J. Nonlinear Funct. Anal. 2022 (2022) 41 https://doi.org/10.23952/jnfa.2022.41

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

A NEW TWO-GRID P_0^2-P_1 MIXED FINITE ELEMENT ALGORITHM FOR GENERAL ELLIPTIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, a new two-grid mixed finite element scheme for distributed optimal control governed by general elliptic equations is presented. P_0^2-P_1 mixed finite elements and piecewise constant functions are used for spatial discretization. Convergence of the proposed two-grid algorithm is discussed. In the two-grid scheme, the solution of the elliptic optimal control problem on a fine grid is reduced to the solution of the elliptic optimal control problem on a much coarser grid and the solution of a symmetric linear algebraic system on the fine grid and the resulting solution still maintains an asymptotically optimal accuracy.

Keywords. Error estimates; General elliptic optimal control problems; P_0^2-P_1 mixed finite element; Two-grid; Superconvergence.

1. Introduction

We consider the following linear optimal control problems with pointwise constraint:

$$\min_{u\in U_{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y-y_d\|^2+\frac{\nu}{2}\|u\|^2\right\} \tag{1.1}$$

subject to the state equation

$$-\operatorname{div}(a\nabla y+\boldsymbol{b}y)+cy=u,\ \ x\in\Omega, \tag{1.2}$$

which can be written in the form of the first order system

$$\operatorname{div}\boldsymbol{p}+cy=u,\ \ \boldsymbol{p}=-(a\nabla y+\boldsymbol{b}y),\ \ x\in\Omega, \tag{1.3}$$

and the boundary condition

$$y=0,\ \ x\in\partial\Omega, \tag{1.4}$$

where Ω is a convex polygon in $\mathbb{R}^2$, and U^{ad} denotes the admissible set of the control variable, defined by

$$U^{ad}=\{u\in L^2(\Omega):\ u\geq 0,\ \text{a.e. in }\Omega\}.$$

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Received July 7, 2022; Accepted September 10, 2022.

Moreover, we assume that $0 < a_0 \le a \le a^0$, $a \in W^{1,\infty}(\Omega)$, $0 < c \in W^{1,\infty}(\Omega)$, $\boldsymbol{b} = (b_1, b_2)^T \in (W^{1,\infty}(\Omega))^2$, $y_d \in H^1(\Omega)$, $\boldsymbol{p}_d \in (H^1(\Omega))^2$, and ν is a fixed positive number. We also assume that the following condition holds [8]: $b_1^2 + b_2^2 \le 4(1-\gamma)ac$ for some $\gamma \in (0,1)$.

In recent years, numerous numerical methods have been widely applied to various optimal control problems governed by partial differential equations; see, e.g., [5, 21, 22] for standard finite element methods, [3, 16, 17] for mixed finite element methods, [11, 18] for finite volume methods, and [6, 9] for spectral methods. Chen and Liu [2] first used Raviart-Thomas mixed finite element method to solve a class of elliptic optimal control problems, in which objective functional contains the gradient of the state variable. They not only considered a priori error estimates for all variables but also derived the supercolse with order $h^{\frac{3}{2}}$ between average L^2 projection and the approximation of the control variable u. In [3], Chen considered the rectangular mixed finite element approximation for elliptic optimal control problems and obtained the superclose between the centroid interpolation and the numerical solution of the optimal control u with order h^2. Guo, Fu and Zhang [12] proposed a splitting positive definite mixed finite element method for the approximation of convex optimal control problem governed by elliptic equations with control constraints. Hou [16] discussed a priori and a posteriori error estimates of H^1-Galerkin mixed finite element methods for elliptic optimal control problems. Hou, Liu and Yang [17] derived a priori error estimates and superconvergence of P_0^2-P_1 mixed finite element approximation for elliptic optimal control problems. Fu and Rui [10] considered a priori error estimates for least-squares mixed finite element approximation of elliptic optimal control problems.

It is well known that the two-grid method [23, 24] is an effective discretization method for solving nonsymmetric, indefinite, and nonlinear partial differential equations. As far as we know, Liu and Wang [20] first attempted to construct a two-grid finite element scheme of elliptic optimal control problems. Subsequently, Hou and his co-authors [14, 15] designed two-grid mixed finite element schemes for optimal control problems governed by general elliptic equations and Stokes equations respectively.

This paper, motivated by the ideas of the results presented in [14, 20], presents a new two-grid scheme for general elliptic optimal control problems discretized by P_0^2-P_1 mixed finite element [4]. Compared with the two-grid scheme proposed in [14], the linear algebraic system on the fine grid of our two-grid scheme is symmetric. Therefore, more solvers can be selected to solve the problem.

The paper is organized as follows. In Section 2, we consider the P_0^2-P_1 mixed finite element approximation for optimal control problem (1.1)-(1.4). In Section 3, the last section, we present our two-grid algorithm and discuss its convergence.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha| \le m} \|D^\alpha v\|_{L^p(\Omega)}^p$ and the semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$ and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. In addition, C denotes a general positive constant.

2. MIXED METHODS FOR OPTIMAL CONTROL PROBLEMS

In this section, we give P_0^2-P_1 mixed finite element approximation of control problem (1.1)-(1.4). Next, we recall a result from Grisvard [13].

Lemma 2.1. *For every function $\psi \in L^2(\Omega)$, let ϕ be the solution of*

$$-\operatorname{div}(a\nabla\phi) + c\phi = \psi \text{ in } \Omega, \ \phi|_{\partial\Omega} = 0. \tag{2.1}$$

Then (2.1) is solvable and that

$$\|\phi\|_2 \le C\|\psi\|. \tag{2.2}$$

Let

$$\boldsymbol{V} = (L^2(\Omega))^2 \ \text{ and } \ W = H_0^1(\Omega).$$

We recast (1.1)-(1.4) as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{V} \times W \times U_{ad}$ such that

$$\min_{u\in U^{ad}} \left\{ \frac{1}{2}\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \frac{1}{2}\|y - y_d\|^2 + \frac{\nu}{2}\|u\|^2 \right\}, \tag{2.3}$$

$$(\alpha\boldsymbol{p}, \boldsymbol{v}) + (\nabla y, \boldsymbol{v}) + (\boldsymbol{\beta} y, \boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.4}$$

$$-(\boldsymbol{p}, \nabla w) + (cy, w) = (u, w), \ \forall\, w \in W, \tag{2.5}$$

where $\alpha = a^{-1}$, $\boldsymbol{\beta} = \alpha\boldsymbol{b}$, and $(\cdot,\cdot)$ is the inner product of $L^2(\Omega)$.

It follows from [19] that optimal control problem (2.3)-(2.5) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is the solution of (2.3)-(2.5) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{V} \times W$ such that $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(\alpha\boldsymbol{p}, \boldsymbol{v}) + (\nabla y, \boldsymbol{v}) + (\boldsymbol{\beta} y, \boldsymbol{v}) = 0, \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.6}$$

$$-(\boldsymbol{p}, \nabla w) + (cy, w) = (u, w), \ \forall\, w \in W, \tag{2.7}$$

$$(\alpha\boldsymbol{q}, \boldsymbol{v}) + (\nabla z, \boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}), \ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8}$$

$$-(\boldsymbol{q}, \nabla w) - (\boldsymbol{\beta}\cdot\boldsymbol{q}, w) + (cz, w) = (y - y_d, w), \ \forall\, w \in W, \tag{2.9}$$

$$(\nu u + z, \tilde{u} - u) \ge 0, \ \forall\, \tilde{u} \in U^{ad}. \tag{2.10}$$

The inequality (2.10) can be expressed as

$$u = \max\{0, -z\}/\nu. \tag{2.11}$$

Let $\mathscr{T}_h$ denote a regular triangulation of the polygonal domain Ω, h_T denote the diameter of T, and $h = \max_{T\in\mathscr{T}_h} h_T$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ be defined by the following finite element pair P_0^2-P_1 [4]:

$$\boldsymbol{V}_h = \{\boldsymbol{v}_h = (\boldsymbol{v}_{1h}, \boldsymbol{v}_{2h}) \in \boldsymbol{V} | \boldsymbol{v}_{1h}, \boldsymbol{v}_{2h} \in P_0(T), \ \forall\, T \in \mathscr{T}_h\},$$

and

$$W_h = \{w_h \in C^0(\Omega) \cap W | w_h \in P_1(T), \ \forall\, T \in \mathscr{T}_h\},$$

where $P_m(T)$ indicates the space of polynomials of degree no more than m on T.

Moreover, let

$$V_h := \{v_h \in L^2(\Omega) : \forall\, T \in \mathscr{T}_h, \ v_h|_T = \text{constant}\},$$

and $U_h^{ad} = V_h \cap U^{ad}$.

Before the P_0^2-P_1 mixed finite element scheme is given, we introduce three projection operators. First, we define the standard elliptic projection [7] $P_h:\ W\to W_h$, which satisfies, for any $\phi\in W$,

$$(\nabla(\phi-P_h\phi),\nabla w_h)=0,\ \ \forall\, w_h\in W_h, \tag{2.12}$$
$$\|\phi-P_h\phi\|_s\le Ch^{2-s}\|\phi\|_2,\ s=0,1,\ \forall\,\phi\in H^2(\Omega). \tag{2.13}$$

Second, we define the standard L^2 projection [1] $\Pi_h:\ \boldsymbol{V}\to\boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q}\in\boldsymbol{V}$

$$(\boldsymbol{q}-\Pi_h\boldsymbol{q},\boldsymbol{v}_h)=0,\ \ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.14}$$
$$\|\Pi_h\boldsymbol{q}\|\le C\|\boldsymbol{q}\|, \tag{2.15}$$
$$\|\boldsymbol{q}-\Pi_h\boldsymbol{q}\|\le Ch\|\boldsymbol{q}\|_1,\ \ \forall\,\boldsymbol{q}\in(H^1(\Omega))^2. \tag{2.16}$$

Last, for any $\psi\in L^2(\Omega)$ and $T\in\mathscr{T}_h$, we define the element average operator $\pi_h: L^2(\Omega)\to V_h$ by

$$\pi_h\psi|_T=\int_T\psi\mathrm{d}x/|T|,$$

where $|T|$ is the area of the element T.

The following approximation property holds

$$\|\psi-\pi_h\psi\|_{-s,2}\le Ch^{1+s}|\psi|_{1,2},\ s=0,1,\ \forall\,\psi\in H^1(\Omega). \tag{2.17}$$

Then, the mixed finite element discretization of (2.3)-(2.5) is as follows: find $(\boldsymbol{p}_h,y_h,u_h)\in\boldsymbol{V}_h\times W_h\times U_h^{ad}$ such that

$$\begin{aligned}&\min_{u_h\in U_h^{ad}}\left\{\frac{1}{2}\|\boldsymbol{p}_h-\boldsymbol{p}_d\|^2+\frac{1}{2}\|y_h-y_d\|^2+\frac{\nu}{2}\|u_h\|^2\right\},\\&(\alpha\boldsymbol{p}_h,\boldsymbol{v}_h)+(\nabla y_h,\boldsymbol{v}_h)+(\boldsymbol{\beta}y_h,\boldsymbol{v}_h)=0,\ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h,\\&-(\boldsymbol{p}_h,\nabla w_h)+(cy_h,w_h)=(u_h,w_h),\ \forall\,w_h\in W_h.\end{aligned}$$

The above optimal control problem has a unique solution $(\boldsymbol{p}_h,y_h,u_h)$ and there is a co-state $(\boldsymbol{q}_h,z_h)\in\boldsymbol{V}_h\times W_h$ such that $(\boldsymbol{p}_h,y_h,\boldsymbol{q}_h,z_h,u_h)$ satisfies the following optimality conditions:

$$(\alpha\boldsymbol{p}_h,\boldsymbol{v}_h)+(\nabla y_h,\boldsymbol{v}_h)+(\boldsymbol{\beta}y_h,\boldsymbol{v}_h)=0,\ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.18}$$
$$-(\boldsymbol{p}_h,\nabla w_h)+(cy_h,w_h)=(u_h,w_h),\ \forall\,w_h\in W_h, \tag{2.19}$$
$$(\alpha\boldsymbol{q}_h,\boldsymbol{v}_h)+(\nabla z_h,\boldsymbol{v}_h)=-(\boldsymbol{p}_h-\boldsymbol{p}_d,\boldsymbol{v}_h),\ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.20}$$
$$-(\boldsymbol{q}_h,\nabla w_h)-(\boldsymbol{\beta}\cdot\boldsymbol{q}_h,w_h)+(cz_h,w_h)=(y_h-y_d,w_h),\ \forall\,w_h\in W_h, \tag{2.21}$$
$$(\nu u_h+z_h,\tilde{u}_h-u_h)\ge 0,\ \forall\,\tilde{u}_h\in U_h^{ad}. \tag{2.22}$$

Similar to (2.11), control inequality (2.22) can be expressed as

$$u_h=\max\{0,-\pi_h z_h\}/\nu.$$

Subtracting (2.18)-(2.21) from (2.6)-(2.9), we easily obtain the following error equations

$$(\alpha(\boldsymbol{p}-\boldsymbol{p}_h),\boldsymbol{v}_h)+(\nabla(y-y_h),\boldsymbol{v}_h)+(\boldsymbol{\beta}(y-y_h),\boldsymbol{v}_h)=0,\ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.23}$$

$$-(\boldsymbol{p}-\boldsymbol{p}_h,\nabla w_h)+(c(y-y_h),w_h)=(u-u_h,w_h),\ \forall\,w_h\in W_h, \tag{2.24}$$

$$(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{v}_h)+(\nabla(z-z_h),\boldsymbol{v}_h)=-(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{v}_h),\ \forall\,\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{2.25}$$

$$-(\boldsymbol{q}-\boldsymbol{q}_h,\nabla w_h)-(\boldsymbol{\beta}\cdot(\boldsymbol{q}-\boldsymbol{q}_h),w_h)+(c(z-z_h),w_h)=(y-y_h,w_h),\ \forall\,w_h\in W_h. \tag{2.26}$$

From [14], we have the following two lemmas.

Lemma 2.2. *Let $(\boldsymbol{p},y,\boldsymbol{q},z,u)$ be the solution to (2.6)-(2.10) and $(\boldsymbol{p}_h,y_h,\boldsymbol{q}_h,z_h,u_h)$ be the solution to (2.18)-(2.22), respectively. Then,*

$$\|y-y_h\|+\|z-z_h\|\le Ch^2, \tag{2.27}$$

$$\|u-u_h\|+\|\nabla(y-y_h)\|+\|\boldsymbol{p}-\boldsymbol{p}_h\|+\|\nabla(z-z_h)\|+\|\boldsymbol{q}-\boldsymbol{q}_h\|\le Ch. \tag{2.28}$$

Lemma 2.3. *Let u be the solution to (2.6)-(2.10) and $\hat{u}_h=\max\{0,-z_h\}/\nu$. Then,*

$$\|u-\hat{u}_h\|\le Ch^2.$$

Now, we derive the following H^{-1}-error estimates.

Lemma 2.4. *Let $\boldsymbol{q}$ and $\boldsymbol{q}_h$ be the solutions to (2.6)-(2.10) and (2.18)-(2.22), respectively. Then*

$$\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1}\le Ch^2.$$

Proof. For $\boldsymbol{\psi}\in(H^1(\Omega))^2$, let $\varphi\in H^2(\Omega)\cap H_0^1(\Omega)$ be the solution of the Dirichlet problem

$$-\operatorname{div}(a\nabla\varphi)=\operatorname{div}\boldsymbol{\psi},\ x\in\Omega,$$
$$\varphi=0,\ x\in\partial\Omega.$$

Then,

$$\|\varphi\|_2\le C\|\operatorname{div}\boldsymbol{\psi}\|\le C\|\boldsymbol{\psi}\|_1. \tag{2.29}$$

Furthermore, $\boldsymbol{\psi}=-a\nabla\varphi+\boldsymbol{\theta}$, where $\operatorname{div}\boldsymbol{\theta}=0$ and $\|\boldsymbol{\theta}\|_1\le C\|\boldsymbol{\psi}\|_1$. Now,

$$(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\psi})=-(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),a\nabla\varphi)+(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}). \tag{2.30}$$

By (2.23)-(2.25), (2.27)-(2.28), (2.29), and Green's formula, we conclude that

$$\begin{aligned}
-(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),a\nabla\varphi) =&(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\Pi_h(a\nabla\varphi)-a\nabla\varphi)\\
&+(\nabla(z-z_h),\Pi_h(a\nabla\varphi)-a\nabla\varphi)+(\boldsymbol{p}-\boldsymbol{p}_h,\Pi_h(a\nabla\varphi)-a\nabla\varphi)\\
&+(\nabla(z-z_h),a\nabla\varphi)+(\boldsymbol{p}-\boldsymbol{p}_h,a\nabla\varphi)\\
=&(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\Pi_h(a\nabla\varphi)-a\nabla\varphi)\\
&+(\nabla(z-z_h),\Pi_h(a\nabla\varphi)-a\nabla\varphi)+(\boldsymbol{p}-\boldsymbol{p}_h,\Pi_h(a\nabla\varphi)-a\nabla\varphi)\\
&-(z-z_h,\mathrm{div}(a\nabla\varphi))+(\alpha(\boldsymbol{p}-\boldsymbol{p}_h),\nabla\varphi-\Pi_h(\nabla\varphi))\\
&-(\nabla(y-y_h),\Pi_h(\nabla\varphi))-(\boldsymbol{\beta}(y-y_h),\Pi_h(\nabla\varphi))\\
=&(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\Pi_h(a\nabla\varphi)-a\nabla\varphi)\\
&+(\nabla(z-z_h),\Pi_h(a\nabla\varphi)-a\nabla\varphi)+(\boldsymbol{p}-\boldsymbol{p}_h,\Pi_h(a\nabla\varphi)-a\nabla\varphi)\\
&-(z-z_h,\mathrm{div}(a\nabla\varphi))+(\alpha(\boldsymbol{p}-\boldsymbol{p}_h),\nabla\varphi-\Pi_h(\nabla\varphi))\\
&+(\nabla(y-y_h),\nabla\varphi-\Pi_h(\nabla\varphi))+(y-y_h,\mathrm{div}(\nabla\varphi))\\
&+(\boldsymbol{\beta}(y-y_h),\nabla\varphi-\Pi_h(\nabla\varphi))-(\boldsymbol{\beta}(y-y_h),\nabla\varphi)\\
\le&Ch(\|\boldsymbol{p}-\boldsymbol{p}_h\|+\|\boldsymbol{q}-\boldsymbol{q}_h\|+\|\nabla(y-y_h)\|+\|\nabla(z-z_h)\|)\|\varphi\|_2\\
&+C(\|y-y_h\|+\|z-z_h\|)\|\varphi\|_2\\
\le&Ch^2\|\boldsymbol{\psi}\|_1.
\end{aligned}\tag{2.31}$$

Using $\mathrm{div}\boldsymbol{\theta}=0$ and the same estimate as (2.31), we find that

$$\begin{aligned}
(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}) =&(\alpha(\boldsymbol{q}-\boldsymbol{q}_h),\boldsymbol{\theta}-\Pi_h\boldsymbol{\theta})+(\nabla(z-z_h),\boldsymbol{\theta}-\Pi_h\boldsymbol{\theta})\\
&+(\boldsymbol{p}-\boldsymbol{p}_h,\boldsymbol{\theta}-\Pi_h\boldsymbol{\theta})+(\alpha(\boldsymbol{p}-\boldsymbol{p}_h),\Pi_h(a\boldsymbol{\theta})-a\boldsymbol{\theta})\\
&+(\nabla(y-y_h),a\boldsymbol{\theta}-\Pi_h(a\boldsymbol{\theta}))+(y-y_h,\mathrm{div}(a\boldsymbol{\theta}))\\
&+(\boldsymbol{\beta}(y-y_h),a\boldsymbol{\theta}-\Pi_h(a\boldsymbol{\theta}))-(\boldsymbol{\beta}(y-y_h),a\boldsymbol{\theta})\\
\le&Ch(\|\boldsymbol{p}-\boldsymbol{p}_h\|+\|\boldsymbol{q}-\boldsymbol{q}_h\|+\|\nabla(y-y_h)\|+\|\nabla(z-z_h)\|)\|\boldsymbol{\theta}\|_1\\
&+C\|y-y_h\|\cdot\|\boldsymbol{\theta}\|_1\\
\le&Ch^2\|\boldsymbol{\psi}\|_1.
\end{aligned}\tag{2.32}$$

By use of (2.30)-(2.32), we have $\|\boldsymbol{q}-\boldsymbol{q}_h\|_{-1}\le Ch^2$. Thus we complete the proof. □

3. Two-Grid Algorithm and Convergence Analysis

In this section, we present our two-grid algorithm based on two triangulations $\mathscr{T}_H$ and $\mathscr{T}_h$, then analyze the convergence of the algorithm.

Two-grid algorithm:

1. Find $(\boldsymbol{p}_H, y_H, \boldsymbol{q}_H, z_H, u_H) \in (\boldsymbol{V}_H \times W_H)^2 \times U_H^{ad}$ such that $(\boldsymbol{p}_H, y_H, \boldsymbol{q}_H, z_H, u_H)$ satisfies the following optimality conditions:

$$(\alpha \boldsymbol{p}_H, \boldsymbol{v}_H) + (\nabla y_H, \boldsymbol{v}_H) + (\boldsymbol{\beta} y_H, \boldsymbol{v}_H) = 0, \ \forall\, \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{3.1}$$

$$-(\boldsymbol{p}_H, \nabla w_H) + (c y_H, w_H) = (u_H, w_H), \ \forall\, w_H \in W_H, \tag{3.2}$$

$$(\alpha \boldsymbol{q}_H, \boldsymbol{v}_H) + (\nabla z_H, \boldsymbol{v}_H) = -(\boldsymbol{p}_H - \boldsymbol{p}_d, \boldsymbol{v}_H), \ \forall\, \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{3.3}$$

$$-(\boldsymbol{q}_H, \nabla w_H) - (\boldsymbol{\beta} \cdot \boldsymbol{q}_H, w_H) + (c z_H, w_H) = (y_H - y_d, w_H), \ \forall\, w_H \in W_H, \tag{3.4}$$

$$(\nu u_H + z_H, \tilde{u}_H - u_H) \geq 0, \ \forall\, \tilde{u}_H \in U_H^{ad}. \tag{3.5}$$

2. Find $(\boldsymbol{p}_h^*, y_h^*, \boldsymbol{q}_h^*, z_h^*, u_h^*) \in (\boldsymbol{V}_h \times W_h)^2 \times U_h^{ad}$ such that

$$(\alpha \boldsymbol{p}_h^*, \boldsymbol{v}_h) + (\nabla y_h^*, \boldsymbol{v}_h) = -(\boldsymbol{\beta} y_H, \boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.6}$$

$$-(\boldsymbol{p}_h^*, \nabla w_h) + (c y_h^*, w_h) = (\hat{u}_H, w_h), \ \forall\, w_h \in W_h, \tag{3.7}$$

$$(\alpha \boldsymbol{q}_h^*, \boldsymbol{v}_h) + (\nabla z_h^*, \boldsymbol{v}_h) = -(\boldsymbol{p}_h^* - \boldsymbol{p}_d, \boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.8}$$

$$-(\boldsymbol{q}_h^*, \nabla w_h) + (c z_h^*, w_h) = (\boldsymbol{\beta} \cdot \boldsymbol{q}_H, w_h) + (y_h^* - y_d, w_h), \ \forall\, w_h \in W_h, \tag{3.9}$$

$$(\nu u_h^* + z_h^*, \tilde{u}_h - u_h^*) \geq 0, \ \forall\, \tilde{u}_h \in U_h^{ad}. \tag{3.10}$$

Theorem 3.1. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ be the solution to (2.6)-(2.10) and $(\boldsymbol{p}_h^*, y_h^*, \boldsymbol{q}_h^*, z_h^*, u_h^*)$ be the solution to (3.1)-(3.10) respectively. Then*

$$\|u - u_h^*\| + \|\nabla(y - y_h^*)\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|\nabla(z - z_h^*)\| + \|\boldsymbol{q} - \boldsymbol{q}_h^*\| \leq C(h + H^2).$$

Proof. For sake of simplicity, we now denote

$$\tau = y - y_h^*, \quad e = z - z_h^*.$$

From equations (2.6)-(2.9) and (3.6)-(3.9), we can easily obtain

$$(\alpha(\boldsymbol{p} - \boldsymbol{p}_h^*), \boldsymbol{v}_h) + (\nabla \tau, \boldsymbol{v}_h) = -(\boldsymbol{\beta}(y - y_H), \boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.11}$$

$$-(\boldsymbol{p} - \boldsymbol{p}_h^*, \nabla w_h) + (c\tau, w_h) = (u - \hat{u}_H, w_h), \ \forall\, w_h \in W_h, \tag{3.12}$$

$$(\alpha(\boldsymbol{q} - \boldsymbol{q}_h^*), \boldsymbol{v}_h) + (\nabla e, \boldsymbol{v}_h) = -(\boldsymbol{p} - \boldsymbol{p}_h^*, \boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.13}$$

$$-(\boldsymbol{q} - \boldsymbol{q}_h^*, \nabla w_h) + (ce, w_h) = (\boldsymbol{\beta} \cdot (\boldsymbol{q} - \boldsymbol{q}_H), w_h) + (\tau, w_h), \ \forall\, w_h \in W_h. \tag{3.14}$$

By $\nabla W_h \subset \boldsymbol{V}_h$ and (2.14), we rewrite (3.11)-(3.14) as

$$\begin{aligned}(\alpha(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*), \boldsymbol{v}_h) + (\nabla(P_h y - y_h^*), \boldsymbol{v}_h) = &-(\boldsymbol{\beta}(y - y_H), \boldsymbol{v}_h) \\ &-(\alpha(\boldsymbol{p} - \Pi_h \boldsymbol{p}), \boldsymbol{v}_h) - (\nabla(y - P_h y), \boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{3.15}$$

$$-(\Pi_h \boldsymbol{p} - \boldsymbol{p}_h^*, \nabla w_h) + (c(P_h y - y_h^*), w_h) = (u - \hat{u}_H, w_h) - (c(y - P_h y), w_h), \ \forall\, w_h \in W_h, \tag{3.16}$$

$$\begin{aligned}(\alpha(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*), \boldsymbol{v}_h) + (\nabla(P_h z - z_h^*), \boldsymbol{v}_h) = &-(\alpha(\boldsymbol{q} - \Pi_h \boldsymbol{q}), \boldsymbol{v}_h) \\ &-(\nabla(z - P_h z), \boldsymbol{v}_h) - (\boldsymbol{p} - \boldsymbol{p}_h^*, \boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h,\end{aligned} \tag{3.17}$$

$$\begin{aligned}-(\Pi_h \boldsymbol{q} - \boldsymbol{q}_h^*, \nabla w_h) + (c(P_h y - y_h^*), w_h) = &(\boldsymbol{\beta} \cdot (\boldsymbol{q} - \boldsymbol{q}_H), w_h) + (\tau, w_h) \\ &-(c(y - P_h y), w_h), \ \forall\, w_h \in W_h.\end{aligned} \tag{3.18}$$

Next, we divide the proof into the following three parts:

Part I. Let ϕ be the solution to (2.1) with $\psi = \tau$. It follows from (2.1), (2.13), (2.16), (3.11)-(3.12), Green's formula, and Cauchy inequality that

$$\begin{aligned}\|\tau\|^2 =&(\tau, -\mathrm{div}(a\nabla\phi)) + (\tau, c\phi)\\ =&(\nabla\tau, a\nabla\phi) + (c\tau, \phi)\\ =&(\nabla\tau, a\nabla\phi - \Pi_h(a\nabla\phi)) - (\alpha(\boldsymbol{p} - \boldsymbol{p}_h^*), \Pi_h(a\nabla\phi))\\ &+ (c\tau, \phi) - (\boldsymbol{\beta}(y - y_H), \Pi_h(a\nabla\phi))\\ =&(\nabla\tau + \alpha(\boldsymbol{p} - \boldsymbol{p}_h^*) + \boldsymbol{\beta}(y - y_H), a\nabla\phi - \Pi_h(a\nabla\phi))\\ &- (\boldsymbol{p} - \boldsymbol{p}_h^*, \nabla\phi) + (c\tau, \phi) - (\boldsymbol{\beta}(y - y_H), a\nabla\phi)\\ =&(\nabla\tau + \alpha(\boldsymbol{p} - \boldsymbol{p}_h^*) + \boldsymbol{\beta}(y - y_H), a\nabla\phi - \Pi_h(a\nabla\phi))\\ &+ (\boldsymbol{p} - \boldsymbol{p}_h^*, \nabla(P_h\phi - \phi)) + (c\tau, \phi - P_h\phi) - (\boldsymbol{\beta}(y - y_H), a\nabla\phi)\\ &+ (u - \hat{u}_H, P_h\phi - \phi) + (u - \hat{u}_H, \phi)\\ \leq& Ch(\|\nabla\tau\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\|)\|\phi\|_2 + C(\|u - \hat{u}_H\| + \|y - y_H\|)\|\phi\|_2. \end{aligned} \tag{3.19}$$

Choosing $\boldsymbol{v}_h = \Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*$ in (3.15) and $w_h = P_hy - y_h^*$ in (3.16), respectively, and adding the two equations, one has

$$\begin{aligned}\|\alpha^{\frac{1}{2}}(\Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*)\|^2 =& -(\alpha(\boldsymbol{p} - \Pi_h\boldsymbol{p}) + \nabla(y - P_hy) + \boldsymbol{\beta}(y - y_H), \Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*)\\ &- (c\tau, P_hy - y_h^*) + (u - \hat{u}_H, P_hy - y_h^*). \end{aligned} \tag{3.20}$$

Using Cauchy inequality, (3.20), (2.13), (2.16), and the assumption on a, we find that

$$\|\Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*\| \leq Ch(\|y\|_2 + \|\boldsymbol{p}\|_1) + C\|\tau\| + C(\|u - \hat{u}_H\| + \|y - y_H\|). \tag{3.21}$$

Letting $\boldsymbol{v}_h = \nabla(P_hy - y_h^*)$ in (3.15), we easily obtain

$$\|\nabla(P_hy - y_h^*)\| \leq Ch(\|y\|_2 + \|\boldsymbol{p}\|_1) + C(\|\Pi_h\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|y - y_H\|). \tag{3.22}$$

Substituting (3.21)-(3.22) into (3.19) and using (2.2), (2.13), and (2.16), for sufficiently small h, we have

$$\|\tau\| \leq Ch(\|y\|_2 + \|\boldsymbol{p}\|_1) + C(\|u - \hat{u}_H\| + \|y - y_H\|). \tag{3.23}$$

Thus it follows from (3.21)-(3.23), (2.13), (2.16), and the triangle inequality that

$$\|\nabla(y - y_h^*)\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*)\| \leq Ch(\|y\|_2 + \|\boldsymbol{p}\|_1) + C(\|u - \hat{u}_H\| + \|y - y_H\|). \tag{3.24}$$

Part II. Let ϕ be the solution to (2.1) with $\psi = e$. Similar to (3.19), we can conclude that

$$\begin{aligned}\|e\|^2 =&(e, -\mathrm{div}(a\nabla\phi)) + (e, c\phi)\\=&(\nabla e, a\nabla\phi) + (ce, \phi)\\=&(\nabla e, a\nabla\phi - \Pi_h(a\nabla\phi)) - (\alpha(\boldsymbol{q} - \boldsymbol{q}_h^*), \Pi_h(a\nabla\phi))\\&+ (ce, \phi) - (\boldsymbol{p} - \boldsymbol{p}_h^*, \Pi_h(a\nabla\phi))\\=&(\nabla e + \alpha(\boldsymbol{q} - \boldsymbol{q}_h^*) + \boldsymbol{p} - \boldsymbol{p}_h^*, a\nabla\phi - \Pi_h(a\nabla\phi))\\&- (\boldsymbol{q} - \boldsymbol{q}_h^*, \nabla\phi) + (ce, \phi) - (\boldsymbol{p} - \boldsymbol{p}_h^*, a\nabla\phi)\\=&(\nabla e + \alpha(\boldsymbol{q} - \boldsymbol{q}_h^*) + \boldsymbol{p} - \boldsymbol{p}_h^*, a\nabla\phi - \Pi_h(a\nabla\phi))\\&+ (\boldsymbol{q} - \boldsymbol{q}_h^*, \nabla(P_h\phi - \phi)) + (ce, \phi - P_h\phi) - (\boldsymbol{p} - \boldsymbol{p}_h^*, a\nabla\phi)\\&+ (\tau + \boldsymbol{\beta}\cdot(\boldsymbol{q} - \boldsymbol{q}_H), P_h\phi - \phi) + (\tau + \boldsymbol{\beta}\cdot(\boldsymbol{q} - \boldsymbol{q}_H), \phi)\\\le& Ch(\|\nabla e\| + \|e\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|\boldsymbol{q} - \boldsymbol{q}_h^*\| + \|\boldsymbol{q} - \boldsymbol{q}_H\|)\|\phi\|_2\\&+ C(\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1} + \|\tau\|)\|\phi\|_1,\end{aligned} \tag{3.25}$$

where we also used the Cauchy inequality and the estimate

$$(\boldsymbol{\beta}\cdot(\boldsymbol{q} - \boldsymbol{q}_H), \phi) = (\boldsymbol{q} - \boldsymbol{q}_H, \boldsymbol{\beta}\phi) \le C\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1}\|\phi\|_1. \tag{3.26}$$

For sufficiently small h, by (3.25), (2.2), and Poincare's inequality, we derive

$$\|e\| \le Ch(\|\nabla e\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|\boldsymbol{q} - \boldsymbol{q}_h^*\| + \|\boldsymbol{q} - \boldsymbol{q}_H\|) + C\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1} + C\|\tau\|. \tag{3.27}$$

Choosing $\boldsymbol{v}_h = \Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*$ in (3.17) and $w_h = P_h z - z_h^*$ in (3.18), respectively and adding the two equations, one has

$$\begin{aligned}\|\alpha^{\frac{1}{2}}(\Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*)\|^2 =& -(\alpha(\boldsymbol{q} - \Pi_h\boldsymbol{q}) + \nabla(z - P_h z) + \boldsymbol{p} - \boldsymbol{p}_h^*, \Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*)\\&+ (\tau - ce, P_h z - z_h^*) + (\boldsymbol{\beta}\cdot(\boldsymbol{q} - \boldsymbol{q}_H), P_h z - z_h^*),\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}(\boldsymbol{\beta}\cdot(\boldsymbol{q} - \boldsymbol{q}_H), P_h z - z_h^*) =& (\boldsymbol{q} - \boldsymbol{q}_H, \boldsymbol{\beta}(P_h z - z_h^*))\\\le& C\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1}\|P_h z - z_h^*\|_1\\\le& C\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1}\|\nabla(P_h z - z_h^*)\|.\end{aligned} \tag{3.29}$$

The Poincare's inequality is used in the last step of (3.29). By virtue of (3.28)-(3.29), (2.16), Cauchy inequality, Poincare's inequality, and the assumption on a, we have

$$\begin{aligned}\|\Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*\| \le& C(\|\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|e\|) + C(\varepsilon)(\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1} + \|\tau\|)\\&+ Ch(\|z\|_2 + \|\boldsymbol{q}\|_1) + \varepsilon\|\nabla(P_h z - z_h^*)\|,\end{aligned} \tag{3.30}$$

where ε is an arbitrary small positive constant. Choosing $\boldsymbol{v}_h = \nabla(P_h z - z_h^*)$ in (3.17), we find that

$$\|\nabla(P_h z - z_h^*)\| \le Ch(\|z\|_2 + \|\boldsymbol{q}\|_1) + C(\|\Pi_h\boldsymbol{q} - \boldsymbol{q}_h^*\| + \|\boldsymbol{p} - \boldsymbol{p}_h^*\|). \tag{3.31}$$

For sufficiently small ε, by using Poincare's inequality, (3.30)-(3.31), (3.27), (2.13), (2.16), and the triangle inequality, we have

$$\begin{aligned}\|\nabla(z - z_h^*)\| + \|\boldsymbol{q} - \boldsymbol{q}_h^*\| \le& Ch(\|z\|_2 + \|\boldsymbol{q}\|_1) + Ch\|\boldsymbol{q} - \boldsymbol{q}_H\|\\&+ C(\|\boldsymbol{q} - \boldsymbol{q}_H\|_{-1} + \|\boldsymbol{p} - \boldsymbol{p}_h^*\| + \|\tau\|).\end{aligned} \tag{3.32}$$

Part III. From (3.10), we know that

$$u_h^* = \max\{0, -\pi_h z_h^*\}/\nu. \tag{3.33}$$

Thus it follows from (2.11), (2.17) and (3.33) that

$$\begin{aligned}\|u-u_h^*\| =&\|\max\{0,-z\}/\nu - \max\{0,-\pi_h z_h^*\}/\nu\| \\ \leq& C\|z-\pi_h z_h^*\| \\ \leq& C\|z-\pi_h z\| + C\|\pi_h(z-z_h^*)\| \\ \leq& Ch\|z\|_1 + C\|z-z_h^*\|. \end{aligned} \tag{3.34}$$

Combining (3.24), (3.32), (3.34), and Lemmas 2.2-2.4 with Poincare's inequality, we complete the proof of theorem. □

Acknowledgements

The author was supported by the National Natural Science Foundation of China (11601014) and the Science and Technology Research Project of Jilin Provincial Department of Education (JJKH20210028KJ).

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