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ON THE NONLINEAR EIGENVALUE PROBLEMS INVOLVING THE FRACTIONAL p-LAPLACIAN OPERATOR WITH SINGULAR WEIGHT

H. HARCHA*, O. CHAKRONE, N. TSOULI

Department of Mathematics, Faculty of Sciences, Mohammed First University, Oujda, Morocco

Abstract. The aim of this paper is to study the following problem with the p-Laplacian fractional involving singular weights

$$\begin{cases} -(\Delta_a)_p^s u + h_b(x)|u|^{p-2}u = \lambda h_c(x)|u|^{p-2}u + h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^{\mathbb{N}} \backslash \Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. The existence and the properties of the principal eigenvalue, such as simplicity, isolation, and corresponding eigenfunctions are obtained. Finally, we study the nonexistence of solutions by using a type version of Picone's identity.

Keywords. Compact embedding theorem; First eigenvalue and eigenfunction; Nonlinear eigenvalue problem; Picone's identity; Variational methods.

1. Introduction

In this paper, our aim is to establish the existence, and properties of the principal eigenvalue and corresponding eigenfunctions for the following nonlinear homogeneous eigenvalue problem

$$\begin{cases} -(\Delta_a)_p^s u + h_b(x)|u|^{p-2}u = \lambda h_c(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^{\mathbb{N}} \backslash \Omega, \end{cases} \tag{P_s}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $(N \geq 3)$ with Lipschitzian boundary $\partial\Omega$, $1 < p < N$, $0 < s < 1$, and a, b, c are three nonnegative parameters such that $a \geq 0$, $0 \leq b \leq a + p$, and $0 \leq c < a + p$. We denote by $d(.)$ the distance function up to the boundary $\partial\Omega$ that means $d(x) = dist(x, \partial\Omega) = \inf_{y \in \partial\Omega} | x - y | \; \forall x \in \Omega$, and λ is a real parameter. Moreover, we denote

*Corresponding author.

E-mail address: harchahanan2maa@gmail.com (H. Harcha), chakrone@yahoo.fr (O. Chakrone), tsouli@hotmail.fr (N. Tsouli).

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$h_a(x) = d^{-a}$, $h_b(x) = d^{-b}$, and $h_c(x) = d^{-c}$. The weighted fractional p-Laplacian operator is given by

$$(-\Delta_a)_p^s u(x) = pv \int_\Omega h_a(x) \frac{| u(x) - u(y) |^{p-2} (u(x) - u(y))}{| x - y |^{N+sp}} dy, \ \forall x \in \Omega,$$

where pv refers to the Cauchy principle value; see [11] for more details.

The study of the problem involving fractional and non-local operators, such as (P_s), has been a great interest in various research fields related to PDEs with nonlocal terms. This interest is also justified by its applications in many fields, such as continuum mechanics, phase transition phenomena, population dynamics, and game theory. The problem is the typical outcome of stochastically stabilization of Levy processes; see [14] and the references therein. For results on non-local operators and their applications, we refer the reader to [2, 5, 6, 8, 9, 10, 11, 20] and the references therein. For the basic properties and the continuous compact theorem of fractional Sobolev spaces, we refer the reader to [11]

When $h_a = 1$, $h_b = 0$, and $h_c = 1$, Franzina and Palatucci [14] presented some basic properties of the eigenfunctions of nonlocal operators of fractional p-Laplacian with order $s \in (0,1)$. They studied the weak solutions for the following class of equations

$$\begin{cases} -(\Delta)_p^s u = \lambda |u|^{p-2} & u \ in \ \Omega, \\ u = 0 & on \ \mathbb{R}^N \backslash \Omega, \end{cases} \tag{P_1}$$

where $p > 1$, $N \geq 2$, and $(-\Delta)_p^s$ is the fractional p-laplacian defined by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)}{|x-y|^{N+sp}} dy.$$

Quasilinear eigenvalue problems with the p-Laplacian involving singular weights represents a starting point in analyzing more complicated equations. A first contribution in this sense is due to Drábek and Hernández [12]) in which the following eigenvalue problem was considered

$$\begin{cases} -div(\frac{1}{d^\alpha} |\nabla u|^{p-2} \nabla u) + \frac{1}{d^\beta} |u|^{p-2} u = \frac{\lambda}{d^\gamma} |u|^{p-2} u & in \ \Omega, \\ u = 0 & on \ \partial\Omega, \end{cases} \tag{P_2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitzian boundary $\partial\Omega$, $1 < p < N$, $\alpha \geq 0$, $0 \leq \beta \leq \alpha + p$, and $0 \leq \gamma \leq \alpha + p$. They proved that λ_1 is the principal eigenvalue to the problem (P_2) with associated eigenfunction u, which has a constant sign. Moreover they showed that λ_1 is simple and isolated.

In this paper, inspired by the results mentioned above, we investigate the properties of the principal eigenvalue λ_1 for problem (P_s), such as the simplicity [4] and associated eigenfunctions. In addition, we prove that λ_1 is isolated from the left and the right-hand side by using the fractional Hardy Sobolev inequality. Finally we study the nonexistence of solutions by adding a function h to our problem and applying a Picone-type identity in the fractional case. Our main aim in this work is to generalize the results concerning the properties of eigenvalues and associated eigenfunctions for problem (P_2) to the fractional case based on [4]. This paper is organized as follows. In Section 2, we start with some basic properties and fundamental results on the theory of the weighted fractional Sobolev spaces ([17]). In Section 3, we prove that there exist a principal eigenvalue $\lambda_1 > 0$ of (P_s) with associated eigenfunction u which does not change sign in Ω. We also prove the simplicity of λ_1, and it is isolated from the left and the right-hand side. Finally, in Section 4, by using a type version of Picone's identity, we study the nonexistence of solutions when we add the function $h > 0$ to this problem, $h_a(x) = 1$.

2. PRELIMINARIES

In order to deal with problem (P_s), we need the theory of weighted fractional Sobolev spaces. Here, we only recall some basic facts which will be used later. Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded domain with smooth boundary $\partial\Omega$. For $\varepsilon \in \mathbb{R}^+$, we let

$$L^p(\Omega; \frac{1}{d^\varepsilon}) = \{u : \Omega \longrightarrow \mathbb{R} \; measurable; \int_\Omega \frac{1}{d^\varepsilon}|u(x)|^p dx < \infty\}$$

to be the weighted Lebesgue space with the norm ([12]) $\|u\|_{p;\varepsilon} = (\int_\Omega \frac{1}{d^\varepsilon}|u(x)|^p dx)^{\frac{1}{p}}$.

Lemma 2.1. *The weighted fractional Sobolev space with* $0 < s < 1$ *defined by*

$$W^{s,p}(\Omega, \frac{1}{d^\varepsilon}) = \{u : \Omega \longrightarrow \mathbb{R} \; measurable; \int_\Omega \int_\Omega \frac{1}{d^\varepsilon} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}} dxdy < \infty\}$$

is a Banach space if it is equipped with the norm $\|u\|_{W^{s,p}(\Omega; \frac{1}{d^\varepsilon})} = (\|u\|^p_{p;\varepsilon} + [u]^p_{s,p;\varepsilon})^{\frac{1}{p}}$, *where* $[u]_{s,p;\varepsilon}$ *is a seminorm defined by* $[u]_{s,p;\varepsilon} = (\int_\Omega \int_\Omega \frac{1}{d^\varepsilon} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy)^{\frac{1}{p}}$.

Proof. Let $(u_n)_n$ be a Cauchy sequence with the norm $\|u\|_{W^{s,p}(\Omega; \frac{1}{d^\varepsilon})}$, so $(u_n)_n$ is a Cauchy sequence in $L^p(\Omega, \frac{1}{d^\varepsilon})$ and converges to u in $L^p(\Omega, \frac{1}{d^\varepsilon})$. Let $(v_n)_n$ with $v_n(x,y) = \frac{u_n(x)-u_n(y)}{d^\varepsilon |x-y|^{s+\frac{N}{p}}}$ be a Cauchy sequence in $L^p(\Omega, \frac{1}{d^\varepsilon})$. It then converges in $L^p(\Omega, \frac{1}{d^\varepsilon})$. Let $(u_{\sigma(n)})_n$ be a subsequence of $(u_n)_n$. According to the dominated convergence theorem, it converges $a.e.$ to $u(x)$ and $(v_{\sigma(n)})_n$ converges $a.e.$ to $v(x,y) = \frac{u(x)-u(y)}{d^\varepsilon |x-y|^{s+\frac{N}{p}}}$. By the Fatou Lemma, we obtain

$$\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{d^\varepsilon |x-y|^{N+sp}} dxdy \leq \liminf_{N \to +\infty} \int_\Omega \int_\Omega \frac{|u_{\sigma(n)}(x) - u_{\sigma(n)}(y)|^p}{d^\varepsilon |x-y|^{N+sp}} dxdy,$$

so $u \in W^{s,p}(\Omega, \frac{1}{d^\varepsilon})$. According to the dominated convergence theorem, we have

$$\frac{|u_n(x) - u_n(y)|^p}{d^\varepsilon |x-y|^{N+sp}} \underset{N \to +\infty}{\longrightarrow} \frac{|u(x) - u(y)|^p}{d^\varepsilon |x-y|^{N+sp}}$$

in $L^p(\Omega \times \Omega, \frac{1}{d^\varepsilon})$, so $u_n \longrightarrow u$ in $W^{s,p}(\Omega, \frac{1}{d^\varepsilon})$. □

Denote $X = W_0^{s,p}(\Omega; \frac{1}{d^a}) = W_0^{s,p,a}(\Omega)$ and X' the dual sound. We introduce an equivalent norm $\|u\|_X = (\int_\Omega \int_\Omega \frac{1}{d^a} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy)^{\frac{1}{p}}$ and define $W_0^{s,p}(\Omega, \frac{1}{d^\varepsilon}) \subset W^{s,p}(\Omega, \frac{1}{d^\varepsilon})$ to be a closure of the set $C_0^\infty(\Omega)$ in $(X, \|.\|_X)$. The weighted fractional p-Laplacian operator is given by $(-\Delta_a)_p^s u(x) = pv \int_\Omega \frac{1}{d^a} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}} dy$ for all $x \in \Omega$.

Lemma 2.2. [17]

(1) *If* $a \geq 0, 0 \leq b \leq a+p$, *and* $0 < s < 1$, *then* $W_0^{s,p}(\Omega; \frac{1}{d^a}) \hookrightarrow L^p(\Omega; \frac{1}{d^b})$.

(2) *If* $a \geq 0, 1 < p < N$, *then* $W_0^{s,p}(\Omega; \frac{1}{d^a}) \hookrightarrow W_0^{s,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$, $p^* = \frac{Np}{N-sp}$

(3) *If* $a \leq c = a + sp < a + p$, *then* $W_0^{s,p}(\Omega; \frac{1}{d^a}) \hookrightarrow\hookrightarrow L^p(\Omega; \frac{1}{d^c})$,

where $\hookrightarrow$ *is the continuous embedding, and* $\hookrightarrow\hookrightarrow$ *is the compact embedding.*

Denote by $L : X \longrightarrow X^{'}$ the operator associated to the $(-\Delta_a)^s_p$. It is defined by

$$\langle L(u), \varphi \rangle = \int_{\Omega\times\Omega} \frac{1}{d^a} \frac{\mid u(x)-u(y) \mid^{p-2} (u(x)-u(y))(\varphi(x)-\varphi(y))}{\mid x-y \mid^{N+sp}} dxdy$$

for all u and φ in X.

Lemma 2.3. *If* $0 < s < 1$*, then the following assertions hold*

(i) *L is a bounded and strictly monotone operator;*
(ii) *L is a mapping of type (S_+), namely $u_n \rightharpoonup u$ and $\limsup_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle \leq 0$ imply $u_n \to u$;*
(iii) *$L : X \to X^{'}$ is a homeomorphism.*

Proof. (**i**) Evidently, L is a bounded operator. Now we show that L is strictly monotone operator. Letting $u \neq v \in X$, we prove that $\langle (-\Delta_a)^s_p u - (-\Delta_a)^s_p v, u-v\rangle > 0$. Observe that $\langle (-\Delta_a)^s_p u - (-\Delta_a)^s_p v, u-v\rangle \geq ([u]^{p-1}_{s,p,a} - [v]^{p-1}_{s,p,a})([u]_{s,p,a} - [v]_{s,p,a})$. If $\langle (-\Delta_a)^s_p u - (-\Delta_a)^s_p v, u-v\rangle = 0$, then $\langle (-\Delta_a)^s_p u - (-\Delta_a)^s_p v, u-v\rangle = ([u]^{p-1}_{s,p,a} - [v]^{p-1}_{s,p,a})([u]_{s,p,a} - [v]_{s,p,a}) = 0$, so $[u]_{s,p,a} = [v]_{s,p,a}$. Moreover, if $\langle (-\Delta_a)^s_p u, v\rangle \leq [u]^{p-1}_{s,p,a}[v]_{s,p,a}$, then $[u]^{p-1}_{s,p,a}$ $[v]_{s,p,a} = \langle (-\Delta_a)^s_p u, v\rangle$ and $[v]^{p-1}_{s,p,a}[u]_{s,p,a} = \langle (-\Delta_a)^s_p v, u\rangle$, (see [18]). For all $\alpha, \beta \geq 0$, nonzero simultaneously, we have $\alpha u = \beta v$, which is a contradiction due to $u \neq v$ for $\alpha = \beta = 1$.

(**ii**) Let $(u_n) \in W^{s,p,a}_0$ be a sequence such that $u_n \rightharpoonup u$ in $W^{s,p,a}_0$ and $\limsup_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle \leq 0$. Then, from **(i)**, we deduce that $\lim_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle = 0$. By the compact embedding, we have

$$u_n(x) \to u(x) \ a.e. \ x \in \Omega \tag{2.1}$$

Let $K(x,y) = d^{-a} \mid x-y \mid^{-(N+sp)}$. By Fatou's Lemma, we obtain

$$\liminf_{n\to+\infty} \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^p K(x,y)dxdy \geq \int_{\Omega\times\Omega} |u(x)-u(y)|^p K(x,y)dxdy, \tag{2.2}$$

Moreover, we have

$$\lim_{n\to+\infty} \langle L(u_n), u_n - u\rangle = \lim_{n\to+\infty} \langle L(u_n) - L(u), u_n - u\rangle = 0. \tag{2.3}$$

Now using Young's inequality, we see that there exists a positive constant γ such that

$$\begin{aligned} &\langle L(u_n), u_n - u\rangle \\ &= \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^p K(x,y)dxdy \\ &\quad - \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))(u(x)-u(y))K(x,y)dxdy \\ &\geq \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^p K(x,y)dxdy - \int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^{p-1}|u(x)-u(y)|K(x,y)dxdy \\ &\geq \gamma(\int_{\Omega\times\Omega} |u_n(x)-u_n(y)|^p K(x,y)dxdy - \int_{\Omega\times\Omega} |u(x)-u(y)|^p K(x,y)dxdy). \end{aligned} \tag{2.4}$$

From (2.2), (2.3), and (2.4), we have

$$\lim_{n\to+\infty}\int_{\Omega\times\Omega}|u_n(x)-u_n(y)|^pK(x,y)dxdy=\int_{\Omega\times\Omega}|u(x)-u(y)|^pK(x,y)dxdy. \tag{2.5}$$

By (2.1) and the Brezis-Lieb Lemma [7], we gave the desired result.

(iii) By **(i)**, we have that L is an injection. In view of the compact embedding, we obtain

$$\lim_{\|u\|\to+\infty}\frac{\langle L(u),u\rangle}{\|u\|}=+\infty.$$

Therefore, L is coercive. In light of Minty-Browder theorem [21], L is a surjection. Hence, L has an inverse mapping $L^{-1}:(W_0^{s,p,a})'\to W_0^{s,p,a}$. It remains to show that L^{-1} is continuous. Indeed, let $(f_n),f\in(W_0^{s,p,a})'$ such that $f_n\to f$ in $(W_0^{s,p,a})'$. Let $u_n=L^{-1}(f_n)$ and $u=L^{-1}(f)$. Then $L(u_n)=f_n$ and $L(u)=f$. In view of the coercivity of L, (u_n) is bounded in $W_0^{s,p,a}$. We may assume that $u_n\rightharpoonup u_0$ in $W_0^{s,p,a}$. It follows that $\lim\limits_{n\to+\infty}\langle L(u_n)-L(u_0),u_n-u_0\rangle=\lim\limits_{n\to+\infty}\langle f_n,u_n-u_0\rangle=0$. Using the fact that L is of type (S^+), we conclude that $u_n\to u_0$ in $W_0^{s,p,a}$. This concludes the proof.

□

3. Main Results

Let us consider the energy functional J_λ corresponding to the problem (P_s), defined by $J_\lambda:W_0^{s,p,a}\longrightarrow\mathbb{R}$, for any $\lambda>0$

$$J_\lambda(u)=\int_{\Omega\times\Omega}\frac{1}{p.d^a}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy+\int_\Omega\frac{1}{p.d^b}|u(x)|^p\,dx-\lambda\int_\Omega\frac{1}{p.d^c}|u(x)|^p\,dx.$$

Definition 3.1. We say that $u\in W_0^{s,p,a}$ is a weak solution to problem (P_s) if, for all $\varphi\in W_0^{s,p,a}$,

$$\int_{\Omega\times\Omega}\frac{1}{d^a}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}dxdy$$
$$+\int_\Omega\frac{1}{d^b}|u(x)|^{p-2}u(x)\varphi(x)dx-\lambda\int_\Omega\frac{1}{d^c}|u(x)|^{p-2}u(x)\varphi(x)dx=0.$$

Moreover, we say that λ is an eigenvalue of problem (P_s) if there exists $u\in W_0^{s,p,a}$ non trivial such that u is the corresponding eigenfunction to λ.

Let us present the following weighted fractional Hardy Sobolev inequality that will be used in the proof of Theorem 3.3. In [19], the authors studied the weighted fractional p-Laplacian and established the following weighted fractional L^p-Hardy inequality

$$C\int_{\mathbb{R}^N}\frac{|u(x)|^p}{d^c}dx\le\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^p}{d^a|x-y|^{N+sp}}dxdy, \tag{3.1}$$

where $u\in C_0^\infty(\mathbb{R}^N)$, $C>0$ is a positive constant, and $c=a+sp<N$.

Lemma 3.2. *Let Ω be a smooth bounded open set in $\mathbb{R}^N$, $s\in]0,1[$, and p satisfy $sp<N$. Then*

(1) *J_λ is well defined;*

(2) $J_\lambda \in C^1(W_0^{s,p},\mathbb{R})$, *and, for all* $u,\varphi \in W_0^{s,p,a}$ *its Gâteaux derivative is given by*

$$\langle J'_\lambda(u),\varphi\rangle = \int_{\Omega\times\Omega} \frac{1}{d^a}\frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{\mid x-y\mid^{N+sp}}dxdy$$
$$+\int_\Omega \frac{1}{d^b}\mid u(x)\mid^{p-2}u(x)\varphi(x)dx - \lambda\int_\Omega \frac{1}{d^c}\mid u(x)\mid^{p-2}u(x)\varphi(x)dx.$$

Proof. (**1**) Let $u\in X$. Then $\|u\|_X^p < +\infty$, $\|u\|_{p;b}^p < +\infty$, and $\|u\|_{p;c} < +\infty$. It follows that $J_\lambda < +\infty$ and J_λ is well defined.

(**2**) The existence of the Gâteaux derivative. We define $\psi(u) = \int_{\Omega\times\Omega}\frac{1}{pd^a}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy$, $\phi(u) = \int_\Omega \frac{1}{pd^b}|u(x)|^p dx$, $\phi_\lambda(u) = \lambda\int_\Omega \frac{1}{pd^c}|u(x)|^p dx$, and $J_\lambda(u) = \psi(u)+\phi(u)-\phi_\lambda(u)$. Then $J'_\lambda(u) = \psi'(u)+\phi'(u)-\phi'_\lambda(u)$. For any $u,\varphi\in X$, we denote $z = \varphi(x)-\varphi(y)$. It follows that

$$\langle\psi'(u),\varphi\rangle = \int_{\Omega\times\Omega}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))z}{d^a|x-y|^{N+sp}}dxdy. \tag{3.2}$$

Indeed,

$$\begin{aligned}\langle\psi'(u),\varphi\rangle &= \lim_{t\to 0}\frac{\Psi(u+t\varphi)-\Psi(u)}{t}\\ &= \lim_{t\to 0}\int_{\Omega\times\Omega}\frac{|(u(x)+t\varphi(x))-(u(y)+t\varphi(y))|^p-|u(x)-u(y)|^p}{tpd^a|x-y|^{N+sp}}dxdy.\end{aligned} \tag{3.3}$$

Let us consider $M:[0,1]\to\mathbb{R}$ defined by $M(a) = \frac{|(u(x)-u(y))+atz|^p}{tpd^a|x-y|^{N+sp}}$. The function M is continuous on $[0,1]$ and differentiable on $]0,1[$. Then by the mean value theorem, there exists $\theta\in]0,1[$ such that $M^{'}(a)(\theta) = M(1)-M(0)$. Thus

$$\begin{aligned}&\frac{|(u(x)-u(y))+\theta tz|^{p-2}[(u(x)-u(y))+t\theta z]z}{d^a|x-y|^{N+sp}}\\ &= f(u,\varphi)\\ &= \frac{|(u(x)-u(y))+tz|^p-|u(x)-u(y)|^p}{td^ap|x-y|N+sp}.\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), we have $\langle\psi^{'}(u),\varphi\rangle = \lim\limits_{t\to 0}\int_{\Omega\times\Omega}f(u,\varphi)dxdy$. Since $t,\theta\in[0,1]$, then $t\theta\leq 1$, which implies

$$f(u,\varphi)\leq\frac{|(u(x)-u(y))+z|^{p-2}[(u(x)-u(y))+z]}{d^a|x-y|^{N+sp}}.$$

On the other hand, we also have

$$f(u,\varphi)\to\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))z}{d^a|x-y|^{N+sp}}.$$

Hence, by the dominated convergence theorem, we obtain (3.3). By the same argument, we have

$$\langle\phi^{'}(u),\varphi\rangle = \int_\Omega\frac{1}{d^b}|u(x)|^{p-2}u(x)\varphi(x)dx,$$

and

$$\langle \phi_\lambda^{'}(u), \varphi \rangle = \lambda \int_\Omega \frac{1}{d^c} |u(x)|^{p-2} u(x) \varphi(x) dx.$$

In view of relation (2.5), we have that the result holds. Letting $u_k \to u$ in W_0, we demonstrate that $\psi^{'}(u_k) \to \psi^{'}(u)$ in $W_0^{'}$. Indeed,

$$\begin{aligned}
&\langle \psi^{'}(u_k) - \psi^{'}(u), \varphi \rangle \\
&= \int_{\Omega\times\Omega} \frac{[|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y)) - |u(x)-u(y)|^{p-2}(u(x)-u(y))]}{d^a|x-y|^{N+sp}} \times z \\
&= \int_{\Omega\times\Omega} [\frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{d^a|x-y|^{(\frac{N}{p}+s)(p-1)}} - \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{d^a|x-y|^{(\frac{N}{p}+s)(p-1)}}] \\
&\quad \times \int_{\Omega\times\Omega} \frac{z}{d^a|x-y|^{\frac{N}{p}+s}} dxdy.
\end{aligned}$$

Set

$$F_k(x,y) = \frac{|u_k(x)-u_k(y)|^{p-2}(u_k(x)-u_k(y))}{d^a|x-y|^{(\frac{N}{p}+s)(p-1)}} \in L^{\hat{p}}(\Omega\times\Omega, \frac{1}{d^a}),$$

$$F(x,y) = \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{d^a|x-y|^{(\frac{N}{p}+s)(p-1)}} \in L^{\hat{p}}(\Omega\times\Omega, \frac{1}{d^a}),$$

and

$$\overline{\varphi}(x,y) = \frac{z}{d^a|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega\times\Omega, \frac{1}{d^a}),$$

where $\frac{1}{p} + \frac{1}{\hat{p}} = 1$. Hence, by the Hölder inequality, we obtain

$$\langle \psi^{'}(u_k) - \psi^{'}(u), \varphi \rangle \le 2\|F_k - F\|_{L^{\hat{p}}(\Omega\times\Omega, \frac{1}{d^a})} \|\overline{\varphi}\|_{L^p(\Omega\times\Omega, \frac{1}{d^a})},$$

Thus

$$\|\psi^{'}(u_k) - \psi^{'}(u)\|_{X^*} \le 2\|F_k - F\|_{L^{\hat{p}}(\Omega\times\Omega, \frac{1}{d^a})}.$$

Now, let

$$v_k(x,y) = \frac{u_k(x)-u_k(y)}{d^a|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega\times\Omega, \frac{1}{d^a})$$

and

$$v(x,y) = \frac{u(x)-u(y)}{d^a|x-y|^{\frac{N}{p}+s}} \in L^p(\Omega\times\Omega, \frac{1}{d^a}).$$

Since $u_k \to u$ in X, then $v_k \to v$ in $L^p(\Omega\times\Omega, \frac{1}{d^a})$. Hence, for a subsequence of $(v_k)_{k\ge 0}$, we have $v_k(x,y) \to v(x,y)$ a.e. in $\Omega\times\Omega$ and $\exists h \in L^p(\Omega\times\Omega, \frac{1}{d^a})$ such that $|v_k(x,y)| \le h(x,y)$. Thus we have $F_k(x,y) \to F(x,y)$ a.e. in $\Omega\times\Omega$ and

$$|F_k(x,y)| = |v_k(x,y)|^{p-1} \le |h(x,y)|^{p-1}.$$

By the dominated convergence theorem, we deduce that $F_k \to F$ in $L^{\hat{p}}(\Omega\times\Omega, \frac{1}{d^a})$. Consequently, $\psi^{'}(u_k) \to \psi^{'}(u)$ in $X^{'} = (W_0^{s,p}(\Omega, \frac{1}{d^a}))^{'}$. By the same argument, we have that $\phi^{'}(u_k) \to \phi^{'}(u)$ in $(L^p(\Omega, \frac{1}{d^\beta}))^{'}$ and $\phi_\lambda^{'}(u_k) \to \phi_\lambda^{'}(u)$ in $(L^p(\Omega, \frac{1}{d^c}))^{'}$. From relation (2.5), we deduce the continuity of $J_\lambda^{'}$ immediately.

□

Our main result is given by the two following theorems. To this end, we denote

$$\lambda_1 = \inf_{u\in W_0^{s,p,a}(\Omega)/\{0\}} \frac{\int_{\Omega\times\Omega} \frac{1}{d^a} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + \int_\Omega \frac{1}{d^b} \mid u(x) \mid^p dx}{\int_\Omega \frac{1}{d^c} \mid u(x) \mid^p dx}.$$

Theorem 3.3. *Let $1 < p < N$, $a \geq 0$, $0 \leq b \leq a+p$, and $0 \leq c < a+p$. Then*

(1) *λ_1 is the principal eigenvalue of problem (P_s), and any value $\lambda < \lambda_1$ can not be an eigenvalue of problem (P_s).*
(2) *if u is an eigenfunction associated to eigenvalue λ_1 of problem (P_s), then u does not change sign in Ω.*
(3) *The principal eigenvalue is simple (see [4]), that is, if u and v are two eigenfunctions associated with λ_1, then there exists a constant $\sigma \in \mathbb{R}$ such that $u = \sigma v$.*
(4) *The eigenvalue λ_1 is isolated from the right-hand side, that is, there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1 + \delta)$ there are no eigenvalues.*

Theorem 3.4. *Let $0 \leq \gamma < 1$ and $v \in X$ be an eigenfunction associated with an eigenvalue $\lambda > 0$, $\lambda \neq \lambda_1$. Then v changes sign in Ω, that is, $v^+ \not\equiv 0$ and $v^- \not\equiv 0$ in Ω, where v^+ and v^- denote the positive and negative part of v, respectively.*

Proof. **[Proof of Theorem 3.3]**

(**1**) To study fractional eigenfunctions is related to the problem of minimizing the following nonlocal Rayleigh quotient

$$\mathfrak{R}(u) = \frac{\|u\|_X^p + \|u\|_{p;b}^p}{\|u\|_{p;c}^p},\ u \in X\backslash\{0\}.$$

By Lemma 2.2, there exists a constant $C_1 > 0$ such that, for all $u \in X$,

$$\|u\|_{p;c} \leq C_1 \|u\|_X.$$

Then, for $u \in X$, $u \neq 0$, we have

$$\mathfrak{R}(u) \geq \frac{\|u\|_X^p}{\|u\|_{p;c}^p} \geq \frac{1}{C_1^p} > 0.$$

First, we show that

$$\lambda_1 = \int_{\Omega\times\Omega} \frac{1}{d^a} \frac{\mid u(x)-u(y) \mid^p}{\mid x-y \mid^{N+sp}} dxdy + \int_\Omega \frac{1}{d^b} \mid u(x) \mid^p dx, u \in \mathfrak{F},$$

where $\mathfrak{F} = \{u \in X : \|u\|_{p;c} = 1\}$. Since $1 < p < \infty$, the norm in the dual space $(L^p(\Omega; \frac{1}{d^c}))'$ is uniformly convex therefore, and the norm $\|u\|_{p;c}$ is uniformly Frêchet differentiable. By [13], it is also of class C^1 on $L^p(\Omega; \frac{1}{d^c})\backslash\{0\}$. Hence, $\mathfrak{F}$ is a C^1-manifold modeled on X due to Lemma 2.2. It is obvious that $\mathfrak{R}$ is bounded from below on $\mathfrak{F}$ by a constant $\frac{1}{C_1^p} > 0$. Assume that $\{u_n\} \subset \mathfrak{F}$ is a minimising sequence for $\mathfrak{R}\backslash\mathfrak{F}$, i.e.,

$$\lim_{n\to\infty} \mathfrak{R}(u_n) = \inf_{u\in\mathfrak{F}} \mathfrak{R}(u). \quad (3.5)$$

Then $\{u_n\}$ is a bounded sequence in $X \hookrightarrow L^p(\Omega; \frac{1}{d^b})$. Since X, $L^p(\Omega; \frac{1}{d^b})$ are uniformly convex Banach spaces, $X \hookrightarrow\hookrightarrow L^p(\Omega; \frac{1}{d^c})$, there exists $u \in X$ such that up to a subsequence

$$u_n \rightharpoonup u \textit{ weakly in } X, \tag{3.6}$$

$$u_n \rightharpoonup u \textit{ weakly in } L^p(\Omega; \frac{1}{d^b}), \tag{3.7}$$

$$u_n \to u \textit{ strongly in } L^p(\Omega; \frac{1}{d^c}), \tag{3.8}$$

and

$$u_n \to u \textit{ a.e. in } \Omega. \tag{3.9}$$

We deduce $u \neq 0$, $\|u\|_{p;c} = 1$. By the lower semicontinuity of the norm in X and $L^p(\Omega; \frac{1}{d^b})$, we have

$$\liminf_{n\to\infty} \|u_n\|_X^p \geq \|u\|_X^p, \liminf_{n\to\infty} \|u_n\|_{p;b}^p \geq \|u\|_{p;b}^p, \tag{3.10}$$

but (3.5) and (3.10) yield

$$\lim_{n\to\infty} \|u_n\|_X = \|u\|_X, \lim_{n\to\infty} \|u_n\|_{p;b} = \|u\|_{p;b}. \tag{3.11}$$

The uniform convexity of X and $L^p(\Omega; \frac{1}{d^b})$ together with (3.6), (3.7), and (3.11) yield

$$u_n \to u \tag{3.12}$$

respectively in X and $L^p(\Omega; \frac{1}{d^b})$. (3.12) together with (3.8) imply $\mathfrak{R}(u) = \inf_{v\in\mathfrak{F}} \mathfrak{R}(v)$. Let w be any other eigenfunction of (P_s) with associated eigenvalue λ. By choosing $\varphi = w$ in Definition 3.1, we arrive at

$$\lambda = \mathfrak{R}(w) \geq \inf_{v\in\mathfrak{F}} \mathfrak{R}(v) = \lambda_1,$$

that is, λ_1 is the principal eigenvalue of (P_s),

$$\lambda = \mathfrak{R}(w) \geq \inf_{v\in\mathfrak{F}} \mathfrak{R}(v) = \lambda_1,$$

so the minimum exists. We pose that $I = \int_{\Omega\times\Omega} \frac{1}{d^a} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy$, $J = \int_\Omega \frac{1}{d^b} \mid u(x) \mid^p dx$, and $K = \int_\Omega \frac{1}{d^c} \mid u(x) \mid^p dx$. We define the quotient $F : W_0^{s,p,a}(\Omega)/\{0\} \longrightarrow \mathbb{R}^{\mathbb{N}}$ such that $F(u) = \frac{I(u)+J(u)}{K(u)}$. Thus $\lambda_1 = \inf_{u\in\mathfrak{F}} F(u)$. Hence, there exist $u \in W_0^{s,p,a}(\Omega)/\{0\}$ and $F(u) = \lambda_1$. We say that functions I, J, and K are differentiable so

$$\langle F^{'}(u), v\rangle = \frac{1}{K(u)\langle K(u), v\rangle} (K(u)\langle (I(u)+J(u))^{'}, v\rangle - (I(u)+J(u))\langle K^{'}(u), v\rangle).$$

We know that u is a minimizer of F. Then $F^{'}(u) = 0$ and $K(u)\langle (I(u)+J(u))^{'}, v\rangle - (I(u)+J(u))\langle K^{'}(u), v\rangle = 0$, which implies that

$$\langle (I(u)+J(u))^{'}, v\rangle = \tfrac{(I(u)+J(u))}{K(u)} \langle K^{'}(u), v\rangle = \lambda_1 \langle K^{'}(u), v\rangle.$$

Thus u is a weak solution of problem (P_s).

(**2**) Since $\mathfrak{R}\backslash\mathfrak{F}$ is a C^1-functional on C^1-manifold $\mathfrak{F}$, then there exists a Lagrange multiplier $\mu \in \mathbb{R}$ such that, for any $\varphi \in X$,

$$\begin{aligned}&p\int_{\Omega\times\Omega}\frac{1}{d^{\alpha}}\frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{\mid x-y\mid^{N+sp}}dxdy\\&\quad+p\int_{\Omega}\frac{1}{d^{\beta}}\mid u(x)\mid^{p-2}u(x)\varphi(x)dx\\&=\mu p\int_{\Omega}\frac{1}{d^{\gamma}}\mid u(x)\mid^{p-2}u(x)\varphi(x)dx.\end{aligned}$$

The special choice $\varphi = u$ leads to

$$\|u\|_X^p + \|u\|_{p;\beta}^p = \mu\|u\|_{p;\gamma}^p,$$

which is equivalent to

$$\mu = \mathfrak{R}(u) = \inf_{v\in\mathfrak{F}} \mathfrak{R}(v).$$

We call $\mu = \lambda_1 > 0$ the principal eigenvalue of (P_s) and $u \in X$ is the corresponding principal eigenfunction. For $u \in X$, we have $|u| \in X$ (see Gilbarg and Trudinger [15, Lemma 7.6]) and $\mathfrak{R}(u) = \mathfrak{R}(|u|)$. We may assume that $u \geq 0$ a.e. in Ω. Applying the strong maximum principle for the fractional case inspired by [1], we have $u > 0$ in Ω.

(**3**) Let u and v be eigenfunctions associated with λ_1 such that $\|u\|_{p,c} = \|v\|_{p,c} = 1$. In Theorem 3.3, u and v minimize $\mathfrak{R}(u)$. We may assume that $u > 0$ and $v > 0$ in Ω. Consider the function test defined by $\theta = (\frac{u^p+v^p}{2})^{\frac{1}{p}}$. It follows that $\int_\Omega \frac{1}{d^c}\theta^p dx = \frac{1}{2}(\int_\Omega \frac{1}{d^c}u^p dx + \int_\Omega \frac{1}{d^c}v^p dx) = 1$. Let $\theta \in \mathfrak{F}$. Then

$$\begin{aligned}&|\theta(x)-\theta(y)|^p\\&=(\frac{u^p+v^p}{2})^{1-p}|\frac{1}{2}(u^{p-1}(u(x)-u(y)+v^{p-1}(v(x)-v(y))|^p\\&=\frac{u^p+v^p}{2}|\frac{1}{2}(\frac{u^p}{\frac{u^p+v^p}{2}}.\frac{(u(x)-u(y)}{u}+\frac{v^p}{\frac{u^p+v^p}{2}}.\frac{(v(x)-v(y)}{v})|^p\\&=\frac{u^p+v^p}{2}|\frac{u^p}{u^p+v^p}.\frac{(u(x)-u(y)}{u}+(1-\frac{u^p}{u^p+v^p}).\frac{(v(x)-v(y)}{v})|^p\\&\leq\frac{u^p+v^p}{2}(\frac{u^p}{u^p+v^p}|\frac{(u(x)-u(y)}{u}|^p+(1-\frac{u^p}{u^p+v^p})|\frac{(v(x)-v(y)}{v}|^p)\\&=\frac{1}{2}(u^p\mid\frac{u(x)-u(y)}{u}\mid^p+v^p\mid\frac{v(x)-v(y)}{v}\mid^p)\\&=\frac{1}{2}(\mid u(x)-u(y)\mid^p+\mid v(x)-v(y)\mid^p),\end{aligned}\tag{3.13}$$

which implies that, for all $y \in \Omega$,

$$\begin{aligned}&\int_\Omega\frac{1}{d^a}\mid\theta(x)-\theta(y)\mid^p dx\\&\leq\frac{1}{2}(\int_\Omega\frac{1}{d^a}\mid u(x)-u(y)\mid^p+\int_\Omega\frac{1}{d^a}\mid v(x)-v(y)\mid^p dx).\end{aligned}\tag{3.14}$$

The strict convexity of the function $t \longrightarrow\mid t \mid^p$, $p > 1$ then implies that $\frac{u(x)-u(y)}{u} = \frac{v(x)-v(y)}{v}$. Hence, there exists a constant $\eta > 0$ such that $u = \eta.v$, so the principal eigenvalue of the problem (P_s) is simple.

(**4**) Let λ be the non-negative eigenvalue and the corresponding eigenfunction be u. It is clear that $\lambda = 0$ is not an eigenvalue. If $\lambda > 0$, we find

$$\int_{\Omega\times\Omega} \frac{\mid u(x)-u(y)\mid^p}{\mid x-y\mid^{N+sp}}dxdy + \int_\Omega h_b(x)\mid u\mid^p dx \geq \lambda_1 \int_\Omega h_c(x)\mid u\mid^p dx$$

and

$$\int_{\Omega\times\Omega} \frac{\mid u(x)-u(y)\mid^p}{\mid x-y\mid^{N+sp}}dxdy + \int_\Omega h_b(x)\mid u\mid^p dx = \lambda \int_\Omega h_c(x)\mid u\mid^p dx$$

that $\lambda \geq \lambda_1$. Thus λ_1 is isolated from the left-hand side. Assume that there exist a sequence of eigenvalues λ_n and the corresponding eigenfunctions u_n such that $\lambda_n > \lambda_1$ and $\lambda_n \longrightarrow \lambda_1$. Let $u_n \in X$ be a solution to problem (P_s). So $-(\Delta_a)^s_p u_n + h_b(x)|u_n|^{p-2}u_n = \lambda_n h_c(x)|u_n|^{p-2}u_n$ in Ω, and $u_n = 0$ on $\mathbb{R}^{\mathbb{N}}\backslash\Omega$ with $\| u_n \|_{p,c}= 1$. Then

$$\int_{\Omega\times\Omega} \frac{\mid u_n(x)-u_n(y)\mid^p}{d^a\mid x-y\mid^{N+sp}}dxdy + \int_\Omega \frac{\mid u_n\mid^p}{d^b}dx = \lambda_n \int_\Omega \frac{\mid u_n\mid^p}{d^c}dx.$$

Using the fractional Hardy-Sobolev inequality (3.1), we obtain that u_n is bounded sequence in X. So a subsequence $u_n \rightharpoonup u$ weakly in X and $u_n \longrightarrow u$ strongly in $L^p(\Omega, \frac{1}{d^c})$. We obtain $-(\Delta_a)^s_p u + h_b(x)|u|^{p-2}u = \lambda h_c(x)|u|^{p-2}u$ in Ω, and $u = 0$ on $\mathbb{R}^{\mathbb{N}}\backslash\Omega$. If $u_n > 0$, then $\Omega_n^- = \{x\in\Omega : u_n < 0\}$ and

$$|\Omega_n^-| \longrightarrow 0. \tag{3.15}$$

Taking u_n^- as the test function above, we obtain

$$\begin{aligned}\int_{\Omega\times\Omega} &\frac{\mid u_n(x)-u_n(y)\mid^{p-2}(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{d^a\mid x-y\mid^{N+sp}} + \int_\Omega \frac{\mid u_n\mid^{p-2}u_nu_n^-}{d^b}\\ &= \lambda_n \int_\Omega \frac{\mid u_n\mid^{p-2}u_nu_n^-}{d^c}.\end{aligned} \tag{3.16}$$

By using the Strong Comparison Principles inspired by [16] for u_n in (3.16), and the Hölders inequality, we have

$$\begin{aligned}&\int_{\Omega\times\Omega} \frac{\mid u_n(x)-u_n(y)\mid^p}{d^a\mid x-y\mid^{N+sp}}dxdy + \int_\Omega \frac{\mid u_n\mid^p}{d^b}dx\\ \leq\ &\int_{\Omega\times\Omega} \frac{\mid u_n(x)-u_n(y)\mid^{p-2}(u_n(x)-u_n(y))(u_n^-(x)-u_n^-(y))}{d^a\mid x-y\mid^{N+sp}} + \int_\Omega \frac{\mid u_n\mid^{p-2}u_nu_n^-}{d^b}\\ =\ &\lambda_n\int_\Omega \frac{\mid u_n\mid^{p-2}u_nu_n^-}{d^c}.\end{aligned}$$

Applying the fractional Sobolev embedding, we obtain

$$\lambda_n \int_\Omega \frac{|u_n|^{p-2}u_nu_n^-}{d^c} \leq \lambda_n C|\Omega_n^-|^{1-\frac{p}{q}} \parallel u_n^- \parallel_X^p$$

with a constant $C > 0$ and $p < q \leq p^* = \frac{Np}{N-sp}$. So we conclude that $|\Omega_n^-|^{1-\frac{p}{q}} \geq \lambda_1^{-1}C^{-1} > 0$, which is a contradiction to estimation (3.15).

□

Proof. **[Proof of Theorem 3.4]** Let u and v be the eigenfunctions corresponding to λ_1 and λ respectively. Then $u \in X$ and $v \in X$ satisfies

$$(-\Delta_a)^s_p u + h_b \mid u \mid^{p-2} u = \lambda_1 h_c \mid u \mid^{p-2} u \tag{3.17}$$

and

$$(-\Delta_a)^s_p v + h_b \mid v \mid^{p-2} v = \lambda h_c \mid v \mid^{p-2} v, \tag{3.18}$$

respectively. Suppose that v does not changes the sign. Then we may assume $u \geq 0$. Let $\{\varphi_\varepsilon\}$ be a sequence in $C_c^\infty(\mathbb{R}^{\mathbb{N}})$ such that $\{\varphi_\varepsilon\} = 0$ in $\mathbb{R}^{\mathbb{N}} \setminus \Omega$, $\varphi_\varepsilon \longrightarrow u$ as $\varepsilon \longrightarrow 0$. Now we consider the test functions $w_1 = u$ and $w_2 = \frac{\varphi_\varepsilon^p}{(u+\varepsilon)^{p-1}}$. Then $w_1, w_2 \in X_0$. Taking w_1 and w_2 as the test functions in (3.17) and (3.18), respectively, we obtain

$$\int_{\Omega\times\Omega} \frac{\mid u(x) - u(y) \mid^p}{\mid x - y \mid^{N+sp}} dxdy + \int_\Omega h_b \mid u \mid^p dx = \lambda_1 \int_\Omega h_c \mid u \mid^p dx \tag{3.19}$$

and

$$\int_{\Omega\times\Omega} \frac{\mid u(x) - u(y) \mid^{p-2} (u(x) - u(y))}{\mid x - y \mid^{N+sp}} (\frac{\varphi_\varepsilon^p}{(u+\varepsilon)^{p-1}}(x) - \frac{\varphi_\varepsilon^p}{(u+\varepsilon)^{p-1}}(y)) dxdy$$
$$+ \int_\Omega h_b \mid u \mid^{p-2} u \frac{\varphi_\varepsilon^p}{(u+\varepsilon)^{p-1}} dx = \lambda_1 \int_\Omega h_c \mid u \mid^{p-2} u \frac{\varphi_\varepsilon^p}{(u+\varepsilon)^{p-1}} dx.$$

Using the discrete Picone-type identity giving in Lemma 4.1, we have $L(\varphi_\varepsilon, u+\varepsilon) \geq 0$. It follows that

$$\int_{\Omega\times\Omega} \frac{\mid \varphi_\varepsilon(x) - \varphi_\varepsilon(y) \mid^p}{\mid x - y \mid^{N+sp}} dxdy + \int_\Omega (h_b - \lambda h_c) \varphi_\varepsilon^p (\frac{u}{u+\varepsilon})^{p-1} dx \geq 0. \tag{3.20}$$

Subtracting (3.19) from (3.20) and taking the limit as $\varepsilon \longrightarrow 0$, we obtain $(\lambda - \lambda_1) \int_\Omega h_c \mid u \mid^p \leq 0$, which gives a contradiction since $\lambda > \lambda_1$ which proves our Theorem 3.4. Finally, we conclude that v can not have a constant sign in Ω. □

Example 3.5. Let $\Omega = B(0,1) \subset \mathbb{R}^{\mathbb{N}}$ and $d(x) = \mid x \mid^p$. Define

$$(P) \begin{cases} -(\Delta_a)^s_p u + \mid x \mid^{-bp} |u|^{p-2} u = \lambda \mid x \mid^{-cp} |u|^{p-2} u & in\ B(0,1), \\ u = 0 & on\ \mathbb{R}^{\mathbb{N}} \backslash B(0,1). \end{cases}$$

Then

$$\lambda_1 = \inf_{u \in W_0^{s,p,a}(B(0,1))/\{0\}} \frac{\int_{B(0,1)\times B(0,1)} \frac{1}{|x|^{ap}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} dxdy + \int_{B(0,1)} \frac{1}{|x|^{bp}} |u(x)|^p dx}{\int_{B(0,1)} \frac{1}{|x|^{cp}} |u(x)|^p dx}.$$

4. Nonexistence of Solutions

In this section, we present nonexistence results of problem (P_s) when $h > 0$,

$$\begin{cases} -(\Delta)^s_p u + h_b(x)|u|^{p-2} u = \lambda h_c(x) |u|^{p-2} u + h(x)\ in\ \Omega, \\ u = 0\ on\ \mathbb{R}^{\mathbb{N}} \backslash \Omega, \end{cases} \tag{P_s'}$$

where $h > 0$ and $h_a(x) = 1$. We study the nonexistence of solutions in the fractional p-Laplacian, where h_c and h in $L^\infty(\Omega)$. Solutions of (P_s') belongs to $L^\infty(\Omega) \bigcap X$. There are two principal eigenvalues : $\lambda_1(h_c)$ and $\lambda_{-1}(h_c) = -\lambda_1(-h_c)$, where

$$\lambda_1(h_c) = \inf_{u\in X}\{\int_{\Omega\times\Omega} \frac{|\, u(x)-u(y)\,|^p}{|\, x-y\,|^{N+sp}}dxdy + \int_\Omega h_b(x) \mid u\mid^p dx : \int_\Omega h_c(x)\mid u \mid^p dx = 1\}.$$

These eigenvalues are simple and the corresponding eigenfunctions can be taken > 0 in Ω. In order to prove the nonexistence of solutions and the simplicity of two eigenvalues $\lambda_1(h_c)$ and $\lambda_{-1}(h_c)$, we need the following Picone-type identity (see [3, Lemma 6.2])

Lemma 4.1. *Let $p \in (1,+\infty)$. For $u,v : \Omega \subset \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$ such that $u \geq 0$ and $v > 0$, we have $L(u,v) \geq 0$ in $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, where $L(u,v)(x,y) =\mid u(x)-u(y)\mid^p - \mid v(x)-v(y)\mid^{p-2}(v(x)-v(y))(\frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}})$. The equality holds if and only if $u = kv$ a.e. for some constant k.*

Proposition 4.2. *If $\lambda \notin [\lambda_{-1}(h_c), \lambda_1(h_c)]$, then the problem ($P_s'$) with $h > 0$ has no solution $u > 0$.*

The proof of this proposition is based on the following lemma.

Lemma 4.3. *Let $u > 0$ be a solution to (P_s') with $h > 0$ in Ω. Then, for any $\varphi \in X \bigcap L^\infty(\Omega) \bigcap C^1(\Omega)$ with $\varphi \geq 0$, $\frac{h\varphi^p}{u^{p-1}} \in L^1(\Omega)$ and*

$$\int_\Omega (\lambda h_c(x) - h_b(x))\varphi^p dx + \int_\Omega \frac{h\varphi^p}{u^{p-1}}dx \leq \int_{\Omega\times\Omega} \frac{|\,\varphi(x)-\varphi(y)\,|^p}{|\,x-y\,|^{N+sp}}dxdy. \tag{4.1}$$

Moreover equality holds in (4.1) if and only if φ is a multiple of u.

Proof. **[Proof of Lemma 4.3]** Using lemma 4.1, we have

$$L(u,v)(x,y) =\mid u(x)-u(y)\mid^p - \mid v(x)-v(y)\mid^{p-2}(v(x)-v(y))(\frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}}) \geq 0.$$

Replace u by φ and v by u and dividing by $\mid x-y\mid^{N+sp}$, we obtain

$$0 \leq \int_{\Omega\times\Omega} \frac{|\,\varphi(x)-\varphi(y)\,|^p}{|\,x-y\,|^{N+sp}}dxdy - \int_\Omega \frac{|\,u(x)-u(y)\,|^{p-2}(u(x)-u(y))(\frac{\varphi(x)^p}{u(x)^{p-1}} - \frac{\varphi(y)^p}{u(y)^{p-1}})}{|\,x-y\,|^{N+sp}}dxdy.$$

According to problem (P_s'), we have

$$\int_{\Omega\times\Omega} \frac{|\,u(x)-u(y)\,|^{p-2}(u(x)-u(y))(\frac{\varphi(x)^p}{u(x)^{p-1}} - \frac{\varphi(y)^p}{u(y)^{p-1}})}{|\,x-y\,|^{N+sp}}dxdy + \int_\Omega h_b(x)\mid u\mid^{p-2} u\frac{\varphi^p}{u^{p-1}}dx$$
$$= \lambda \int_\Omega h_c(x)\mid u\mid^{p-2} u\frac{\varphi^p}{u^{p-1}}dx + \int_\Omega h\frac{\varphi^p}{u^{p-1}}dx.$$

Hence,

$$\int_\Omega (\lambda h_c(x) - h_b(x))\varphi^p dx + \int_\Omega \frac{h\varphi^p}{u^{p-1}}dx \leq \int_{\Omega\times\Omega} \frac{|\,\varphi(x)-\varphi(y)\,|^p}{|\,x-y\,|^{N+sp}}dxdy.$$

Moreover, if the equality holds in (4.1), by lemma4.3, we have that φ is a multiple of u. This completes the proof. □

Proof. **[Proof of Proposition 4.2]** Assume that there exists a solution $u > 0$ of (P_s') for some $\lambda \in \mathbb{R}$ and some $h > 0$. Applying the strong maximum principle for the fractional case inspired by [1], we have $u > 0$ in Ω. So lemma 4.3 can be applied. This gives

$$\lambda \int_\Omega h_c(x)\varphi^p dx \leq \int_{\Omega\times\Omega} \frac{|\varphi(x)-\varphi(y)|^p}{|x-y|^{N+sp}} dxdy + \int_\Omega h_b(x)\varphi^p dx$$

for all $\varphi \in X \bigcap L^\infty(\Omega) \bigcap C^1(\Omega)$ with $\varphi \geq 0$. By density, this inequality still holds for all $\varphi \in X$. This implies $\lambda \leq \lambda_1(h_c)$ as well as $-\lambda \leq \lambda_1(-h_c)$. We conclude $\lambda \in [\lambda_{-1}(h_c), \lambda_1(h_c)]$, which reaches a contradiction. Thus problem (P_s') has no solution $u > 0$. □

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