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A STRONG CONVERGENCE THEOREM FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS WITH PSEUDO-MONOTONE AND LIPSCHITZ MAPPINGS

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Abstract. In this paper, we propose a Mann-type self-adaptive projected reflected subgradient extragradient algorithm for solving the classical variational inequality problem with Lipschitz continuous and pseudo-monotone mappings in a real Hilbert space. The strong convergence of the proposed algorithm is proven without the prior knowledge of the Lipschitz constant of the cost function. Finally, we give some numerical examples to illustrate the superiority of our proposed algorithm.

Keywords. Pseudo-monotone mapping; Strong convergence; Subgradient extragradient; Variational inequality problem.

1. Introduction

Let $\mathscr{H}$ be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty, closed, and convex set in $\mathscr{H}$.

Fichera [10, 11] introduced the *variational inequality problem* (shortly, VIP):

Problem 1.1. Find a point $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \ \ \forall x \in C,$$

where $A : \mathscr{H} \to \mathscr{H}$ is a single-valued mapping.

The solution set of Problem 1.1 is denoted by $VI(C,A)$. Due to the wide applications of the variational inequality problem in economics, mathematical programming, transportation, optimization, and other fields, it has attracted extensive attention; see, e.g., [1, 2, 19, 20, 21, 34] and the references therein. Recently, a number of authors proposed various methods for solving the variational inequality problem. The simplest one is gradient projection method:

$$x^{k+1} = P_C(x^k - \lambda Ax^k),$$

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where P_C denotes the metric projection from $\mathscr{H}$ onto C. As is known to all, the assumptions that guarantee the convergence of this method are that the operator A is L-Lipschitz continuous and α-strongly monotone (or inverse strongly). If the strong monotonicity is reduced to the monotonicity, then this method may not converge.

In order to deal with this situation, Korpelevich [23] in a finite dimensional Euclidean space $\mathbb{R}^n$ proposed the extragradient method :

$$\begin{cases} x^0 \in \mathbb{R}^n, \\ y^k = P_C(x^k - \lambda A x^k), \\ x^{k+1} = P_C(x^k - \lambda A y^k), \end{cases}$$

where $A : \mathbb{R}^n \to \mathbb{R}^n$ is a monotone and L-Lipschitz continuous operator, and λ is a constant in $(0, \frac{1}{L})$. It was proved that the sequence $\{x^k\}$ generated by this algorithm converges to an element of the solution set of Problem 1.1.

It is noted that the extragradient method needs to calculate the projection onto the feasible set C twice in each iteration. As everyone knows, when C is a general closed convex set, the evaluation of the projection operator onto C is computationally expensive, which may seriously affect the computational efficiency of the extragradiet method. Therefore, many authors have considered how to improve the extragradient method so that one only needs to calculate the projection onto C once in each iteration. As far as we know, there are two most commonly used improvement methods. The first one is Tseng's extragradient method proposed by Tseng [38]:

$$\begin{cases} y^k = P_C(x^k - \lambda A x^k), \\ x^{k+1} = y^k - \lambda (A y^k - A x^k), \end{cases}$$

where A is a monotone and L-Lipschitz continuous mapping, and λ is a constant in $(0, \frac{1}{L})$. The generated sequence $\{x^k\}$ by Tseng's extragradient is weakly convergent in the setting of infinite dimensional Hilbert spaces.

The second one is the subgradient extragradient method proposed by Censor *et al.* [5, 6, 7]:

$$\begin{cases} y^k = P_C(x^k - \lambda A x^k), \\ T_k = \{x \in \mathscr{H} : \langle x^k - \lambda A x^k - y^k, x - y^k \rangle \leq 0\}, \\ x^{k+1} = P_{T_k}(x^k - \lambda A y^k), \end{cases}$$

where A is a monotone and L-Lipschitz continuous mapping, and λ is a constant in $(0, \frac{1}{L})$. This method replaces the second projection onto the closed and convex subset C by the projection onto a half-space. They proved that the sequence generated by subgradient extragradient method weakly converges to the unique solution of Problem 1.1.

Because Tseng's extragradient method and subgradient extragradient method only need to calculate the projection onto the feasible set C once in each iteration, they have been extensively investigated; see, e.g., [4, 9, 14, 24, 28, 31, 32, 33, 36, 37, 39] and the references therein.

The pseudo-monotone mappings in the sense of Karamardian were introduced in [18] as a generalization of the monotone mappings. The concept of the pseudo-monotone mapping has many applications in variational inequalities and economics. Recently, Gibali *et al.* [12] proposed an adaptive projected reflected subgradient extragradient method for solving the variation inequality problem with A being a pseudo-monotone and L-Lipschitz continuous mapping.

Algorithm 1.1 (Adaptive projected reflected subgradient extragradient method)

Step 0. Give $\lambda_0 > 0$, $\mu \in (0,1)$. Let $x^0, x^1 \in \mathscr{H}$ be arbitrary.

Given the current iterates x^k, calculate x^{k+1} as follows:

Step 1.

$$\begin{cases} w^k = 2x^k - x^{k-1}, \\ y^k = P_C(w^k - \lambda_k A w^k). \end{cases}$$

If $x^k = w^k = y^k = x^{k+1}$, then stop. Otherwise:

Step 2. Compute:

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k P_{T_k}(w^k),$$

where

$$T_k := \{x \in \mathscr{H} : h_k(x) \le 0\}$$

and

$$h_k(x) = \langle w^k - y^k - \lambda_k(Aw^k - Ay^k), x - y^k \rangle.$$

Update:

$$\lambda_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu\|w^k - y^k\|}{\|Aw^k - Ay^k\|}, \lambda_k \right\}, & \text{if } Aw^k \neq Ay^k, \\ \lambda_k, & \text{otherwise.} \end{cases}$$

Set $k := k+1$, and go to **Step 1.**

Algorithm 1.1 uses self-adaptive step sizes, and the convergence of this algorithm was proven without any assumption of prior knowledge of the Lipschitz constant of the cost function. However, this algorithm only weakly converges to the solution of Problem 1.1.

In this paper, motivated and inspired by the above works, we introduce a Mann-type self-adaptive projected reflected subgradient extragradient algorithm for solving Problem 1.1 in real Hilbert spaces with A being a pseudo-monotone and L-Lipschitz mapping. Like algorithm 1.1, our algorithm does not need to know the Lipschitz constant of the mapping. Under some conditions, we prove that the iterative sequence generated by our algorithm strongly converges to a solution of Problem 1.1. Some numerical experiments are provided to support the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we recalls some preliminary results and lemmas for further use. In Section 3, the algorithm is given and its convergence is analyzed. In Section 4, some numerical examples are presented to illustrate the numerical behavior of the proposed algorithm and compare it with some existing ones. In the last section, Section 5, a concluding remark is given.

2. Preliminaries

The weak convergence of a sequence $\{x^k\}_{k=1}^{\infty}$ to x as $k \to \infty$ is denoted by $x^k \rightharpoonup x$ while the strong convergence of $\{x^k\}_{k=1}^{\infty}$ to x as $k \to \infty$ is denoted by $x^k \to x$.

Definition 2.1. Let $A : \mathscr{H} \to \mathscr{H}$ be an operator. Then

(a) A is said to be *L-Lipschitz continuous* with Lipschitz constant $L > 0$ if

$$\|Fx - Fy\| \le L\|x - y\|, \quad \forall x, y \in \mathscr{H}.$$

(b) A is said to be *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \ \ \forall x, y \in \mathscr{H}.$$

(c) A is said to be *pseudo-monotone* if

$$\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0, \ \ \forall x, y \in \mathscr{H}.$$

(d) A is said to be *sequentially weakly continuous* if, for each sequence $\{x^k\}$, $\{x^k\}$ converges weakly to x implies that $\{Ax^k\}$ converges weakly to Ax.

Lemma 2.2. [13, 22] *Let C be a closed and convex subset of a real Hilbert spaces $\mathscr{H}$, and $x \in \mathscr{H}$. Then the following inequalities are true:*
(a) $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \ \ \forall y \in \mathscr{H}$.
(b) $\|P_C(x) - y\|^2 \leq \|x - y\|^2 - \|x - P_C(x)\|^2, \ \ \forall y \in C$.
(c) $\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \ \ \forall y \in C$.

Lemma 2.3. *The following statements hold in any real Hilbert space $\mathscr{H}$:*
(a) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \ \ \forall x, y \in \mathscr{H}$.
(b) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \ \ \forall x, y \in \mathscr{H}$.
(c) $\left\| \sum_{i=1}^{m} t_i x_i \right\|^2 = \sum_{i=1}^{m} t_i \|x_i\|^2 - \sum_{i \neq j} t_i t_j \|x_i - x_j\|^2$, *where* $t_i \geq 0$ *and* $\sum_{i=1}^{m} t_i = 1$, $\forall x_i \in \mathscr{H}$, $1 \leq i \leq m$.

In [15], the following lemma was given in n-dimensional Euclidean spaces. Similarly, we can show this lemma in real Hilbert space.

Lemma 2.4. *Let h be a real-valued function on a real Hilbert space $\mathscr{H}$, and define $K := \{x \in \mathscr{H} : h(x) \leq 0\}$. If h is Lipschitz continuous on $\mathscr{H}$ with modulus $\theta > 0$, and K is nonempty, then*

$$dist(x, K) \geq \frac{1}{\theta} \max\{0, h(x)\}, \ \ \forall x \in \mathscr{H},$$

where $dist(x, K)$ is the distance function from x to K.

Lemma 2.5. [26] *Let $\{a_k\}$ be a non-negative real number sequence, which satisfies*

$$a_{k+1} \leq (1 - \alpha_k) a_k + \alpha_k b_k, \ \ \forall k \geq 1,$$

where $\{\alpha_k\} \subset (0, 1)$ and $\{b_k\}$ are two sequences such that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\limsup_{k \to \infty} b_k \leq 0$. Then $\lim_{k \to \infty} a_k = 0$.

Lemma 2.6. [8] *Let $A : C \to \mathscr{H}$ be a continuous and pseudo-monotone mapping, where C is a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$. Then, x^* is a solution of the VIP if and only if $\langle Ax, x - x^*\rangle \geq 0$, $\forall x \in C$.*

3. Main Results

In this section, we propose a Mann-type self-adaptive projected reflected subgradient extragradient algorithm for solving the variational inequality problem and show its strong convergence. In order to state the main results, we need the following assumptions.

Condition 3.1. The feasible set C is a nonempty, closed, and convex subset of $\mathscr{H}$.

Condition 3.2. The operator $A : \mathscr{H} \to \mathscr{H}$ is pseudo-monotone, sequentially weakly and Lipschitz continuous on a real Hilbert space $\mathscr{H}$.

Condition 3.3. The solution set of Problem 1.1 is nonempty, that is, $VI(C,A) \neq \emptyset$.

Condition 3.4. Let $\{\varepsilon_k\}$ be a positive sequence such that $\lim_{k\to\infty} \dfrac{\varepsilon_k}{\alpha_k} = 0$, where $\{\alpha_k\} \subset (0,1)$ is with the restrictions that $\sum_{k=1}^{\infty} \alpha_k = \infty$ and $\lim_{k\to\infty} \alpha_k = 0$. Let $\{\beta_k\} \subset (a,b) \subset (0,1-\alpha_k)$ for some $a > 0$, $b > 0$.

Algorithm 3.1 (Mann-type self-adaptive projected reflected subgradient extragradient algorithm)

Step 0. Give $\theta > 0$, $\lambda_1 > 0$, $\mu \in (0,1)$. Choose a nonnegative real sequence $\{\xi_k\}$ such that $\sum_{k=1}^{\infty} \xi_k < +\infty$. Let $x^0, x^1 \in \mathscr{H}$ be arbitrary.

Step 1. Given the current iterates x^{k-1} and x^k, set

$$w^k = x^k + \theta_k(x^k - x^{k-1}),$$

where

$$\theta_k = \begin{cases} \min\left\{ \dfrac{\varepsilon_k}{\|x^k - x^{k-1}\|}, \theta \right\}, & \text{if } x^k \neq x^{k-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$y^k = P_C(w^k - \lambda_k A w^k).$$

If $x^k = y^k$, then stop, and y^k is a solution of Problem 1.1. Otherwise, go to **Step 3.**

Step 3. Compute

$$x^{k+1} = (1 - \alpha_k - \beta_k) w^k + \beta_k P_{T_k}(w^k), \tag{3.1}$$

where $T_k := \{x \in \mathscr{H} : h_k(x) \leq 0\}$ and

$$h_k(x) = \langle w^k - y^k - \lambda_k (A w^k - A y^k), x - y^k \rangle. \tag{3.2}$$

Update

$$\lambda_{k+1} = \begin{cases} \min\left\{ \dfrac{\mu \|w^k - y^k\|}{\|A w^k - A y^k\|}, \lambda_k + \xi_k \right\}, & \text{if } A w^k \neq A y^k, \\ \lambda_k + \xi_k, & \text{otherwise.} \end{cases} \tag{3.3}$$

Set $k := k+1$, and go to **Step 1.**

Remark 3.5. It follows from Algorithm 3.1 that

$$\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} \|x^k - x^{k-1}\| = 0.$$

In fact, whether $x^k = x^{k-1}$ or $x^k \neq x^{k-1}$, the definition of $\{\theta_k\}$ implies that $\theta_k \|x^k - x^{k-1}\| \leq \varepsilon_k$ for all $k \geq 1$. From $\lim_{k\to\infty} \dfrac{\varepsilon_k}{\alpha_k} = 0$, we have

$$\lim_{k\to\infty} \frac{\theta_k}{\alpha_k} \|x^k - x^{k-1}\| \leq \lim_{k\to\infty} \frac{\varepsilon_k}{\alpha_k} = 0.$$

The following lemmas are quite useful for proving the convergence of Algorithm 3.1.

Lemma 3.6. [25] *Let $\{\lambda_k\}$ be the sequence generated by (3.3). Then $\lim_{k\to\infty}\lambda_k = \lambda$ and*

$$\lambda \in \left[\min\left\{\frac{\mu}{L}, \lambda_1\right\}, \lambda_1 + \xi\right],$$

where $\xi = \sum_{k=1}^{\infty} \xi_k$.

Lemma 3.7. *Assume that Condition 3.1-3.3 hold. Let p be a solution of Problem 1.1, and the function $\{h_k\}$ be defined by (3.2). Then, $h_k(p) \le 0$, and there exists $n_0 \in \mathbb{N}$ such that*

$$h_k(w^k) \ge \frac{1-\mu}{2}\|w^k - y^k\|^2, \quad \forall k \ge n_0.$$

In particular, if $w^k \ne y^k$, then $h_k(w^k) > 0$.

Proof. Because p is a solution of Problem 1.1, we conclude from Lemma 2.6 that

$$\langle Ay^k, p - y^k\rangle \le 0.$$

It follows from the definition of y^k and Lemma 2.4 that

$$\begin{aligned} h_k(p) =& \langle w^k - y^k - \lambda_k(Aw^k - Ay^k), p - y^k\rangle \\ =& \langle w^k - y^k - \lambda_k Aw^k, p - y^k\rangle + \lambda_k\langle Ay^k, p - y^k\rangle \\ \le& 0. \end{aligned}$$

Hence, the proof of $h_k(p) \le 0$ is achieved.

Next, we prove

$$h_k(w^k) \ge \frac{1-\mu}{2}\|w^k - y^k\|^2, \quad \forall k \ge n_0.$$

Clearly, using (3.3), we obtain

$$\|Aw^k - Ay^k\| \le \frac{\mu}{\lambda_{k+1}}\|w^k - y^k\|, \ \forall k \ge 1. \tag{3.4}$$

Obviously, if $Aw^k = Ay^k$, then (3.4) must be true. In fact, (3.4) is satisfied if $Aw^k \ne Ay^k$. From the definition of λ_{k+1}, it is easy to see

$$\lambda_{k+1} = \min\left\{\frac{\mu\|w^k - y^k\|}{\|Aw^k - Ay^k\|}, \lambda_k + \xi_k\right\} \le \frac{\mu\|w^k - y^k\|}{\|Aw^k - Ay^k\|}.$$

By (3.4), we have

$$\begin{aligned} h_k(w^k) =& \langle w^k - y^k - \lambda_k(Aw^k - Ay^k), w^k - y^k\rangle \\ =& \|w^k - y^k\|^2 - \lambda_k\langle Aw^k - Ay^k, w^k - y^k\rangle \\ \ge& \|w^k - y^k\|^2 - \lambda_k\|Aw^k - Ay^k\|\|w^k - y^k\| \\ \ge& \|w^k - y^k\|^2 - \mu\frac{\lambda_k}{\lambda_{k+1}}\|w^k - y^k\|^2 \\ =& \left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right)\|w^k - y^k\|^2, \ \forall k \ge 1. \end{aligned} \tag{3.5}$$

From Lemma 3.6, we know

$$\lim_{k\to\infty}\left(1 - \mu\frac{\lambda_k}{\lambda_{k+1}}\right) = 1 - \mu > 0.$$

Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$1-\mu\frac{\lambda_k}{\lambda_{k+1}} > \frac{1-\mu}{2} > 0, \ \ \forall k \geq n_0.$$

From (3.5), we have

$$h_k(w^k) \geq \frac{1-\mu}{2}\|w^k - y^k\|^2, \quad \forall k \geq n_0.$$

This completes the proof. □

The proof of the following lemma is the same as [12, Lemma 11], and we omit it.

Lemma 3.8. *Let $\{w^k\}$ be a sequence generated by Algorithm 3.1 and assume that conditions 3.1-3.3 hold. If there exists $\{w^{k_j}\}$, a subsequence of $\{w^k\}$, such that $\{w^{k_j}\}$ converges weakly to $z \in \mathscr{H}$ and $\lim_{j\to\infty}\|w^{k_j} - y^{k_j}\| = 0$, then $z \in VI(C,A)$.*

Remark 3.9. Imposing the sequential weak continuity on A is not necessary when A is a monotone operator; see [9].

Theorem 3.10. *Assume that Condition 3.1-3.4 hold. Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to $p \in VI(C,A)$ in norm, where*

$$\|p\| = \min\{\|z\| : z \in VI(C,A)\}.$$

Proof. Define $u^k := P_{T_k}(w^k)$. From Lemma 2.2, we have

$$\begin{aligned}\|u^k - p\|^2 &= \|P_{T_k}(w^k) - p\|^2 \\ &\leq \|w^k - p\|^2 - \|w^k - u^k\|^2 \\ &= \|w^k - p\|^2 - dist^2(w^k, T_k),\end{aligned} \tag{3.6}$$

which implies that

$$\|u^k - p\| \leq \|w^k - p\|. \tag{3.7}$$

Claim 1. We first prove that $\{x^k\}$ is a bounded sequence. Combining (3.1) and (3.7), we have

$$\begin{aligned}\|x^{k+1} - p\| &= \|((1-\alpha_k-\beta_k)w^k + \beta_k u^k - p\| \\ &= \|(1-\alpha_k-\beta_k)(w^k - p) + \beta_k(u^k - p) - \alpha_k p\| \\ &\leq (1-\alpha_k-\beta_k)\|w^k - p\| + \beta_k\|u^k - p\| + \alpha_k\|p\| \\ &\leq (1-\alpha_k)\|w^k - p\| + \alpha_k\|p\|.\end{aligned} \tag{3.8}$$

Note that Remark 3.5 implies that $\dfrac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \to 0$. Therefore, there exists a constant $M_1 > 0$ such that

$$\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq M_1, \ \ \forall k \geq 1. \tag{3.9}$$

Using the definition of w^k and (3.9), it is easy to show

$$\begin{aligned}\|w^k-p\| &= \|x^k+\theta_k(x^k-x^{k-1})-p\| \\ &\le \|x^k-p\|+\theta_k\|x^k-x^{k-1}\| \\ &= \|x^k-p\|+\alpha_k\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\| \\ &\le \|x^k-p\|+\alpha_k M_1, \quad \forall k\ge 1.\end{aligned} \tag{3.10}$$

From (3.8) and (3.10), we find that

$$\begin{aligned}\|x^{k+1}-p\| \le&(1-\alpha_k)\|w^k-p\|+\alpha_k\|p\| \\ \le&(1-\alpha_k)\|x^k-p\|+\alpha_k(\|p\|+M_1) \\ \le&\max\left\{\|x^k-p\|,\|p\|+M_1\right\} \\ &\vdots \\ \le&\max\left\{\|x^1-p\|,\|p\|+M_1\right\},\end{aligned}$$

which implies that $\{x^k\}$ is bounded. Therefore, the sequences $\{w^k\}$ and $\{u^k\}$ are also bounded.

Claim 2. There exists $M>0$ such that

$$\begin{aligned}&\beta_k\left[\left(\frac{1}{M}\frac{1-\mu}{2}\|w^k-y^k\|^2\right)^2+M_3\|w^k-u^k\|^2\right] \\ \le&\|x^k-p\|^2-\|x^{k+1}-p\|^2+\alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\|p\|^2\right), \quad \forall k\ge n_0,\end{aligned}$$

where $M_2=\sup_{k\in\mathbb{N}}\left\{\|x^k-p\|,\theta_k\|x^k-x^{k-1}\|\right\}$ and $M_3>0$. Indeed, by Lemma 2.3, we obtain

$$\begin{aligned}&\|x^{k+1}-p\|^2 \\ =&\|(1-\alpha_k-\beta_k)(w^k-p)+\beta_k(u^k-p)+\alpha_k(-p)\|^2 \\ =&(1-\alpha_k-\beta_k)\|w^k-p\|^2+\beta_k\|u^k-p\|^2+\alpha_k\|p\|^2 \\ &-\beta_k(1-\alpha_k-\beta_k)\|w^k-u^k\|^2-\alpha_k(1-\alpha_k-\beta_k)\|w^k\|^2-\alpha_k\beta_k\|u^k\|^2 \\ \le&(1-\alpha_k-\beta_k)\|w^k-p\|^2+\beta_k\|u^k-p\|^2+\alpha_k\|p\|^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-u^k\|^2.\end{aligned} \tag{3.11}$$

Using Lemma 2.4, Lemma 3.7, and (3.6), we know that exists a modulus $M>0$ such that

$$\|u^k-p\|^2\le\|w^k-p\|^2-\left(\frac{1}{M}\frac{1-\mu}{2}\|w^k-y^k\|^2\right)^2, \quad \forall k\ge n_0. \tag{3.12}$$

It is easy to know

$$\begin{aligned}&\|w^k-p\|^2 \\ =&\|x^k+\theta_k(x^k-x^{k-1})-p\|^2 \\ \le&\|x^k-p\|^2+2\theta_k\|x^k-p\|\|x^k-x^{k-1}\|+\theta_k^2\|x^k-x^{k-1}\|^2 \\ \le&\|x^k-p\|^2+3M_2\theta_k\|x^k-x^{k-1}\|, \quad \forall k\ge 1,\end{aligned} \tag{3.13}$$

where $M_2 = \sup_{k\in\mathbb{N}}\left\{\|x^k - p\|, \theta_k\|x^k - x^{k-1}\|\right\}$. Combining (3.11), (3.12), and (3.13), we have

$$\begin{aligned}
&\|x^{k+1}-p\|^2\\
\le&(1-\alpha_k-\beta_k)\|w^k-p\|^2+\beta_k\|w^k-p\|^2-\beta_k\left(\frac{1}{M}\frac{1-\mu}{2}\|w^k-y^k\|^2\right)^2\\
&+\alpha_k\|p\|^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-u^k\|^2\\
\le&\|w^k-p\|^2+\alpha_k\|p\|^2-\beta_k\left(\frac{1}{M}\frac{1-\mu}{2}\|w^k-y^k\|^2\right)^2\\
&-\beta_k(1-\alpha_k-\beta_k)\|w^k-u^k\|^2\\
\le&\|x^k-p\|^2+\alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\|p\|^2\right)\\
&-\beta_k\left(\frac{1}{M}\frac{1-\mu}{2}\|w^k-y^k\|^2\right)^2-\beta_k(1-\alpha_k-\beta_k)\|w^k-u^k\|^2,\ \ \forall k\ge n_0.
\end{aligned}\tag{3.14}$$

From Condition 3.4, we know that $\beta_k < b \le 1-\alpha_k$. It is easy to see that there exists a constant $M_3 > 0$ such that $\beta_k + M_3 < b \le 1-\alpha_k$, which implies that

$$1-\alpha_k-\beta_k \ge M_3.$$

Then (3.14) can be written as

$$\begin{aligned}
\|x^{k+1}-p\|^2\le&\|x^k-p\|^2+\alpha_k\left(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+\|p\|^2\right)\\
&-\beta_k\left(\frac{1}{M}\frac{1-\mu}{2}\|w^k-y^k\|^2\right)^2-\beta_kM_3\|w^k-u^k\|^2,\ \ \forall k\ge n_0.
\end{aligned}$$

where $M_2 = \sup_{k\in\mathbb{N}}\left\{\|x^k - p\|, \theta_k\|x^k - x^{k-1}\|\right\}$ and $M_3 > 0$.

Claim 3. Note that

$$a^{k+1} \le (1-\alpha_k)a^k + \alpha_k b^k,$$

where $a^k = \|x^k - p\|^2$ and

$$b^k = 3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|+2\beta_k\|u^k-w^k\|\|x^{k+1}-p\|+2\langle p, p-x^{k+1}\rangle,$$

where $M_2 = \sup_{k\in\mathbb{N}}\left\{\|x^k - p\|, \theta_k\|x^k - x^{k-1}\|\right\}$. Let $t^k = (1-\beta_k)w^k + \beta_k u^k$. Then

$$\|t^k - w^k\| = \beta_k\|u^k - w^k\|. \tag{3.15}$$

From the definition of t^k and (3.7), we obtain

$$\begin{aligned}
\|t^k-p\| &= \|(1-\beta_k)(w^k-p)+\beta_k(u^k-p)\|\\
&\le (1-\beta_k)\|w^k-p\|+\beta_k\|u^k-p\|\\
&\le \|w^k-p\|.
\end{aligned}\tag{3.16}$$

Using (3.1), (3.16), and Lemma 2.3, we have

$$
\begin{aligned}
&\|x^{k+1}-p\|^2\\
=&\|t^k-\alpha_k w^k-p\|^2\\
=&\|(1-\alpha_k)(t^k-p)+\alpha_k(t^k-w^k-p)\|^2\\
\le&(1-\alpha_k)\|t^k-p\|^2+2\alpha_k\langle t^k-w^k-p,x^{k+1}-p\rangle\\
\le&(1-\alpha_k)\|t^k-p\|^2+2\alpha_k\|t^k-w^k\|\|x^{k+1}-p\|+2\alpha_k\langle p,p-x^{k+1}\rangle\\
\le&(1-\alpha_k)\|w^k-p\|^2+2\alpha_k\|t^k-w^k\|\|x^{k+1}-p\|+2\alpha_k\langle p,p-x^{k+1}\rangle.
\end{aligned}
\tag{3.17}
$$

Combining (3.13), (3.15), and (3.17), we obtain

$$
\begin{aligned}
&\|x^{k+1}-p\|^2\\
\le&(1-\alpha_k)\|x^k-p\|^2+3M_2(1-\alpha_k)\theta_k\|x^k-x^{k-1}\|\\
&+2\alpha_k\|t^k-w^k\|\|x^{k+1}-p\|+2\alpha_k\langle p,p-x^{k+1}\rangle\\
\le&(1-\alpha_k)\|x^k-p\|^2+\alpha_k\Big(3M_2\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|\\
&+2\beta_k\|u^k-w^k\|\|x^{k+1}-p\|+2\langle p,p-x^{k+1}\rangle\Big),
\end{aligned}
$$

where $M_2=\sup_{k\in\mathbb{N}}\{\|x^k-p\|,\theta_k\|x^k-x^{k-1}\|\}$.

Claim 4. We prove that $\{\|x^k-p\|^2\}$ converges to zero by considering two cases on the sequence $\{\|x^k-p\|^2\}$.

Case 1. There exists an $N\in\mathbb{N}$ such that $\|x^{k+1}-p\|^2\le\|x^k-p\|^2$ for all $k\ge\mathbb{N}$. This implies that $\lim_{k\to\infty}\|x^k-p\|^2$ exists. By Lemma 2.5 and Claim 3, we just need to show that $\limsup_{k\to\infty}b^k\le 0$. From the boundedness of $\{x^k\}$ and

$$
\lim_{k\to\infty}\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|=0,
$$

we need to show $\limsup_{k\to\infty}\|u^k-w^k\|\le 0$ and $\limsup_{k\to\infty}\langle p,p-x^{k+1}\rangle\le 0$. By Claim 2 and $\mu\in(0,1)$, we obtain

$$
\lim_{k\to 0}\|w^k-y^k\|=0,\quad \lim_{k\to 0}\|w^k-u^k\|=0.
\tag{3.18}
$$

According the definition of w^k, we have

$$
\lim_{k\to 0}\|x^k-w^k\|=\lim_{k\to 0}\alpha_k\frac{\theta_k}{\alpha_k}\|x^k-x^{k-1}\|=0.
\tag{3.19}
$$

On the other hand, we see that

$$
\lim_{k\to 0}\|x^{k+1}-w^k\|=\lim_{k\to 0}\alpha_k\|w^k\|+\lim_{k\to 0}\beta_k\|w^k-u^k\|=0.
$$

This together with (3.19) obtains

$$
\lim_{k\to\infty}\|x^{k+1}-x^k\|=0.
\tag{3.20}
$$

Since $\{x^k\}$ is bounded, it follows that there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$, which converges weakly to some $q \in \mathscr{H}$ such that

$$\limsup_{k\to\infty}\langle p, p - x^k\rangle = \lim_{j\to\infty}\langle p, p - x^{k_j}\rangle = \langle p, p - q\rangle.$$

Due to $\|w^k - x^k\| \to 0$, we know that w^{k_j} converges weakly to q. From $\|w^{k_j} - y^{k_j}\| \to 0$ and Lemma 3.8, we have $q \in VI(C,A)$. Since $\|p\| = \min\{\|z\| : z \in VI(C,A)\}$, that is, $p = P_{VI(C,A)}0$, we obtain

$$\limsup_{k\to\infty}\langle p, p - x^k\rangle = \langle p, p - q\rangle \le 0. \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$\limsup_{k\to\infty}\langle p, p - x^{k+1}\rangle \le \limsup_{k\to\infty}\langle p, p - x^k\rangle = \langle p, p - q\rangle \le 0. \tag{3.22}$$

Hence, it follows from (3.18), (3.22), Claim 3, and Lemma 2.5 that

$$\lim_{k\to\infty}\|x^k - p\| = 0.$$

Case 2. Assume that there is no $N \in \mathbb{N}$ such that $\{\|x^k - p\|\}$ is monotonically decreasing. The technique of proof used here is adapted from [29, 30]. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $k \ge \mathbb{N}$ (for some N large enough) by

$$\tau(k) := \max\{n \in \mathbb{N} : n \le k,\ \|x^n - p\| \le \|x^{n+1} - p\|\},$$

i.e., $\tau(k)$ is the largest number n in $\{1,2,...,k\}$ such that $\|x^n - p\|$ increases at $n = \tau(k)$. Note that, in view of Case 2, this $\tau(k)$ is well-defined for all sufficiently large k. Clearly, τ is a non-decreasing sequence such that $\tau(k) \to \infty$ as $k \to \infty$ and

$$\|x^{\tau(k)} - p\| \le \|x^{\tau(k)+1} - p\|, \quad \forall k \ge N. \tag{3.23}$$

From Claim 2, we have

$$\begin{aligned}
&\beta_{\tau(k)}\left[\left(\frac{1}{M}\frac{1-\mu}{2}\|w^{\tau(k)} - y^{\tau(k)}\|^2\right)^2 + M_3\|w^{\tau(k)} - u^{\tau(k)}\|^2\right] \\
\le&\|x^{\tau(k)} - p\|^2 - \|x^{\tau(k)+1} - p\|^2 + \alpha_{\tau(k)}\left(3M_2\frac{\theta_{\tau(k)}}{\alpha_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| + \|p\|^2\right) \\
\le&\alpha_{\tau(k)}\left(3M_2\frac{\theta_{\tau(k)}}{\alpha_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| + \|p\|^2\right), \quad \forall \tau(k) > n_0.
\end{aligned}$$

Since $\lim_{k\to\infty}\alpha_k = 0$, we know

$$\lim_{k\to\infty}\|w^{\tau(k)} - y^{\tau(k)}\| = 0, \quad \lim_{k\to\infty}\|w^{\tau(k)} - u^{\tau(k)}\| = 0.$$

As proved in the first case, it is easy to see $\|x^{\tau(k)+1} - x^{\tau(k)}\| \to 0$ and $\limsup_{k\to\infty}\langle p, p - x^{\tau(k)+1}\rangle \le 0$. Using Claim 3, we obtain

$$\begin{aligned}
\|x^{\tau(k)+1} - p\|^2 \le&(1 - \alpha_{\tau(k)})\|x^{\tau(k)} - p\|^2 + \alpha_{\tau(k)}\Big(3M_2\frac{\theta_{\tau(k)}}{\alpha_{\tau(k)}}\|x^{\tau(k)} - x^{\tau(k)-1}\| \\
&+ 2\beta_{\tau(k)}\|u^{\tau(k)} - w^{\tau(k)}\|\|x^{\tau(k)+1} - p\| + 2\langle p, p - x^{\tau(k)+1}\rangle\Big).
\end{aligned}$$

From (3.23), we obtain

$$\|x^{\tau(k)}-p\|^2 \leq 3M_2\frac{\theta_{\tau(k)}}{\alpha_{\tau(k)}}\|x^{\tau(k)}-x^{\tau(k)-1}\|+2\langle p,p-x^{\tau(k)+1}\rangle + 2\beta_{\tau(k)}\|u^{\tau(k)}-w^{\tau(k)}\|\|x^{\tau(k)+1}-p\|. \tag{3.24}$$

Using Remark 3.5, $\lim_{k\to\infty}\|u^{\tau(k)}-w^{\tau(k)}\|=0$, and $\limsup_{k\to\infty}\langle p,p-x^{\tau(k)+1}\rangle\leq 0$, (3.24) implies $\lim_{k\to\infty}\|x^{\tau(k)}-p\|=0$. Therefore, $\lim_{k\to\infty}\|x^{\tau(k)+1}-p\|=0$.

Furthermore, for $k\geq\mathbb{N}$, it is easy to see $\|x^k-p\|\leq\|x^{\tau(k)+1}-p\|$. Next, we prove this. Because $\tau(k)\leq k$, we consider the following three cases: $\tau(k)=k$, $\tau(k)=k-1$, and $\tau(k)\leq k-2$. For the first and second cases, it is obvious that $\|x^k-p\|\leq\|x^{\tau(k)+1}-p\|$, for $k\geq\mathbb{N}$. For the third case $\tau(k)\leq k-2$, we have from the definition of $\tau(k)$ and for any integer $k\geq N$ that $\|x^j-p\|>\|x^{j+1}-p\|$ for $\tau(k)+1\leq j\leq k-1$. Thus,

$$\|x^{\tau(k)+1}-p\|\geq\|x^{\tau(k)+2}-p\|\geq\ldots\geq\|x^{k-1}-p\|\geq\|x^k-p\|.$$

As a sequence, we obtain for all sufficiently large k that $0\leq\|x^k-p\|\leq\|x^{\tau(k)+1}-p\|$. Hence $\lim_{k\to\infty}\|x^k-p\|=0$. Therefore, $\{x^k\}$ converges strongly to p. This completes the proof. □

4. Numerical Experiments

In this section, we provide three numerical examples to test the proposed algorithm. We show the practicability of our proposed algorithm and compare them with the algorithm 3.1 and the algorithm 3.2 in [35] and the algorithm 1 in [27]. All the codes were written in Matlab (R2016a) and run on PC with Intel(R) Core(TM) i3-370M Processor 2.40 GHz.

Take $\theta=0.3$, $\lambda_1=4$, $\mu=0.8$, $\alpha_k=\frac{1}{k}$, $\beta_k=0.9(1-\alpha_k)$, and $\varepsilon_k=\frac{1}{k^2}$ in Algorithm 3.1 and, the algorithms 3.1 and 3.2 in [35]. Choose $\xi_k=\frac{1}{k^2}$ in Algorithm 3.1 and $\lambda_k=\frac{0.8}{L}$, $\alpha_k=\frac{1}{k^2}$, $\mu_k=\frac{1}{k}$ and $\nu_k=0.9(1-\mu_k)$ in the algorithm 1 in [27].

Example 4.1. Consider $C:=\{x\in H:\|x\|\leq 2\}$. Let $g:C\to\mathbb{R}$ be defined by

$$g(u):=\frac{1}{1+\|u\|^2}.$$

Observe that g is L_g-Lipchitz continuous with $L_g=\dfrac{16}{25}$ and $0.2\leq g(u)\leq 1$, $\forall\, u\in C$. Define the Volterra integral operator $F:L^2([0,1])\to L^2([0,1])$ by

$$F(u)(t):=\int_0^t u(s)ds,\ \ \forall\, u\in L^2([0,1]),\ t\in[0,1].$$

Then F is bounded linear monotone (see [3, Exercise 20.12]) and $\|F\|=\dfrac{2}{\pi}$. Now, define $A:C\to L^2([0,1])$ by

$$A(u)(t):=g(u)F(u)(t),\ \ \forall\, u\in C,\ t\in[0,1].$$

Thus, A is pseudo-monotone and L_A-Lipschitz-continuous with $L_A=\frac{82}{\pi}$.

Let $x^0=x^1=\sin(2\pi t^2)$ and take $\|x^k-x^{k-1}\|\leq 10^{-15}$ as the stopping criterion in Figure 1.

We compared Algorithm 3.1, the algorithms 3.1 and 3.2 in [35], and the algorithm 1 in [27]. The numerical result is described in Figure 1. This illustrates that the performance of Algorithm 3.1 is better than others.

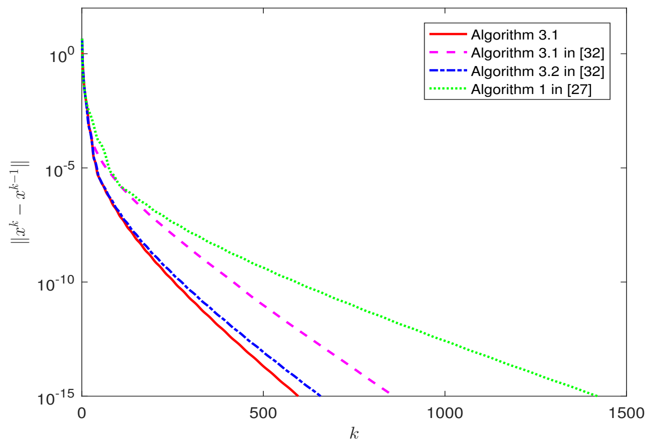

FIGURE 1. Comparison results of this algorithms in Example 4.1.

Example 4.2. [16] Let $H = L^2([0,1])$ with norm $\|x\| := \left(\int_0^1 |x(t)|^2 dt\right)^{\frac{1}{2}}$ and inner product $\langle x,y\rangle := \int_0^1 x(t)y(t)dt,\ \forall x,y \in H$. Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball. Define an operator $A : C \to H$ by

$$A(x)(t) = \int_0^1 (x(t) - f(t,s)g(x(s)))ds + h(t),\ \ \forall x \in C,\ t \in [0,1],$$

where

$$f(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}},\ g(x) = \cos x,\ \ h(t) = \frac{2te^t}{e\sqrt{e^2-1}}.$$

It is known that A is monotone (hence pseudo-monotone) and L-Lipschitz-continuous with $L = 2$, and $\{0\}$ is the solution of the corresponding variational inequality problem.

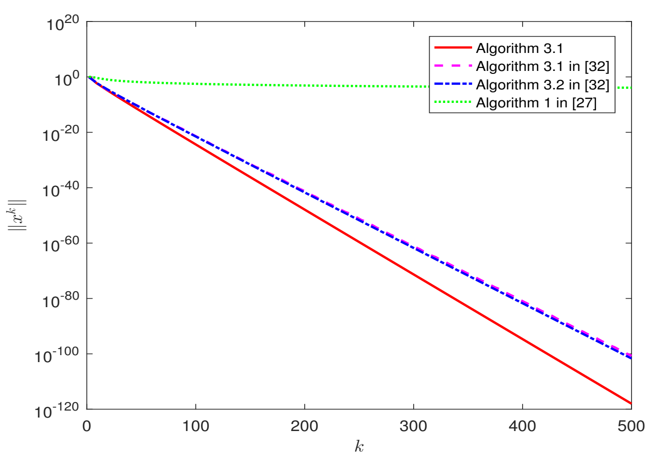

FIGURE 2. Comparison results of this algorithms in Example 4.2.

Let $x^0 = [2,-10]$ and take $k = 500$ as the stopping criterion in Figure 2.

Figure 2 shows that when the number of iteration steps is the same, the error of Algorithm 3.1 is smaller that of the algorithms 3.1 and 3.2 in [35] and the algorithm 1 in [27].

Example 4.3. Consider a two-dimensional variational inequality problem. Let us define

$$A(x) = \begin{pmatrix} (x_1^2 + (x_2-1)^2)(1+x_2) \\ -x_1^3 - x_1(x_2-1)^2 \end{pmatrix}$$

and $C := \{x \in \mathrm{R}^2 : (x_1-2)^2 + (x_2-2)^2 \leq 3\}$. It is easy to see that A is not a monotone map. However, using the Monte Carlo approach (see [17]), it can be shown that A is pseudo-monotone.

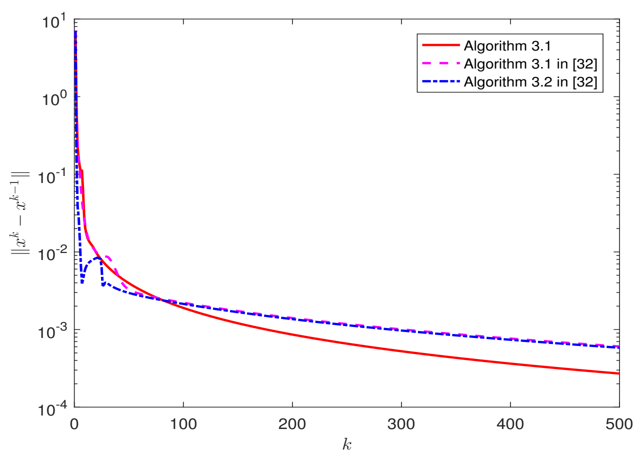

FIGURE 3. Comparison results of this algorithms in Example 4.3.

The initial point x^0 and x^1 are randomly chosen. Take $k = 500$ as the stopping criterion in Figure 3.

The numerical result is described in Figure 3, which illustrates that the performance of Algorithm 3.1 is better than that of the algorithms 3.1 and 3.2 in [35].

5. CONCLUSIONS

In this paper, we introduced a Mann-type self-adaptive projected reflected subgradient extragradient algorithm for solving the variational inequality problem with a pseudo-monotone and Lipschitz contiuous mapping in real Hilbert spaces. For this method, we do not need to know the Lipschitz constant of the involved operator. We proved that the sequence $\{x^k\}$ generated by the proposed algorithm converges to $p \in VI(C,A)$. Finally, three numerical examples show that the proposed algorithm is better than some existing algorithms.

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