J. Nonlinear Funct. Anal. 2022 (2022) 36 https://doi.org/10.23952/jnfa.2022.36

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

AN OPTIMIZATION PROBLEM RELATED TO A QUASILINEAR EQUATION

CHONG QIU[1], QIANSHENG FENG[1,*], ZENG LIU[2], SHAOCHUN JI[1]

[1]Faculty of Mathematics and Physics of Huaiyin Institute of Technology, Huaian 223001, China
[2]Department of Mathematics of Suzhou University of Science and Technology, Suzhou 215000, China

Abstract. In this paper, we consider a rearrangement minimization problem related to a quasilinear equation with two independent data functions. First, we demonstrate that the equation has a unique ground state solution for each pair of the given data functions. Next, we prove that the corresponding rearrangement minimization problem is solvable. Finally, we prove the uniqueness and symmetry of the solution of the minimization problem if the domain is a ball centered at the origin.

Keywords. Ground state solution; Quasilinear equation; Rearrangement minimization problem.

1. Introduction

For a given measurable function $f(x):\Omega\to\mathbb{R}$, where $\Omega\subset\mathbb{R}^N (N\geq 3)$ is a bounded smooth domain, we say that $g(x):\Omega\to\mathbb{R}$ is a rearrangement of f if and only if

$$meas(\{x\in\Omega : g(x)\geq a\}) = meas(\{x\in\Omega : f(x)\geq a\}),\quad \forall a\in\mathbb{R},$$

and the set of all the rearrangements of f is often denoted by $\mathscr{R}(f)$. A rearrangement optimization problem is often referred to as an optimization problem which takes all the rearrangements of a given function as the admissible set. These problems have been investigated by many authors since Burton's fundamental work [5, 6]; see, e.g., [1, 2, 4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 25] and the references therein.

Cuccu and Porru [9] and Marras [17] considered two different rearrangement optimization problems respectively related to the same following boundary value problem

$$(\mathscr{P}_f)\qquad \begin{cases} -\Delta_p u = f(x) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

*Correspoinding author.
E-mail address: chongqiu@hyit.edu.cn (C. Qiu), fqs2008@hyit.edu.cn (Q. Feng), zliu@mail.usts.edu.cn (Z. Liu), jiscmath@163.com (S. Ji).

Received August 18, 2021; Accepted August 22, 2022.

The solvability of the both rearrangement optimization problems were obtained respectively. Inspired by these results, in [22], Qiu et al. considered two rearrangement optimization problems related to the following quasilinear elliptic boundary value problem:

$$(\mathscr{P}) \qquad \begin{cases} -\Delta_p u + h(x,u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $1 < p < \infty$, and $h(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. The authors proved that the minimum and maximum optimization problems related to $(\mathscr{P})$ are solvable in both cases of $1 < p \leq N$ and $p > N$, which extended the corresponding results in [5, 6] with $p = 2$ and [9, 17] with $1 < p < \infty$. In [22], an essential assumption is that $h(x,t)$ is non-decreasing with respect to the second variable. However, in the present paper, we will consider a rearrangement optimization problem related to the following quasilinear equation:

$$(\mathscr{P}_{h,f}) \qquad \begin{cases} -\Delta_p u - \lambda h(x)|u|^{p-2}u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $h(x) > 0$. Obviously, the term $-\lambda h(x)|u|^{p-2}u$ in the problem $(\mathscr{P}_{h,f})$ would be decreasing with respect to u and then it violates the essential assumption given in [22]. By using a convex inequality, we can prove that the problem $(\mathscr{P}_{h,f})$ has a unique ground state solution for the given two functions f and h when λ is small enough (cf. Proposition 3.1 of Section 3). Let $I_{h,f} : W_0^{1,p}(\Omega) \to \mathbb{R}$ be the energy functional corresponding to the problem $(\mathscr{P}_{h,f})$, which is given by

$$I_{h,f}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega h|u|^p dx - \int_\Omega f u dx, \tag{1.1}$$

and let $0 < h_0 \in L^\infty(\Omega)$ and $f_0 \in L^r(\Omega)$ for some positive r be two given functions. Then we study the following minimum optimization problem:

$$(OPT) \qquad \inf_{h \in \mathscr{R}(h_0), f \in \mathscr{R}(f_0)} I_{h,f}(u_{h,f}),$$

where $\mathscr{R}(h_0)$ and $\mathscr{R}(f_0)$ denote the sets of all rearrangement of h_0 and f_0, respectively, and $u_{h,f}$ is the unique ground state solution to $(\mathscr{P}_{h,f})$.

We show that there exists $\lambda_* > 0$ such that, for all $\lambda \in (0, \lambda_*)$, problem (OPT) is solvable.

We note that the minimum optimization problem considered here is constrained by two rearrangement sets generated by two fixed independent data functions. However, the optimization problem considered in all the papers mentioned above was constrained by a rearrangement set which was generated by just one fixed function. Therefore our case needs special handling. To the best of our knowledge, the results obtained in this paper are new.

This paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we show that the problem $(\mathscr{P}_{h,f})$ has a unique ground state solution. Section 4 is devoted to discuss the minimization problem (OPT) in detail.

2. PRELIMINARIES

We denote by $L^r(\Omega)$ $(1 \leq r \leq \infty)$ and $W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ the usual Sobolev spaces endowed with the norms $\|u\|_{L^r} = (\int_\Omega |u|^r dx)^{1/r}$ if $1 \leq r < \infty$, $\|u\|_\infty = \text{ess sup}_{x\in\Omega}|u(x)|$ if $r = \infty$, and $\|u\| =$

$(\int_\Omega |\nabla u|^p dx)^{1/p}$, respectively. Throughout the paper, C denotes a positive (possibly different) constant.

By a solution u of problem $(\mathscr{P}_{h,f})$, we mean that $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v - \lambda h|u|^{p-2}uv - fv\right) dx = 0,\ \forall v \in W_0^{1,p}(\Omega).$$

Let $I_{h,f}$ be given in (1.1). It is easy to check that $I_{h,f} \in C^1(W_0^{1,p}(\Omega),\mathbb{R})$ and

$$I'_{h,f}(u)v = \int_\Omega \left(|\nabla u|^{p-2}\nabla u \nabla v - \lambda h|u|^{p-2}uv - fv\right) dx,\ \forall v \in W_0^{1,p}(\Omega).$$

Therefore, $u \in W_0^{1,p}(\Omega)$ is a weak solution if and only if $I'_{h,f}(u)v = 0$, $\forall v \in W_0^{1,p}(\Omega)$.

Let $f : \Omega \mapsto [0,\infty)$ be a measurable function. The Schwarz symmetric decreasing rearrangement of f is the function $f^* : B(0,r) \mapsto [0,\infty)$ defined by ([16, Definition 16.5])

$$f^*(x) = \inf\left\{t \in [0,\infty) : \mu_f(t) \le \omega_N |x|^N\right\}, \forall x \in B(0,r)$$

where ω_N denotes the volume of the unit ball in N-dimensions, $r := (\text{meas}(\Omega)/\omega_N)^{1/N}$, and $\mu_f : \mathbb{R} \mapsto [0,\infty)$ is the distribution function of f defined by

$$\mu_f(t) = \text{meas}(\{x \in \Omega : f(x) > t\}).$$

It is well known that $f^* = g^*$ for each $g \in \mathscr{R}(f)$. The following lemmas are used through the proofs of our main results.

Lemma 2.1. [6, Lemma 2.1] *Let $1 \le r < \infty$ and $f \in L^r(\Omega)$. Then, for any $g \in \mathscr{R}(f)$, $g \in L^r(\Omega)$ and $\|g\|_{L^r} = \|f\|_{L^r}$.*

Lemma 2.2. [5, Theorem 5] *For any $1 \le r < \infty$, define $r' = \frac{r}{r-1}$ if $r > 1$ and $r' = \infty$ if $r = 1$. Let $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$. If the linear functional $L(l) = \int_\Omega lg dx$ has a unique maximizer (minimizer) $\hat{f}$ relative to $\mathscr{R}(f)$, then there exists an increasing (decreasing) function $\varphi : \mathbb{R} \mapsto \mathbb{R}$ such that $\varphi \circ g = \hat{f}$.*

Lemma 2.3. [14, Lemma 2.3] *Let $f \in L^r(\Omega)$ and $g \in L^{r'}(\Omega)$. Then there exists $\hat{f} \in \mathscr{R}(f)$ which maximizes (minimizes) the linear functional $\int_\Omega hg dx$, relative to $h \in \overline{\mathscr{R}(f)^w}$, where $\overline{\mathscr{R}(f)^w}$ denotes the weak closure of $\mathscr{R}(f)$ in $L^r(\Omega)$.*

Lemma 2.4. [16, Theorem 16.9] *If B is a ball centered at the origin, then $\int_B fg dx \le \int_B f^* g^* dx$, for any non-negative measurable functions f and g.*

The following result can be deduced from Lemmas 2.3 and 3.2 and Theorem 1.1 of [3].

Lemma 2.5. *Suppose that B is a ball centered at the origin. If $u \in W_0^{1,p}(B)$ with $1 < p < \infty$ and $u \ge 0$, then $u^* \in W_0^{1,p}(\Omega)$ and*

$$\int_B |\nabla u|^p dx \ge \int_B |\nabla u^*|^p dx. \tag{2.1}$$

If the equality holds in (2.1), then $u^{-1}(\alpha,\infty)$ is a translation of $u^{*-1}(\alpha,\infty)$ for every $\alpha \in [0, \text{ess sup}_{x\in B} u(x))$. Moreover, if the set

$$\left\{x \in B : \nabla u(x) = 0, 0 < u(x) < \operatorname*{ess\,sup}_{y\in B} u(y)\right\}$$

has zero measure, then $u = u^*$.

It is well known that the first eigenvalue $\lambda_1(h)$ of the following eigenvalue problem

$$(\mathscr{L}_h) \qquad \begin{cases} -\Delta_p u = \lambda h(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

can be characterized by

$$\lambda_1(h) = \inf_{v \in W_0^{1,p}(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega h|v|^p dx}. \tag{2.2}$$

By [10, Theorem 3.1], if $0 < h_0(x) \in L^\infty(\Omega)$, then there exists $\bar{h} \in \mathscr{R}(h_0)$ such that

$$0 < \lambda_* := \lambda_1(\bar{h}) = \inf_{h \in \mathscr{R}(h_0)} \lambda_1(h) = \inf_{h \in \mathscr{R}(h_0)} \inf_{v \in W_0^{1,p}(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega h|v|^p dx}. \tag{2.3}$$

3. Uniqueness for the Ground State Solution of Problem $(\mathscr{P}_{h,f})$

In this section, we investigate the existence and uniqueness for the ground state solution of problem $(\mathscr{P}_{h,f})$.

Proposition 3.1. *Fix $0 < h(x) \in L^\infty(\Omega)$, and $f \in L^r(\Omega)$ with $r > r^*$ and $0 < \lambda < \lambda_1(h)$, where $r^* = 1$ if $p \geq N$; $r^* = pN/(pN - N + p)$ if $1 < p < N$, and $\lambda_1(h)$ is the first eigenvalue of the problem $(\mathscr{L}_h)$. Then problem $(\mathscr{P}_{h,f})$ has a unique ground state solution $u_{h,f} \in W_0^{1,p}(\Omega)$, i.e.,*

$$I'_{h,f}(u_{h,f})v = 0, \forall v \in W_0^{1,p}(\Omega)$$

and

$$I_{h,f}(u_{h,f}) = \inf_{v \in W_0^{1,p}(\Omega)} I_{h,f}(v).$$

Moreover, if, in addition, $f(x) > 0$, a.e. $x \in \Omega$, then $u_{h,f} > 0$.

Proof. First, we show that the problem $(\mathscr{P}_{h,f})$ has a ground state solution.

By the Hölder inequality and the Sobolev embedding inequality, we have

$$\left| \int_\Omega f u dx \right| \leq \|f\|_{L_r} \|u\|_{L_{r'}} \leq C\|u\| \tag{3.1}$$

for all $u \in W_0^{1,p}(\Omega)$ since now $1 < r' := r/(r-1) < p^*$, where $p^* := pN/(N-p)$, if $1 < p < N$; $p^* := \infty$, if $p \geq N$, is the critical Sobolev exponent.

Then we can deduce from (1.1), (2.2), and (3.1) that

$$I_{h,f}(u) \geq \frac{1}{p}\left(1 - \frac{\lambda}{\lambda_1(h)}\right)\|u\|^p - C\|u\| \to \infty$$

as $\|u\| \to \infty$, which demonstrates that the functional $I_{h,f}$ is coercive. On the other hand, it is easy to see that the functional $I_{h,f}$ is weakly lower semi-continuous so that the functional $I_{h,f}$ has a global minimizer $u_{h,f} \in W_0^{1,p}(\Omega)$, i.e., $I_{h,f}(u_{h,f}) = \inf_{v \in W_0^{1,p}(\Omega)} I_{h,f}(v)$. Moveover, by a standard argument (cf. [24, Lemma 2.16]), we can easily prove that $I_{h,f} \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Thus $u_{h,f}$ will be a critical point of $I_{h,f}$, and then a ground state solution of problem $(\mathscr{P}_{h,f})$.

Next, we prove that $u_{h,f}$ is actually the unique ground state solution of problem $(\mathscr{P}_{h,f})$.

Denote by $m := \inf_{v\in W_0^{1,p}(\Omega)} I_{h,f}(v)$, then $I_{h,f}(u_{h,f}) = m$. Since $0 < \lambda < \lambda_1(h)$, we have

$$\int_\Omega |\nabla u|^p dx \geq \int_\Omega |\nabla u|^p dx - \lambda \int_\Omega h|u|^p dx \geq \frac{\lambda_1(h)-\lambda}{\lambda_1(h)} \int_\Omega |\nabla u|^p dx.$$

It follows that $\|u\|_\lambda := (\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega h|u|^p dx)^{1/p}$ will be an equivalent norm in $W_0^{1,p}(\Omega)$. By the triangle inequality for the norm, we have that, for any $t \in (0,1), u, v \in W_0^{1,p}(\Omega)$,

$$\|tu + (1-t)v\|_\lambda \leq t\|u\|_\lambda + (1-t)\|v\|_\lambda.$$

Thus

$$\begin{aligned} I_{h,f}(tu+(1-t)v) &= \frac{1}{p}\|tu+(1-t)v\|_\lambda^p - \int_\Omega f(tu+(1-t)v)dx \\ &\leq \frac{1}{p}(t\|u\|_\lambda + (1-t)\|v\|_\lambda)^p - t\int_\Omega fudx - (1-t)\int_\Omega fvdx \\ &< \frac{t}{p}\|u\|_\lambda^p + \frac{1-t}{p}\|v\|_\lambda^p - t\int_\Omega fudx - (1-t)\int_\Omega fvdx \\ &= tI_{h,f}(u) + (1-t)I_{h,f}(v), \end{aligned}$$

where the above strict inequality comes from the strict convexity of the power function s^p.

Assume that $v_{h,f} \in W_0^{1,p}(\Omega)$ is another ground state solution to problem $(\mathscr{P}_{h,f})$, i.e., $I_{h,f}(v_{h,f}) = m$, and $u_{h,f} \neq v_{h,f}$. Then, for any $t \in (0,1)$, we have $tu_{h,f} + (1-t)v_{h,f} \in W_0^{1,p}(\Omega)$ and

$$I_{h,f}(tu_{h,f} + (1-t)v_{h,f}) < tI_{h,f}(u_{h,f}) + (1-t)I_{h,f}(v_{h,f}) = m,$$

a contradiction. Thus we have proved that $u_{h,f}$ is the unique ground state solution to problem $(\mathscr{P}_{h,f})$.

Finally, if $f(x) > 0$, then we can easily check that $I_{h,f}(|u_{h,f}|) \leq I_{h,f}(u_{h,f})$, which shows that $|u_{h,f}|$ is also a global minimizer of $I_{h,f}$ and thus a ground state solution to problem $(\mathscr{P}_{h,f})$. Then, $u_{h,f} = |u_{h,f}| \geq 0$, by the uniqueness of the ground state solution. Since

$$-\Delta_p u_{h,f}(x) = f(x) + \lambda h(x)|u_{h,f}(x)|^{p-2}u_{h,f}(x) > 0, \text{ a.e. } x \in \Omega,$$

we have $u_{h,f}(x) > 0$, a.e. $x \in \Omega$ (cf. [23, Theorem 5]). This completes the proof. □

Remark 3.2. In the case of $\lambda \geq \lambda_1(h)$, we suppose that φ is the eigenfunction corresponding to $\lambda_1(h)$ of $(\mathscr{L}_h)$. Then, for any $t \in R$,

$$I_{h,f}(t\varphi) = \int_\Omega (|\nabla(t\varphi)|^p - \lambda h|t\varphi|^p - tf\varphi)\,dx \leq -t\int_\Omega (f\varphi)\,dx.$$

It is easy to see that $I_{h,f}(t\varphi)$ would tend to minus infinity if $\int_\Omega (f\varphi)\,dx \neq 0$. Thus $I_{h,f}$ has no global minimizer, and the equation $(\mathscr{P}_{h,f})$ has no ground state solution.

4. Solvability of the Problem (OPT)

Theorem 4.1. *Suppose that $0 < h_0(x) \in L^\infty(\Omega)$, $f_0 \in L^r(\Omega)$ with $r > r^*$, and $0 < \lambda < \lambda_*$, where λ_* is given by (2.3). Then there exists $\hat{h} \in \mathscr{R}(h_0), \hat{f} \in \mathscr{R}(f_0)$ which solves the problem (OPT), i.e.,*

$$I(\hat{u}) = \inf_{h\in\mathscr{R}(h_0), f\in\mathscr{R}(f_0)} I_{h,f}(u_{h,f}),$$

where $\hat{u} := u_{\hat{h},\hat{f}}$ is the unique solution of $(\mathscr{P}_{\hat{h},\hat{f}})$ and $I(\hat{u}) := I_{\hat{h},\hat{f}}(\hat{u})$ for short.

Proof. By (2.3), we have $\lambda_* \le \lambda_1(h), \forall h \in \mathscr{R}(h_0)$. It follows from Proposition 3.1 that problem $(\mathscr{P}_{h,f})$ has a unique ground state solution $u_{h,f} \in W_0^{1,p}(\Omega)$ for each pair of given functions h and f. Let

$$A = \inf_{h\in\mathscr{R}(h_0), f\in\mathscr{R}(f_0)} I_{h,f}(u_{h,f}).$$

One easily check that A is well-defined, i.e., $A > -\infty$. In fact, for each $h \in \mathscr{R}(h_0), f \in \mathscr{R}(f_0)$, from (2.2), we have

$$\lambda_* \int_\Omega h|u_{h,f}|^p dx \le \lambda_1(h) \int_\Omega h|u_{h,f}|^p dx \le \int_\Omega |\nabla u_{h,f}|^p dx,$$

and then

$$\begin{aligned} I_{h,f}(u_{h,f}) &= \frac{1}{p}\int_\Omega |\nabla u_{h,f}|^p dx - \frac{\lambda}{p}\int_\Omega h|u_{h,f}|^p dx - \int_\Omega f u_{h,f} dx \\ &\ge \frac{1}{p}(1-\frac{\lambda}{\lambda_*})\|u_{h,f}\|^p - C\|f\|_{L_r}\|u_{h,f}\|. \end{aligned} \tag{4.1}$$

By Lemma 2.1, $\|f\|_{L_r} \equiv \|f_0\|_{L_r}$, we can then deduce from (4.1) that A must be finite.

Now, choose a minimizing sequence $\{(h_i, f_i)\}$, i.e.,

$$h_i \in \mathscr{R}(h_0),\ f_i \in \mathscr{R}(f_0), \forall i \in \mathbb{N}$$

and

$$A = \lim_{i\to\infty} I(u_i),$$

where $u_i := u_{h_i,f_i}$ and $I(u_i) := I_{h_i,f_i}(u_i)$ for short. It follows from (4.1) that $\{u_i\}$ must be bounded in $W_0^{1,p}(\Omega)$. Then it has a subsequence (still denoted $\{u_i\}$) which weakly converges to $u \in W_0^{1,p}(\Omega)$ and strongly converges to u in $L^{r'}(\Omega)$ with $1 < r' = r/(r-1) < p^*$. Since $\|f_i\|_{L_r} \equiv \|f_0\|_{L_r}$, $\{f_i\}$ contains a subsequence (still denoted $\{f_i\}$) converging weakly to some $\bar{f} \in \overline{\mathscr{R}(f_0)^w}$, the weak closure of $\mathscr{R}(f_0)$ in $L^r(\Omega)$. Then

$$\left|\int_\Omega (f_i - \bar{f})u dx\right| \to 0 \text{ as } i \to \infty,$$

since $u \in L^{r'}(\Omega)$. Combining with the Hölder inequality, we have that

$$\begin{aligned} \left|\int_\Omega (f_i u_i - \bar{f}u)dx\right| &\le \left|\int_\Omega f_i(u_i - u)dx\right| + \left|\int_\Omega (f_i - \bar{f})udx\right| \\ &\le \|f_i\|_{L^r}\|u_i - u\|_{L^{r'}} + \left|\int_\Omega (f_i - \bar{f})udx\right| \to 0, \end{aligned} \tag{4.2}$$

as $i \to \infty$. Since $\|h_i\|_\infty \equiv \|h_0\|_\infty$, $\{h_i\}$ is bounded in $L^\infty(\Omega)$, it must contain a subsequence (still denoted $\{h_i\}$) converging weakly to some $\bar{h} \in \overline{\mathscr{R}(h_0)^w}$, the weak closure of $\mathscr{R}(h_0)$ in $L^\infty(\Omega)$. Similarly as (4.2) we have

$$\lim_{i\to\infty} \int_\Omega (h_i|u_i|^p - \bar{h}|u|^p)dx = 0. \tag{4.3}$$

By (4.2), (4.3), and the weak lower semi-continuity of the norm in $W_0^{1,p}(\Omega)$, we obtain that

$$A = \lim_{i\to\infty} I(u_i) \ge \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \bar{h}|u|^p dx - \int_\Omega \bar{f}udx. \tag{4.4}$$

By Lemma 2.3, there exists $\hat{f} \in \mathscr{R}(f_0)$ which maximizes the linear functional $\int_\Omega l u dx$, relative to $l \in \overline{\mathscr{R}(f_0)^w}$. As a consequence,

$$\int_\Omega \bar{f} u dx \leq \int_\Omega \hat{f} u dx.$$

Similarly, there exists $\hat{h} \in \mathscr{R}(h_0)$ which maximizes the linear functional $\int_\Omega l|u|^p dx$, relative to $l \in \overline{\mathscr{R}(h_0)^w}$. It follows that

$$\int_\Omega \bar{h}|u|^p dx \leq \int_\Omega \hat{h}|u|^p dx.$$

Then we can see from (4.4) that

$$A \geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \hat{h}|u|^p dx - \int_\Omega \hat{f} u dx. \tag{4.5}$$

By Proposition 3.1, we have

$$\begin{aligned} I(\hat{u}) &= \inf_{v\in W_0^{1,p}(\Omega)} \int_\Omega \left(\frac{1}{p}|\nabla v|^p - \frac{\lambda}{p}\hat{h}|v|^p dx - \hat{f} v\right) dx \\ &\leq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega \hat{h}|u|^p dx - \int_\Omega \hat{f} u dx. \end{aligned} \tag{4.6}$$

It follows from (4.5) and (4.6) that $I(\hat{u}) \leq A$.

On the other hand, recalling that $A = \inf_{h\in\mathscr{R}(h_0), f\in\mathscr{R}(f_0)} I_{h,f}(u_{h,f})$, we must have $A \leq I(\hat{u})$. Thus $A = I(\hat{u})$. □

We now obtain a representation result of the optimal solution $(\hat{h}, \hat{f})$ for problem (OPT).

Theorem 4.2. *Under the assumptions of Theorem 4.1, suppose that meas*$(\{x \in \Omega : f_0(x) = 0\}) = 0$*. Then there exist increasing functions* ϕ *and* φ *such that*

$$\begin{aligned} \hat{h} &= \phi(|\hat{u}|^p) \quad a.e.\ in\ \Omega, \\ \hat{f} &= \varphi(\hat{u}) \qquad a.e.\ in\ \Omega, \end{aligned} \tag{4.7}$$

where $\hat{u} = u_{\hat{h},\hat{f}}$ *is the unique ground state solution to* $(\mathscr{P}_{\hat{h},\hat{f}})$.

Proof. It is easy to see that $I_{\hat{h},\hat{f}}(\hat{u}) \leq I_{h,\hat{f}}(u_{h,\hat{f}}) \leq I_{h,\hat{f}}(\hat{u}), \forall h \in \mathscr{R}(h_0)$. Thus

$$\frac{1}{p}\int_\Omega |\nabla \hat{u}|^p dx - \frac{\lambda}{p}\int_\Omega \hat{h}|\hat{u}|^p dx - \int_\Omega \hat{f}\hat{u} dx \leq \frac{1}{p}\int_\Omega |\nabla \hat{u}|^p dx - \frac{\lambda}{p}\int_\Omega h|\hat{u}|^p dx - \int_\Omega \hat{f}\hat{u} dx,$$

which implies that

$$\int_\Omega h|\hat{u}|^p dx \leq \int_\Omega \hat{h}|\hat{u}|^p dx, \quad \forall h \in \mathscr{R}(h_0).$$

So that $\hat{h}$ is a maximizer of the linear functional $L(h) := \int_\Omega h|\hat{u}|^p dx$, relative to $h \in \mathscr{R}(h_0)$. In fact, $\hat{h}$ is the unique maximizer of $L(h)$. To prove this, suppose that $\bar{h} \in \mathscr{R}(h_0)$ is also a maximizer of $L(h)$, and we show in the following that $\bar{h} = \hat{h}$. Clearly,

$$\int_\Omega \hat{h}|\hat{u}|^p dx = \int_\Omega \bar{h}|\hat{u}|^p dx.$$

Thus

$$\begin{aligned}
I(\hat{u}) &= \frac{1}{p}\int_\Omega |\nabla \hat{u}|^p dx - \frac{\lambda}{p}\int_\Omega \hat{h}|\hat{u}|^p dx - \int_\Omega \hat{f}\hat{u}dx \\
&= \frac{1}{p}\int_\Omega |\nabla \hat{u}|^p dx - \frac{\lambda}{p}\int_\Omega \bar{h}|\hat{u}|^p dx - \int_\Omega \hat{f}\hat{u}dx \\
&= I_{\bar{h},\hat{f}}(\hat{u}) \\
&\geq I_{\bar{h},\hat{f}}(u_{\bar{h},\hat{f}}) \\
&\geq I(\hat{u}).
\end{aligned}$$

We can see that all the above inequalities turn to be equalities. Thus it holds that

$$I_{\bar{h},\hat{f}}(\hat{u}) = I_{\bar{h},\hat{f}}(u_{\bar{h},\hat{f}}).$$

It must has $\hat{u} = u_{\bar{h},\hat{f}}$, by the uniqueness of the global minimizer of the functional $I_{\bar{h},\hat{f}}$. Thus $\hat{u}$ is the solution of problems $(\mathscr{P}_{\bar{h},\hat{f}})$ and $(\mathscr{P}_{\hat{h},\hat{f}})$, i.e.,

$$\int_\Omega |\nabla \hat{u}|^{p-2}\nabla \hat{u}\nabla v dx - \lambda \int_\Omega \bar{h}|\hat{u}|^{p-2}\hat{u}v dx = \int_\Omega \hat{f}v dx, \quad \forall v \in W_0^{1,p}(\Omega).$$
$$\int_\Omega |\nabla \hat{u}|^{p-2}\nabla \hat{u}\nabla v dx - \lambda \int_\Omega \hat{h}|\hat{u}|^{p-2}\hat{u}v dx = \int_\Omega \hat{f}v dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

Hence,

$$\int_\Omega (\bar{h} - \hat{h})|\hat{u}|^{p-2}\hat{u}v dx = 0, \quad \forall v \in W_0^{1,p}(\Omega),$$

which implies that

$$(\bar{h}(x) - \hat{h}(x))|\hat{u}|^p = 0, \ a.e.\ x \in \Omega. \tag{4.8}$$

By the assumption, $meas(\{x \in \Omega : f_0(x) = 0\}) = 0$, we have $meas(\{x \in \Omega : \hat{f}(x) = 0\}) = 0$ thanks to $\hat{f} \in \mathscr{R}(f_0)$. Thus $meas(\{x \in \Omega : \hat{u}(x) = 0\}) = 0$. Combining with (4.8), we have $\bar{h}(x) = \hat{h}(x)$, a.e. $x \in \Omega$. Thus $\hat{h}$ is the unique maximizer of $L(h)$. Note that $L(h) = \int_\Omega h|\hat{u}|^p dx$. By using Lemma 2.2, we have that there exists an increasing function ϕ such that

$$\phi(|\hat{u}|^p) = \hat{h}, \text{ a.e. in } \Omega.$$

Similarly, we can show that $\hat{f}$ is the unique maximizer of the linear functional $l(f) := \int_\Omega f\hat{u}dx$, relative to $f \in \mathscr{R}(f_0)$. Also from Lemma 2.2, there exists an increasing function φ such that

$$\varphi(\hat{u}) = \hat{f}, \text{ a.e. in } \Omega.$$

This completes the proof. □

Theorem 4.3. *Assume that the assumptions of Theorem 4.1 hold, Ω is a ball centered at the origin, and $f_0(x) > 0$, a.e. $x \in \Omega$. Then problem (OPT) has a unique solution $(\hat{h}, \hat{f})$ and $\hat{h} = h_0^*$, $\hat{f} = f_0^*$, where h_0^* (f_0^*) is the Schwarz symmetric decreasing rearrangement of h_0 (f_0).*

Proof. Denote by $\hat{u}^*$ the Schwarz symmetric decreasing rearrangement of $\hat{u}$, where $\hat{u} = u_{\hat{h},\hat{f}}$ is the ground state solution of $(\mathscr{P}_{\hat{h},\hat{f}})$. First, we claim that

$$\int_\Omega |\nabla \hat{u}^*|^p dx = \int_\Omega |\nabla \hat{u}|^p dx. \tag{4.9}$$

Indeed, since $I(\hat{u}) \leq I_{h_0^*,f_0^*}(u_{h_0^*,f_0^*}) \leq I_{h_0^*,f_0^*}(\hat{u}^*)$, i.e.,

$$\frac{1}{p}\int_\Omega |\nabla \hat{u}|^p dx - \int_\Omega (\frac{\lambda}{p}\hat{h}\hat{u}^p + \hat{f}\hat{u})dx \leq \frac{1}{p}\int_\Omega |\nabla u_{h_0^*,f_0^*}|^p dx - \int_\Omega (\frac{\lambda}{p} h_0^* u^p_{h_0^*,f_0^*} + f_0^* u_{h_0^*,f_0^*})dx$$
$$\leq \frac{1}{p}\int_\Omega |\nabla \hat{u}^*|^p dx - \int_\Omega (\frac{\lambda}{p} h_0^*(\hat{u}^*)^p + f_0^*\hat{u}^*)dx,$$

it follows from Lemma 2.4 that

$$\frac{1}{p}\int_\Omega (|\nabla \hat{u}^*|^p - |\nabla \hat{u}|^p)dx \geq \int_\Omega (\frac{\lambda}{p} h_0^*(\hat{u}^*)^p - \frac{\lambda}{p}\hat{h}\hat{u}^p + f_0^*\hat{u}^* - \hat{f}\hat{u})dx \geq 0,$$

which, together with (2.1), implies that (4.9) holds. Then we claim that

$$\text{meas}\left(\left\{x \in \Omega : \nabla \hat{u} = 0,\ 0 < \hat{u}(x) < \operatorname*{ess\ sup}_{y\in\Omega} \hat{u}(y)\right\}\right) = 0. \tag{4.10}$$

In fact, for each $x_0 \in \Omega$ satisfying $0 < \hat{u}(x_0) < \text{ess sup}_{x\in\Omega} \hat{u}(x)$, we set $S = \{x \in \Omega : \hat{u}(x) \geq \hat{u}(x_0)\}$, which is then a closed ball by Lemma 2.5. If we define $u(x) = \hat{u}(x) - \hat{u}(x_0)$, then $-\Delta_p u(x) = -\Delta_p \hat{u}(x) > 0$, a.e. $x \in \Omega$. By the strong maximum principle (cf. [23, Theorem 5]), we deduce that $u(x) > 0$ is in the interior $\overset{\circ}{S}$ of S. Hence, $\hat{u}(x) > \hat{u}(x_0)$ for all $x \in \overset{\circ}{S}$, and x_0 must be a boundary point of S. By the Hopf boundary lemma, we derive $\frac{\partial u}{\partial \nu}(x_0) = \frac{\partial \hat{u}}{\partial \nu}(x_0) \neq 0$, where ν stands for the outward unit normal to ∂S at x_0. This means that

$$\left\{x \in \Omega : \nabla \hat{u} = 0,\ 0 < \hat{u}(x) < \operatorname*{ess\ sup}_{y\in\Omega} \hat{u}(y)\right\} = \emptyset,$$

which indicates that (4.10) is true. Now, by using Lemma 2.5 and noting (4.9) and (4.10), we see that $\hat{u} = \hat{u}^*$. By (4.7), we have that $\hat{h} = \phi \circ (\hat{u}^*)^p$ and $\hat{f} = \varphi \circ \hat{u}^*$ are spherically symmetric decreasing functions. It follows that $\hat{h}$ coincides its Schwarz rearrangement, i.e., $\hat{h} = \hat{h}^* = h_0^*$, so is $\hat{f}$. We complete the proof. □

Funding

This research was funded by Natural Science Foundation of China (grants numbers 11771319 and 11901418) and Natural Science Foundation of Jiangsu Province (grant number BK20170590).

REFERENCES

[1] N. Amiri, M. Zivari-Rezapour, Maximization and minimization problems related to an equation with the p-Laplacian, Indian J. Pure Appl. Math. 51 (2020) 777-788.

[2] C. Anedda, Maximization and minimization in problems involving the bi-Laplacian, Annali di Matematica 190 (2011) 145-156.

[3] J. Brothers, W. Ziemer, Minimal rearrangements of Sobolev functions, J. Reine Angew. Math. 384 (1988) 153-79.

[4] J. Bonder, A. Silva, J. Spedaletti, Uniqueness of minimal energy solutions for a semilinear problem involving the fractional laplacian, Proc. Amer. Math. Soc. 147 (2019) 2925-2936.

[5] G. Burton, Rearrangements of functions, maximization of convex functionals and vortex rings, Math. Ann. 276 (1987) 225-253.

[6] G. Burton, Variational problems on classes of rearrangements and multiple configurations for steady vortices, Ann. Inst. Henri Poincaré, 6 (1989) 295-319.

[7] S. Chanillo, D. Grieser, K. Kurata, The free boundary problem in the optimization of composite membranes, Contemp. Math. 268 (2000) 61-81.

[8] S. Chanillo, C. Kenig, Weak uniqueness and partial regularity for the composite membrane problem, J. Eur. Math. Soc. 10 (2008) 705–737.

[9] F. Cuccu, B. Emamizadeh, G. Porru, Nonlinear elastic membrane involving the p-Laplacian operator, Electron. J. Differential Equations, 2006 (2006) 1-10.

[10] F. Cuccu, B. Emamizadeh, G. Porru, Optimization of the first eigenvalue in problems involving the p-Laplacian, Proc. Amer. Math. Soc. 137 (2009), 1677-1687.

[11] F. Cuccu, B. Emamizadeh, G. Porru, Optimization problems for an elastic plate, J. Math. Phys. 47 (2006) 1-12.

[12] B. Emamizadeh, M. Zivari-Rezapour, Rearrangement optimization for some elliptic equations, J. Optim. Theory Appl. 135 (2007) 367-379.

[13] B. Emamizadeh, R. Fernandes, Optimization of the principal eigenvalue of the one-Dimensional Schrödinger operator, Electron. J. Differential Equations, 2008 (2008) 1-11.

[14] B. Emamizadeh, J. Prajapat, Symmetry in rearrangemet optimization problems, Electron. J. Differential Equations, 2009 (2009) 1-10.

[15] K. Kurata, M. Shibata, S. Sakamoto, Symmetry-breaking phenomena in an optimization problem for some nonlinear elliptic equation, Appl. Math. Optim. 50 (2004) 259-278.

[16] G. Leoni, A First Course in Sobolev Spaces, Grad. Studies in Math. American Mathematical Society: Providence, 2009.

[17] M. Marras, Optimization in problems involving the p-Laplacian, Electron. J. Differential Equations, 2010 (2010) 1-10.

[18] M. Marras, G. Porru, V. Stella, Optimization problems for eigenvalues of p-Laplace equations, J. Math. Anal. Appl. 398 (2013) 766-775.

[19] J. Nycander, B. Emamizadeh, Variational problem for vortices attached to seamounts, Nonlinear Anal. 55 (2003) 15-24.

[20] L. Pezzo, J. Bonder, Some Optimization problems for p-Laplacian type equations, Appl. Math. Optim. 59 (2009) 365-381.

[21] L. Pezzo, J. Bonder, An optimization problem for the first weighted eigenvalue problem plus a potential, Proc. Amer. Math. Soc. 138 (2010), 3551-3567.

[22] C. Qiu, Y. Huang, Y. Zhou, A class of rearrangement optimization problems involving the p-Laplacian, Nonlinear Anal. 122 (2015) 30-42.

[23] J. Vázquez, A strong maximum principle for some quasilinear elliptic equations, Appl. Math. Optim. 12 (1984) 191-202.

[24] M. Willem, Minimax Theorems, Birkhauser, Basel, 1996.

[25] M. Zivari-Rezapour, Maximax rearrangement optimization related to a homogeneous Dirichlet problem, Arab. J. Math. 2 (2013) 427-433.