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BOUNDARY VALUE PROBLEMS FOR HILFER-HADAMARD FRACTIONAL DIFFERENTIAL INCLUSIONS WITH NONLOCAL INTEGRO-MULTI-POINT BOUNDARY CONDITIONS

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Abstract. In this paper, we study boundary value problems for differential inclusions involving Hilfer-Hadamard fractional derivative and nonlocal integro-multi-point boundary conditions. Both convex valued and non-convex valued multi-valued maps involved in the problem are considered. The existence result dealing with convex valued multi-maps is derived by applying Leray-Schauder nonlinear alternative for multi-valued maps. In the non-convex case, we apply Covitz-Nadler fixed point theorem to contractive multi-valued maps. Illustrating examples are also presented.

Keywords. Boundary value problems; Hilfer-Hadamard fractional derivative; Nonlocal integro-multi-point boundary conditions.

1. Introduction

Fractional calculus and fractional differential equations have been of great interest because they play a vital role in describing many real world processes from applied sciences (such as biology, physics, chemistry, economics, ecology, control theory, and so on) as compared to classical integer order differential equations. For the basic theory of fractional calculus and fractional differential equations, we refer to the monographs by Diethelm [9], Kilbas et al. [15], Miller and Ross [19], Podlubny [20] and Ahmed et al. [1]. Boundary value problems for fractional differential equations have been studied by many researchers by using different

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kinds of boundary conditions. Thus integral and multi-point boundary conditions were studied in [24], anti-periodic boundary conditions in [2], coupled integral boundary conditions in [3], integral boundary conditions with sequential Riemman-Liouville and Caputo fractional derivatives were studied in [16], multi-point strip boundary conditions in [6] and separated boundary conditions in [23]. For a variety of boundary value problems, we refer to [4]. There are several kinds of fractional derivatives in the literature, such as Riemann–Liouville, Caputo, Hadamard, Hilfer, and Katugampola, to name a few. Hilfer fractional derivative [12] interpolate Riemann–Liouville and Caputo fractional derivatives. Many applications of Hilfer fractional differential equations can be found in many fields of mathematics, physics, etc (see [22] and the references cited therein.

Recently, in [21], the authors studied the existence and uniqueness of solutions for boundary value problems for Hilfer-Hadamard fractional differential equations with nonlocal integro-multi-point boundary conditions

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}x(t) = f(t,x(t)), \quad t \in [1,T], \\ x(1)=0,\ \displaystyle\sum_{i=1}^{m}\theta_i x(\xi_i) = \lambda\ {}^{H}I_1^{\delta}x(\eta), \end{cases} \tag{1.1}$$

where ${}^{HH}D_1^{\alpha,\beta}$ is the Hilfer-Hadamard fractional derivative of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$, $\theta_i, \lambda \in \mathbb{R}$, $i=1,2,\ldots,m$, are given constants, and $f:[1,T]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, ${}^{H}I^{\delta}$ is the Hadamard fractional integral of order $\delta>0$, and $\eta,\xi_i \in (1,T)$, $i=1,2,\ldots,m$, are given points. Existence and uniqueness results were proved by using a variety of fixed point theorems. The results of [21] was extended in [5] to Hilfer-Hadamard fractional boundary value problems with nonlocal mixed boundary conditions.

In this paper, we introduce and study a new boundary value problem for multi-valued fractional differential equations, consisting of the Hilfer-Hadamard fractional derivative and nonlocal integro-multi-point boundary conditions. We emphasize that the fractional integral which appears in the boundary conditions is of Riemann-Liouville type, while in [21] and [5], the corresponding integrals were of Hadamard type. To be more precise, in this paper we investigate the following boundary value problem of Hilfer-Hadamard fractional differential inclusions with nonlocal integro-multi-point boundary conditions

$$\begin{cases} {}^{HH}D_1^{\alpha,\beta}x(t) \in F(t,x(t)), \quad t \in [1,T], \\ x(1)=0,\ x(T) = \displaystyle\sum_{i=1}^{m}\theta_i x(\xi_i) + \sum_{j=1}^{p}\lambda_j\ {}^{RL}I_1^{\delta_j}x(\eta_j), \end{cases} \tag{1.2}$$

where $F:[1,T]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multivalued map, ($\mathscr{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$), ${}^{RL}I_1^{\delta_j}$ is the Riemann-Liouville fractional integral of order $\delta_j>0$, stating at a point $t=1$, for $j=1,2,\cdots,p$ and the others parameters are as in problem (1.1).

Multivalued differential equations (differential inclusions) are a generalization of single-valued differential equations, and arise in the mathematical modelling of certain problems in control theory, optimization, mathematical economics, sweeping process, stochastic analysis, and in other fields; see, for example, [17].

We obtain existence results for problem (1.2) by applying the nonlinear alternative of Leray-Schauder type when the right hand side is convex valued and a fixed point theorem, due to Covitz and Nadler in the case of non-convex right hand side. Illustrative examples are also constructed. We emphasize that our results are new in the context of nonlocal integro-multi-point Hilfer-Hadamard fractional fractional differential inclusions and contribute significantly to the topic. The method used is well known, however their exposition in the framework of problem (1.2) is new. The rest of this paper is organized as follows. In Section 2, we recall some preliminary concepts of fractional calculus and multivalued analysis. The main results are presented in Section 3. Section 4, the last section, is devoted to constructing illustrative examples.

2. Preliminaries

In this section, some notations, definitions, and lemmas from fractional calculus and multi-valued analysis are recalled.

2.1. **Fractional calculus.**

Definition 2.1. (Hadamard fractional integral [15]). The Hadamard fractional integral of order $\alpha > 0$ for a function $f : [a,\infty) \to \mathbb{R}$ is defined as

$$^{H}I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)}{\tau} d\tau, \quad t > a, \tag{2.1}$$

provided that the integral exists, where $\log(.) = \log_e(.)$.

Definition 2.2. (Hadamard fractional derivative [15]). The Hadamard fractional derivative of order $\alpha > 0$, applied to the fuction $f : [a,\infty) \to \mathbb{R}$, is defined as follows:

$$_{H}D_a^{\alpha} f(t) = \delta^n \left(^{H}I_a^{n-\alpha} f\right)(t), \qquad n = [\alpha]+1, \tag{2.2}$$

where $\delta^n = (t\frac{d}{dt})^n$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. The Riemann-Liouville fractional integral of order α for a function $f : (a,\infty) \to \mathbb{R}$ is given by

$$^{RL}I_a^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \qquad \alpha > 0,$$

provided that the right-hand side is pointwise defined on (a,∞).

Lemma 2.4. [15, Property 2.24] *If $a, \alpha, \beta > 0$, then*

$$\left(_{H}D_a^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\left(\log\frac{x}{a}\right)^{\beta-\alpha-1}.$$

and

$$\left(^{H}I^{\alpha}\left(\log\frac{t}{a}\right)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\left(\log\frac{x}{a}\right)^{\beta+\alpha-1}.$$

Lemma 2.5. [15] *Let $q > 0$ and $x \in C[1,\infty) \cap L^1[1,\infty)$. Then the Hadamard fractional differential equation $_{H}D^q x(t) = 0$ has the solutions $x(t) = \sum_{i=1}^{n} c_i (\log t)^{q-i}$, and the following formula holds:*

$$^{H}I^q \left(_{H}D^q x\right)(t) = x(t) + \sum_{i=1}^{n} c_i (\log t)^{q-i},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$, and $n-1 < q < n$.

Definition 2.6. (Hilfer-Hadamard fractional derivative [13]). Let $n-1<\alpha<n$, $0\leq\beta\leq 1$, and $f\in L^1(a,b)$. The Hilfer-Hadamard fractional derivative of order α and tybe β of f is defined as

$$\begin{aligned}({}^{HH}D_a^{\alpha,\beta}f)(t) &= ({}^{H}I_a^{\beta(n-\alpha)}\delta^n\,{}^{H}I_a^{(n-\alpha)(1-\beta)}f)(t)\\ &= ({}^{H}I_a^{\beta(n-\alpha)}\delta^n\,{}^{H}I_a^{(n-\gamma)}f)(t); \quad \gamma=\alpha+n\beta-\alpha\beta,\\ &= ({}^{H}I_a^{\beta(n-\alpha)}\,{}_{H}D_a^{\gamma}f)(t),\end{aligned}$$

where ${}^{H}I_a^{(.)}$ and ${}_{H}D_a^{(.)}$ is the Hadamard fractional integral and derivative defined by (2.1) and (2.2), respectively.

The Hilfer-Hadamard fractional derivative interpolates between Hadamard and Caputo-Hadamard fractional derivatives since for $\beta=0$ this derivative reduces to the Hadamard fractional derivative. For $\beta=1$, it leads the Caputo-Hadamard fractional derivative defined by

$${}_{H}^{C}D_a^{\alpha}f(t)=\left({}^{H}I_a^{n-\alpha}\delta^n f\right)(t), \qquad n=[\alpha]+1.$$

We recall the following known theorem by Kilbas *et al.* [15] which will be used in the following.

Theorem 2.7. *([15]). Let $\alpha>0$, $0\leq\beta\leq 1$, $\gamma=\alpha+n\beta-\alpha\beta$, and $n=[\alpha]+1$ and $0<a<b<\infty$. If $f\in L^1(a,b)$ and $({}^{H}I_a^{n-\gamma}f)(t)\in AC_\delta^n[1,T]$, then*

$$\begin{aligned}{}^{H}I_a^{\alpha}({}^{HH}D_a^{\alpha,\beta}f)(t) &= {}^{H}I_a^{\gamma}({}^{HH}D_a^{\gamma}f)(t)\\ &= f(t)-\sum_{j=0}^{n-1}\frac{(\delta^{(n-j-1)}({}^{H}I_a^{n-\gamma}f))(a)}{\Gamma(\gamma-j)}\left(\log\frac{t}{a}\right)^{\gamma-j-1}.\end{aligned}$$

Since $\gamma\in[\alpha,n]$, then the $\Gamma(\gamma-j)$ exists for all $j=1,2,\ldots,n-1$.

The following lemma dealing with a linear variant of the problem (1.2) is the basic tool for transforming the problem (1.2) into a fixed point problem.

Lemma 2.8. *Let $h\in C([1,T],\mathbb{R})$ and*

$$\Lambda:=(\log T)^{\gamma-1}-\sum_{i=1}^{m}\theta_i(\log\xi_i)^{\gamma-1}-\sum_{j=1}^{p}\lambda_j\Phi_j^{\gamma-1}\neq 0,$$

where

$$\Phi_j^{\gamma-1}=\frac{1}{\Gamma(\delta_j)}\int_1^{\eta_j}(\eta_j-s)^{\delta_j-1}(\log s)^{\gamma-1}ds. \tag{2.3}$$

Then x is a solution to the following linear Hilfer-Hadamard fractional differential equation

$${}^{HH}D_1^{\alpha,\beta}x(t)=h(t),\ \ 1<\alpha\leq 2,\ \ t\in[1,T], \tag{2.4}$$

supplemented with the integro-multi-point boundary conditions in (1.2) if and only if

$$\begin{aligned}x(t) &= {}^{H}I_1^{\alpha}h(t)+\frac{(\log t)^{\gamma-1}}{\Lambda}\Big\{\sum_{i=1}^{m}\theta_i\left({}^{H}I_1^{\alpha}h\right)(\xi_i)+\sum_{j=1}^{p}\lambda_j\left({}^{RL}I_1^{\delta_j H}I_1^{\alpha}h\right)(\eta_j)\\ &\quad -{}^{H}I_1^{\alpha}h(T)\Big\},\ t\in[1,T].\end{aligned} \tag{2.5}$$

Proof. By taking the Hadamard fractional integral of order α from 1 to t on both sides of (2.4), and using Theorem 2.7, it follows that

$$x(t)-\sum_{j=0}^{1}\frac{\left(\delta^{(2-j-1)}({}_{H}I_{1^+}^{2-\gamma}x)\right)(1)}{\Gamma(\gamma-j)}(\log t)^{\gamma-j-1}={}^{H}I_{1}^{\alpha}h(t).$$

Then,

$$x(t)-\frac{\delta({}_{H}I_{1^+}^{2-\gamma}x)(1)}{\Gamma(\gamma)}(\log t)^{\gamma-1}-\frac{({}_{H}I_{1^+}^{2-\gamma}x)(1)}{\Gamma(\gamma-1)}(\log t)^{\gamma-2}={}^{H}I_{1}^{\alpha}h(t). \tag{2.6}$$

The equation (2.6) can be rewritten by

$$x(t)=\ c_0(\log t)^{\gamma-1}+c_1(\log\ t)^{\gamma-2}+{}^{H}I_{1}^{\alpha}h(t), \tag{2.7}$$

where c_0, c_1 are arbitrary constants. Now, the first boundary condition $x(1)=0$ together with (2.7) yields $c_1=0$ since $\gamma\in[\alpha,2]$. Putting $c_1=0$ in (2.7), we have

$$x(t)=c_0(\log\ t)^{\gamma-1}+{}^{H}I_{1}^{\alpha}h(t). \tag{2.8}$$

Thus

$$\begin{aligned}
&x(T)=c_0\left(\log T\right)^{\gamma-1}+{}^{H}I_{1}^{\alpha}h(T),\\
&\sum_{i=1}^{m}\theta_i x(\xi_i)=c_0\sum_{i=1}^{m}\theta_i\left(\log\xi_i\right)^{\gamma-1}+\sum_{i=1}^{m}\theta_i{}^{H}I_{1}^{\alpha}h(\xi_i),\\
&\sum_{j=1}^{p}\lambda_j\,{}^{RL}I_{1}^{\delta_j}x(\eta_j)=c_0\sum_{j=1}^{p}\lambda_j\,{}^{RL}I_{1}^{\delta_j}(\log\ t)^{\gamma-1}(\eta_j)+\sum_{j=1}^{p}\lambda_j\,{}^{RL}I_{1}^{\delta_j H}I_{1}^{\alpha}h(\eta_j).
\end{aligned}$$

From above computations and the second boundary condition of (1.2), we have

$$c_0=\frac{1}{\Lambda}\left\{\sum_{i=1}^{m}\theta_i{}^{H}I_{1}^{\alpha}h(\xi_i)+\sum_{j=1}^{p}\lambda_j\,{}^{RL}I_{1}^{\delta_j H}I_{1}^{\alpha}h(\eta_j)-{}^{H}I_{1}^{\alpha}h(T)\right\}.$$

Substituting the value of c_0 into (2.8), we have equation (2.5) as desired. The converse follows by direct computation. The proof is completed. □

2.2. **Multi-valued analysis.** Let us recall some basic definitions on multi-valued maps. For more details, see Deimling [10] and Hu and Papageorgiou [14].

For a normed space $(X,\|\cdot\|)$, let $\mathscr{P}_{cl}(X)=\{Y\in\mathscr{P}(X):Y$ is closed$\}$, $\mathscr{P}_b(X)=\{Y\in\mathscr{P}(X):$ Y is bounded$\}$, and $\mathscr{P}_{cp,c}(X)=\{Y\in\mathscr{P}(X):Y$ is compact and convex$\}$.

Definition 2.9. A multi-valued map $G:X\to\mathscr{P}(X)$ is said to be upper semi-continuous (u.s.c.) on X if, for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if, for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathscr{N}_0$ of x_0 such that $G(\mathscr{N}_0)\subseteq N$.

Definition 2.10. A multi-valued map $G:X\to\mathscr{P}(X)$ is measurable if the function $t\longmapsto d(y,G(t))$ $=\inf\{|y-z|:z\in G(t)\}$ is measurable for every $y\in\mathbb{R}$.

Definition 2.11. A map $F:[1,T]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is coming as Carathéodory multi-function if

(i) $t\longmapsto F(t,x)$ is measurable for each $x\in\mathbb{R}$;
(ii) $x\longmapsto F(t,x)$ is upper semicontinuous for almost all $t\in[1,T]$;

Also it is L^1-Carathéodory if in addition of (i) and (ii) satisfies

(iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([1,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [1,T]$.

Definition 2.12. (a) The set of the selections of F, for each $x \in C([1,T],\mathbb{R})$, is defined by

$$S_{F,y} := \{v \in L^1([1,T],\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in [1,T]\}.$$

(b) The graph of F is defined be the set $Gr(F) = \{(x,y) \in X \times Y, y \in F(x)\}$.

3. Existence Results

Let $C([1,T],\mathbb{R})$ be the Banach space of all continuous functions from $[1,T]$ into $\mathbb{R}$ endowed with the norm

$$\|x\| := \sup\{|x(t)| : t \in [1,T]\}.$$

Also let $L^1([1,T],\mathbb{R})$ be the space of functions $x : [1,T] \to \mathbb{R}$ such that $\|x\|_{L^1} = \int_a^b |x(t)|dt$.

Definition 3.1. A solution to problem (1.2) is a function $x \in C([1,T],\mathbb{R})$ if there exists a function $v \in L^1([1,T],\mathbb{R})$ with $v(t) \in F(t,x)$ for a.e. $t \in [1,T]$ such that x satisfies the differential equation ${}^{HH}D_1^{\alpha,\beta}x(t) = v(t)$ on $[1,T]$ and the boundary conditions $x(1) = 0$, $x(T) = \sum_{i=1}^m \theta_i x(\xi_i) + \sum_{j=1}^p \lambda_j\, {}^{RL}I_1^{\delta_j} x(\eta_j)$.

For the computational convenience, we set

$$\Omega = \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=1}^m |\theta_i|\frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^p \frac{|\lambda_j|\Phi_j^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right\}, \tag{3.1}$$

where Φ_j^α is defined by (2.3).

Consider first the case when F has convex values. Assuming that F is L^1-Carathéodory, we prove an existence result by applying Leray-Schauder nonlinear alternative for multi-valued maps ([11]).

Theorem 3.2. *Assume that*

(H_1) $F : [1,T] \times \mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory;*
(H_2) *there exist* $\psi \in C([0,\infty),(0,\infty))$ *which is nondecreasing and* $p \in C([1,T],\mathbb{R}^+)$ *such that*

$$\|F(t,x)\|_{\mathscr{P}} := \sup\{|x| : x \in F(t,x)\} \leq p(t)\psi(\|x\|) \text{ for each } (t,x) \in [1,T]\times\mathbb{R};$$

(H_3) *there exists a positive real constant* M *such that*

$$\frac{M}{\|p\|\psi(M)\Omega} > 1.$$

Then the nonlocal integro-multi-point Hilfer-Hadamard fractional boundary value problem (1.2) has at least one solution on $[1,T]$.

Proof. To transform problem (1.2) into a fixed point problem, we define an operator $\mathscr{F}$: $C([1,T],\mathbb{R}) \longrightarrow \mathscr{P}(C([1,T],\mathbb{R}))$ by

$$\mathscr{F}(x) = \left\{ \begin{array}{l} h \in C([1,T],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} {}^H I_1^\alpha v(t) + \dfrac{(\log t)^{\gamma-1}}{\Lambda} \Big\{ \displaystyle\sum_{i=1}^m \theta_i \big({}^H I_1^\alpha v\big)(\xi_i) \\ + \displaystyle\sum_{j=1}^p \lambda_j \left({}^{RL}I_1^{\delta_j H} I_1^\alpha v\right)(\eta_j) - {}^H I_1^\alpha v(T) \Big\}, \ v \in S_{F,x} \end{array} \right. \end{array} \right\}$$

for $t \in [1,T]$. It is obvious that the fixed points of $\mathscr{F}$ are solutions of Hilfer-Hardamard nonlocal integro-multi-point fractional boundary value problem (1.2). The proof will be given in a series of steps, by applying Leray-Schauder nonlinear alternative ([11]).

Step 1. For each $x \in C([1,T],\mathbb{R})$, $\mathscr{F}(x)$ is convex.
Observe that F has convex values. We omit the proof because it is obvious.

Step 2. Bounded sets in $C([1,T],\mathbb{R})$ are mapped by $\mathscr{F}$ into bounded sets.
Let

$$B_r = \{x \in C([1,T],\mathbb{R}) : \|x\| \le r\}$$

be a bounded set in $C([1,T],\mathbb{R})$. Then, for each $h \in \mathscr{F}(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$h(t) = {}^H I_1^\alpha v(t) + \frac{(\log t)^{\gamma-1}}{\Lambda} \Big\{ \sum_{i=1}^m \theta_i \big({}^H I_1^\alpha v\big)(\xi_i) + \sum_{j=1}^p \lambda_j \left({}^{RL}I_1^{\delta_j H} I_1^\alpha v\right)(\eta_j) - {}^H I_1^\alpha v(T) \Big\}.$$

Then, for $t \in [1,T]$, we have

$$\begin{aligned} & |h(t)| \\ \le\ & {}^H I_1^\alpha |v(t)| + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \Big\{ \sum_{i=1}^m |\theta_i| \, {}^H I_1^\alpha |v|(\xi_i) + \sum_{j=1}^p |\lambda_j| \left({}^{RL}I_1^{\delta_j H} I_1^\alpha |v|\right)(\eta_j) + {}^H I_1^\alpha |v(T)| \Big\} \\ \le\ & \left[\frac{(\log T)^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \sum_{i=1}^m |\theta_i| \frac{(\log \xi_i)^\alpha}{\Gamma(\alpha+1)} + \sum_{j=1}^p \frac{|\lambda_j| \Phi_j^\alpha}{\Gamma(\alpha+1)} + \frac{(\log T)^\alpha}{\Gamma(\alpha+1)} \right\} \right] \\ & \times \|p\| \psi(\|x\|) \\ =\ & \Omega \|p\| \psi(r), \end{aligned}$$

and consequently,

$$\|h\| \le \Omega \|p\| \psi(r).$$

Step 3. Bounded sets are mapped by $\mathscr{F}$ into equicontinuous sets of $C([1,T],\mathbb{R})$.

Let $t_1,t_2 \in [1,T]$ with $t_1 < t_2$ and $u \in B_R$. It follows that

$$\begin{aligned}
&|\mathscr{F}(x)(t_2)-\mathscr{F}(x)(t_1)|\\
\leq\;& \frac{1}{\Gamma(\alpha)}\int_1^{t_1}\left[\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}-\left(\log\frac{t_1}{\tau}\right)^{\alpha-1}\right]\frac{|v(\tau)|}{\tau}d\tau\\
&+\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}\left(\log\frac{t_2}{\tau}\right)^{\alpha-1}\frac{|v(\tau)|}{\tau}d\tau\\
&+\frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|\Lambda|}\left\{\sum_{i=1}^m|\theta_i|\,{}^HI_1^\alpha|v(\xi_i)|+\sum_{j=1}^p|\lambda_j|\left({}^{RL}I_1^{\delta_j}{}^HI_1^\alpha v\right)(\eta_j)+{}^HI_1^\alpha|v(T)|\right\}\\
\leq\;& \frac{\|p\|\psi(r)}{\Gamma(\alpha+1)}\Big[2(\log t_2-\log t_1)^\alpha+|(\log t_2)^\alpha-(\log t_1)^\alpha|\Big]\\
&+\frac{|(\log t_2)^{\gamma-1}-(\log t_1)^{\gamma-1}|}{|\Lambda|}\frac{(\log T)^{\gamma-1}}{|\Lambda|}\left\{\sum_{i=1}^m|\theta_i|\frac{(\log\xi_i)^\alpha}{\Gamma(\alpha+1)}+\sum_{j=1}^p\frac{|\lambda_j|\Phi_j^\alpha}{\Gamma(\alpha+1)}\right.\\
&\left.+\frac{(\log T)^\alpha}{\Gamma(\alpha+1)}\right\}\|p\|\psi(r),
\end{aligned}$$

which tends to zero, independently of $x \in B_R$, as $t_1 \to t_2$. Hence, by the Arzelá-Ascoli theorem, $\mathscr{F}: C([1,T],\mathbb{R}) \to \mathscr{P}(C([1,T],\mathbb{R}))$ is completely continuous.

Next we need to show the upper semicontinuity of the operator $\mathscr{F}$. From Proposition 1.2 of [10] we know that a completely continuous multi-valued map is upper semi-continuous if and only if it has a closed graph. Thus it is enough to show that $\mathscr{F}$ has a closed graph.

Step 4. $\mathscr{F}$ has a closed graph.

Let $x_n \to x_*, h_n \in \mathscr{F}(x_n)$ and $h_n \to h_*$. We show that $h_* \in \mathscr{F}(x_*)$. Let $h_n \in \mathscr{F}(x_n)$. Then there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1,T]$,

$$h_n(t)={}^HI_1^\alpha v_n(t)+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^m\theta_i\left({}^HI_1^\alpha v_n\right)(\xi_i)+\sum_{j=1}^p\lambda_j\left({}^{RL}I_1^{\delta_j}{}^HI_1^\alpha v_n\right)(\eta_j)-{}^HI_1^\alpha v_n(T)\right\}.$$

We next show that there exists $v_* \in S_{F,x_*}$ such that for, each $t \in [1,T]$,

$$h_*(t)={}^HI_1^\alpha v_*(t)+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^m\theta_i\left({}^HI_1^\alpha v_*\right)(\xi_i)+\sum_{j=1}^p\lambda_j\left({}^{RL}I_1^{\delta_j}{}^HI_1^\alpha v_*\right)(\eta_j)-{}^HI_1^\alpha v_*(T)\right\}.$$

Consider the linear operator $\Theta: L^1([1,T],\mathbb{R}) \to C([1,T],\mathbb{R})$ given by

$$v\mapsto\Theta(v)(t)={}^HI_1^\alpha v(t)+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^m\theta_i\left({}^HI_1^\alpha v\right)(\xi_i)+\sum_{j=1}^p\lambda_j\left({}^{RL}I_1^{\delta_j}{}^HI_1^\alpha v\right)(\eta_j)-{}^HI_1^\alpha v(T)\right\}.$$

Observe that $\|h_n(t)-h_*(t)\| \to 0$, as $n \to \infty$. By a Lazota-Opial result on closed graphs [18], we deduce that $\Theta \circ S_F$ is a closed graph operator. Moreover $h_n(t) \in \Theta(S_{F,x_n})$, and since $x_n \to x_*$, we have

$$h_*(t)={}^HI_1^\alpha v_*(t)+\frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^m\theta_i\left({}^HI_1^\alpha v_*\right)(\xi_i)+\sum_{j=1}^p\lambda_j\left({}^{RL}I_1^{\delta_j}{}^HI_1^\alpha v_*\right)(\eta_j)-{}^HI_1^\alpha v_*(T)\right\},$$

for some $v_* \in S_{F,x_*}$.

Step 5. In the final step, we will prove the existence of an open set $U \subseteq C([1,T],\mathbb{R})$ such that $x \notin \nu\mathscr{F}(x)$ for any $\nu \in (0,1)$ and all $x \in \partial U$.

Assume that $\nu \in (0,1)$ and $x \in \nu\mathscr{F}(x)$. Then there exists $v \in L^1([1,T],\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in [1,T]$,

$$x(t) = {}^H I_1^{\alpha} v(t) + \frac{(\log t)^{\gamma-1}}{\Lambda}\Big\{\sum_{i=1}^{m}\theta_i\big({}^H I_1^{\alpha} v\big)(\xi_i) + \sum_{j=1}^{p}\lambda_j\left({}^{RL}I_1^{\delta_j}{}^H I_1^{\alpha} v\right)(\eta_j) - {}^H I_1^{\alpha} v(T)\Big\}.$$

Using the computations of the second step, we have $|x(t)| \le \Omega\|p\|\psi(\|x\|)$. Consequently

$$\frac{\|x\|}{\Omega\|p\|\psi(\|x\|)} \le 1.$$

In view of (H_3), there exists M such that $\|x\| \ne M$. Let

$$U = \{x \in C([1,T],\mathbb{R}) : \|x\| < M\}.$$

The operator $\mathscr{F} : \overline{U} \to \mathscr{P}(C([1,T],\mathbb{R}))$ is a compact multivalued map, upper semicontinuous with convex closed values. Also there is no $x \in \partial U$ such that $x \in \nu\mathscr{F}(x)$ for some $\nu \in (0,1)$, by the choice of U. Hence the operator $\mathscr{F}$ has a fixed point $x \in \overline{U}$, by the Leray-Schauder nonlinear alternative ([11]). Consequently the Hilfer-Hardamard nonlocal integro-multi-point fractional boundary value problem (1.2) has at least one solution on $[1,T]$. The proof is completed. □

In our second result, we show the existence of solutions for the problem (1.2) when F is not necessary nonconvex valued via a fixed point theorem for multivalued contractive maps due to Covitz and Nadler [8].

Theorem 3.3. *Assume that:*

- (A_1) *$F : [1,T]\times\mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ is such that $F(\cdot,x) : [1,T] \to \mathscr{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*
- (A_2) *$H_d(F(t,x),F(t,\bar{x})) \le m(t)|x-\bar{x}|$ for almost all $t \in [1,T]$ and $x,\bar{x} \in \mathbb{R}$ with $m \in C([1,T],\mathbb{R}^+)$ and $d(0,F(t,0)) \le m(t)$ for almost all $t \in [1,T]$.*

Then nonlocal multi-point-integral Hilfer-Hadamard fractional boundary value problem (1.2) *has at least one solution on $[1,T]$, provided that $\Omega\|m\| < 1$, where Ω is defined by (3.1).*

Proof. Consider the operator $\mathscr{F}$ defined at the begin of the proof of Theorem 3.2. By the assumption (A_1), the set $S_{F,x}$ is nonempty for each $x \in C([1,T],\mathbb{R})$. Hence F has a measurable selection (see Theorem III.6 [7]). To show that the operator $\mathscr{F}$ satisfies the assumptions of Covitz and Nadler theorem ([8]), we must first show that $\mathscr{F}(x) \in \mathscr{P}_{cl}(C([1,T],\mathbb{R}))$ for each $x \in C([1,T],\mathbb{R})$. Let $\{u_n\}_{n\ge0} \in \mathscr{F}(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C([1,T],\mathbb{R})$. Then $u \in C([1,T],\mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in [1,T]$,

$$u_n(t) = {}^H I_1^{\alpha} v_n(t) + \frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m}\theta_i\big({}^H I_1^{\alpha} v_n\big)(\xi_i) + \sum_{j=1}^{p}\lambda_j\left({}^{RL}I_1^{\delta_j}{}^H I_1^{\alpha} v_n\right)(\eta_j) - {}^H I_1^{\alpha} v_n(T)\right\}.$$

Note that v_n converges to v in $L^1([1,T],\mathbb{R})$ since F has compact values. Thus, $v \in S_{F,x}$ and for each $t \in J$, we have

$$u_n(t) \to u(t) = {}^H I_1^{\alpha} v(t) + \frac{(\log t)^{\gamma-1}}{\Lambda}\left\{\sum_{i=1}^{m}\theta_i\big({}^H I_1^{\alpha} v\big)(\xi_i) + \sum_{j=1}^{p}\lambda_j\left({}^{RL}I_1^{\delta_j}{}^H I_1^{\alpha} v\right)(\eta_j) - {}^H I_1^{\alpha} v(T)\right\}.$$

Hence, $u \in \mathscr{F}(x)$.

In the second step, we show that there exists δ $(\delta := \Omega\|m\| < 1)$ such that

$$H_d(\mathscr{F}(x), \mathscr{F}(\bar{x})) \leq \delta\|x - \bar{x}\| \text{ for each } x, \bar{x} \in C^2([1,T], \mathbb{R}).$$

Let $x, \bar{x} \in C^2([1,T], \mathbb{R})$ and $h_1 \in \mathscr{F}(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in [1,T]$,

$$h_1(t) = {}^H I_1^{\alpha} v_1(t) + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \sum_{i=1}^{m} \theta_i ({}^H I_1^{\alpha} v_1)(\xi_i) + \sum_{j=1}^{p} \lambda_j \left({}^{RL}I_1^{\delta_j H} I_1^{\alpha} v_1 \right)(\eta_j) - {}^H I_1^{\alpha} v_1(T) \right\}.$$

By (A_2), we have

$$H_d(F(t,x), F(t,\bar{x})) \leq m(t)|x(t) - \bar{x}(t)|.$$

So, there exists $w \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)|, \ t \in [1,T].$$

Define $U : [1,T] \to \mathscr{P}(\mathbb{R})$ by

$$U(t) = \{ w \in \mathbb{R} : |v_1(t) - w| \leq m(t)|x(t) - \bar{x}(t)| \}.$$

There exists a function $v_2(t)$ which is a measurable selection for U, since the multivalued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [7]). Hence $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in [1,T]$, we have $|v_1(t) - v_2(t)| \leq m(t)|x(t) - \bar{x}(t)|$. For each $t \in [1,T]$, let us define

$$h_2(t) = {}^H I_1^{\alpha} v_2(t) + \frac{(\log t)^{\gamma-1}}{\Lambda} \left\{ \sum_{i=1}^{m} \theta_i ({}^H I_1^{\alpha} v_2)(\xi_i) + \sum_{j=1}^{p} \lambda_j \left({}^{RL}I_1^{\delta_j H} I_1^{\alpha} v_2 \right)(\eta_j) - {}^H I_1^{\alpha} v_2(T) \right\}.$$

Thus

$$\begin{aligned}
|h(t)| &\leq {}^H I_1^{\alpha}|v_1 - v_2|(t) + \frac{(\log t)^{\gamma-1}}{|\Lambda|} \Big\{ \sum_{i=1}^{m} |\theta_i| \, {}^H I_1^{\alpha}|v_1 - v_2|(\xi_i) \\
&\quad + \sum_{j=1}^{p} |\lambda_j| \, {}^{RL}I_1^{\delta_j H} I_1^{\alpha}|v_1 - v_2|(\eta_j) + {}^H I_1^{\alpha}|v_1 - v_2|(T) \Big\} \\
&\leq \left[\frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\gamma-1}}{|\Lambda|} \left\{ \sum_{i=1}^{m} |\theta_i| \frac{(\log \xi_i)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{j=1}^{p} \frac{|\lambda_j| \Phi_j^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log T)^{\alpha}}{\Gamma(\alpha+1)} \right\} \right] \\
&\quad \times \|m\| \|x - \bar{x}\| \\
&= \Omega\|m\| \|x - \bar{x}\|.
\end{aligned}$$

Hence $\|h_1 - h_2\| \leq \Omega\|m\| \|x - \bar{x}\|$. Analogously, interchanging the roles of x and $\bar{x}$, we obtain

$$H_d(\mathscr{F}(x), \mathscr{F}(\bar{x})) \leq \Omega\|m\| \|x - \bar{x}\|.$$

So $\mathscr{F}$ is a contraction. Therefore, it follows by Covitz and Nadler theorem ([8]) that $\mathscr{F}$ has a fixed point x which is a solution to Hilfer-Hardamard nonlocal integro-multi-point fractional boundary value problem (1.2). The proof is finished. □

4. EXAMPLES

In this section, we construct examples to illustrate the existence results obtained in the previous section. Consider the following boundary value problem for nonlocal multi-point-integral Hilfer-Hadamard fractional differential inclusions with nonlocal multi-point boundary conditions of the form:

$$\begin{cases} {}^{HH}D_1^{\frac{7}{4},\frac{3}{5}}x(t) \in F(t,x(t)), & t \in \left[1, \frac{15}{7}\right], \\ x(1) = 0, & \\ x\left(\frac{15}{7}\right) = \frac{2}{61}x\left(\frac{8}{7}\right) + \frac{4}{73}x\left(\frac{13}{7}\right) + \frac{6}{89}{}^{RL}I_1^{\frac{1}{3}}x\left(\frac{9}{7}\right) + \frac{8}{97}{}^{RL}I_1^{\frac{2}{3}}x\left(\frac{11}{7}\right) + \frac{5}{79}{}^{RL}I_1^{\frac{4}{3}}x\left(\frac{12}{7}\right). \end{cases} \tag{4.1}$$

Here $\alpha = 7/4$, $\beta = 3/5$, $T = 15/7$, $m = 2$, $\theta_1 = 2/61$, $\theta_2 = 4/73$, $\xi_1 = 8/7$, $\xi_2 = 13/7$, $p = 3$, $\lambda_1 = 6/89$, $\lambda_2 = 8/97$, $\lambda_3 = 5/79$, $\eta_1 = 9/7$, $\eta_2 = 11/7$, $\eta_3 = 12/7$, $\delta_1 = 1/3$, $\delta_2 = 2/3$, $\delta_3 = 4/3$. From these data, we have $\gamma = 19/10$, $\Phi_1^{\frac{9}{10}} \approx 0.1655500906$, $\Phi_2^{\frac{9}{10}} \approx 0.2444334694$, $\Phi_3^{\frac{9}{10}} \approx 0.1546014426$, $|\Lambda| \approx 0.7010794212$, $\Phi_1^{\frac{7}{4}} \approx 0.04471482676$, $\Phi_2^{\frac{7}{4}} \approx 0.09788311165$, $\Phi_3^{\frac{7}{4}} \approx 0.06046790028$, and $\Omega \approx 0.8457554067$.

(*i*) For illustration of the result in Theorem 3.2, let the multifunction $F(t,x)$ be given, for $(t,x) \in [1,15/7] \times \mathbb{R}$, by

$$F(t,x) = \left[\frac{1}{t+2}\left(\frac{x^{136}}{1+x^{134}} + \frac{1}{16}\right), \frac{1}{t+1}\left(\frac{2x^{136}}{1+x^{134}} + \frac{1}{2}\right)\right]. \tag{4.2}$$

From (4.2), we have

$$\|F(t,x)\|_{\mathscr{P}} \le \left(\frac{1}{t+1}\right)\left(2x^2 + \frac{1}{2}\right).$$

Taking $p(t) = 1/(t+1)$ and $\psi(x) = 2x^2 + (1/2)$, we obtain by direct computation that $\|p\| = 1/2$ and there exists a constant $M \in (0.2757471029, 0.9066278395)$ satisfying the inequality in (H_3). Thus, by Theorem 3.2, the boundary value problem (4.1), with F given in (4.2), has at least one solution on $[1,15/7]$.

(*ii*) In order to illustrate the benefit of Theorem 3.3, we assume that $F(t,x)$ is defined, for $(t,x) \in [1,15/7] \times \mathbb{R}$, by

$$F(t,x) = \left[0, e^{-(t-1)^2}\left(\frac{x^2+2|x|}{2(1+|x|)} + \frac{1}{2}\right)\right], \tag{4.3}$$

which is obviously measurable for all $x \in \mathbb{R}$, and such that

$$H_d(F(t,x), F(t,\bar{x})) \le e^{-(t-1)^2}|x - \bar{x}|,$$

for all $x, \bar{x} \in \mathbb{R}$. Choosing $m(t) = e^{-(t-1)^2}$, we have that $\|m\| = 1$ and $d(0, F(t,0)) \le (1/2)m(t) \le m(t)$ for each $t \in [1,15/7]$. Thus

$$\Omega\|m\| \approx 0.8457554067 < 1.$$

Hence, by Theorem 3.3, the boundary value problem (4.1), with F is given by (4.3), has at least one solution on $[1,15/7]$.

5. Conclusions

In this paper, we studied a boundary value problem for nonlocal multi-point-integral Hilfer-Hadamard fractional fractional differential inclusions. Existence results were obtained in the cases when the multi-valued map has convex or nonconvex values. In the case of convex multi-valued map the existence result was based on nonlinear alternative of Leray-Schauder type, while in the case of nonconvex multi-valued map the existence result was established by using Covitz and Nadler fixed point theorem for multivalued contractive maps. We also demonstrated the applications of our results with the aid of examples.

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