N=2, M=2$	The method in [23] with $N=2, M=3$
1/10	$6.30052E-15$	$6.29E-3$	$1.94E-13$
2/10	$5.55112E-16$	$1.63E-2$	$2.58E-14$
3/10	$3.52496E-15$	$6.37E-3$	$6.19E-13$
4/10	$6.02296E-15$	$2.35E-2$	$1.74E-12$
5/10	$6.88338E-15$	$1.74E-2$	$1.25E-11$
6/10	$6.13398E-15$	$1.56E-2$	$2.74E-12$
7/10	$3.74700E-15$	$2.88E-2$	$2.70E-12$
8/10	$1.94289E-16$	$4.80E-2$	$3.81E-12$
9/10	$5.82867E-15$	$4.90E-3$	$5.60E-13$

Example 5.4. Consider the following Fredholm integral equation of the first kind:

$$\int_0^1 \sqrt{t^2+s^2} f(s)\mathrm{d}s=\frac{\left(1+t^2\right)^{3/2}-t^3}{3}, \quad 0\leq t\leq 1,$$

with the exact solution $f(t)=t$. In Table 4, the numerical results of the proposed scheme are compared with the ones of the Chebyshev wavelets method [2].

Example 5.5. Consider the following Fredholm integral equation:

$$\int_0^1 \sqrt{t+s} f(s)\mathrm{d}s=w(t), \quad 0\leq t\leq 1,$$

TABLE 4. Absolute errors for Example 5.4

t	Proposed method with $N=1, M=2$	The method in [2] with $N=2, M=3$	Exact solution
1/10	$1.12965E-14$	$1.80309E-9$	0.1
2/10	$6.93889E-17$	$1.19155E-9$	0.2
3/10	$1.11716E-14$	$4.18620E-9$	0.3
4/10	$2.23987E-14$	$7.18084E-9$	0.4
5/10	$3.25295E-14$	$1.21431E-8$	0.5
6/10	$1.53211E-14$	$8.37393E-9$	0.6
7/10	$9.18016E-15$	$4.60473E-9$	0.7
8/10	$3.03230E-15$	$8.35530E-10$	0.8
9/10	$3.10862E-15$	$2.93367E-9$	0.9

where

$$w(t) = -\frac{16}{105}t^{7/2} + \frac{2}{7}(1+t)^{3/2} - \frac{8}{35}t(1+t)^{3/2} + \frac{16}{105}t^2(1+t)^{3/2},$$

with the exact solution $f(t) = t^2$. The tabulated absolute errors in Table 5 resulted from our method and the approximated results achieved in [27].

TABLE 5. Absolute errors for Example 5.5

t	Proposed method with $N=2, M=1$	The method in [27] with $j=6$	Exact solution
$1/10$	$7.61613E-14$	$1.87149E-2$	0.01
$2/10$	$1.26565E-14$	$2.16759E-3$	0.04
$3/10$	$7.41074E-14$	$1.18126E-2$	0.09
$4/10$	$1.08302E-13$	$1.53514E-2$	0.16
$5/10$	$1.15130E-13$	$1.17887E-2$	0.25
$6/10$	$9.45910E-14$	$1.17545E-2$	0.36
$7/10$	$4.67126E-14$	$1.02953E-2$	0.49
$8/10$	$2.85327E-14$	$2.06099E-2$	0.64
$9/10$	$1.31062E-13$	$6.57294E-3$	0.81

6. CONCLUSION

In this paper, an efficient direct scheme, based on the hybrid block-pulse functions and Euler polynomials, was implemented to find the solution of Fredholm integral equations of the first kind. A linear system of algebraic equations corresponding to the mentioned problem was achieved by employing the special properties of hybrid functions. The numerical simulations demonstrated the high accuracy and broad applicability of our method. Also, the presented examples proved that the estimations with hybrid functions contained a unique suitable precision for non-sufficiently smooth solutions belonging to the class of C^1 and C^2. We achieved an acceptable outcome only with a few hybrid basis functions. Our approach has the potential to be simply expanded and implemented to the Volterra and nonlinear Fredholm integral equations of the first kind.

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