J. Nonlinear Funct. Anal. 2022 (2022) 42 https://doi.org/10.23952/jnfa.2022.42

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

CHANDRABHAN TYPE MAPS WITH WEAKLY SEQUENTIALLY CLOSED GRAPHS

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Abstract. In this paper, we present general fixed point results for the multimaps with weakly sequentially closed graphs, satisfying some weak compactness type condition on countable sets. We also establish fixed point theorems for the weakly condensing multimaps with weakly sequentially closed graphs in Banach spaces.

Keywords. Fixed points; Chandrabhan type maps; Mönch type maps; Weakly condensing maps; Weakly sequentially closed graphs.

1. Introduction

Himmelberg fixed-point theorem [5] received much attention and was employed to solve a variety of problems, such as equilibrium, differential inclusion, variational inequalities, optimization, and integral equations. In 1984, Arino, Gautier and Penot [1] obtained a fixed point theorem for weakly sequentially upper semicontinuous multimaps that could escape the limits of separability or reflexivity on the spaces, and they applied it to differential equations. In 2000, O'Regan and Precup [11] extended Arino et al.'s result [1] to multimaps with weakly sequentially closed graphs. In [2, 3], Amar et al. obtained some fixed point theorems for weakly condensing multimaps and weakly countably condensing multimaps in Banach spaces, respectively.

Recently, O'Regan [8, 9, 10] extended the concept of Mönch type single-valued maps, introduced by Mönch [7], to multimaps and obtained fixed point theorems for these type multimaps with weakly sequentially closed graphs. In addition, Dhage [4] generalized Mönch type multimaps to Chandrabhan multimaps. In 2021, Kim [6] extended the result of [11] to Chandrabhan type multimaps.

From now on, we always refer to multimaps as maps. We, in this paper, generalize the Mönch type results in [9, 10] to Chandrabhan type maps with weakly sequentially closed graphs and establish a new fixed point theorem for the Mönch type maps with weakly sequentially closed graphs in a metrizable locally convex linear topological space. We also establish fixed point theorems for Chandrabhan type maps with weakly sequentially closed graphs in Banach spaces.

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Received June 7, 2022; Accepted December 2, 2022.

These improve the results presented in [10]. Finally, we obtain a fixed point theorem for weakly condensing maps in Banach spaces, which is different from those presented in Amar et al. [2, 3].

The following fixed point theorem in [8] is a crucial tool in this paper.

Proposition 1.1. *Let Q be a nonempty, convex, and weakly compact subset of a metrizable locally convex linear topological space E. Suppose that $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph; where $\mathscr{K}(Q)$ denotes the family of nonempty convex, weakly compact subsets of Q. Then F has a fixed point.*

2. Fixed Point Theorems on a Metrizable Locally Convex Linear Topological Space

The following is a fixed point theorem for Chandrabhan type maps with weakly sequentially closed graphs.

Theorem 2.1. *Assume that E is a metrizable locally convex linear topological space. Assume that Q is a convex, closed, and nonempty subset of E, and assume that B is a relatively weakly compact subset of Q. Assume that $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph and F takes relatively weakly compact sets into relatively weakly compact sets. If*

1. *for a subset A of Q and a countable subset C of A, $A = \overline{\text{co}}(B \cup F(A))$ with $\overline{C^w} = A$ implies A is weakly compact;*
2. *for any relatively weakly compact subset A of E, there exists a countable set $S \subset A$ with $\overline{S^w} = \overline{A^w}$; and*
3. *if A is a weakly compact subset of E, then $\overline{\text{co}}(A)$ is weakly compact,*

then F has a fixed point.

Proof. Put $K_0 = \overline{\text{co}}(B)$, $K_{n+1} = \overline{\text{co}}(B \cup F(K_n))$ for $n = 0, 1, 2, \cdots$, and $K = \bigcup_{n=0}^{\infty} K_n$. By induction, one has $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \cdots$. One can demonstrate that $K = \overline{\text{co}}(B \cup F(K))$. For each n,

$$\overline{\text{co}}(B \cup F(K_n)) \subseteq \overline{\text{co}}(B \cup F(K)),$$

so

$$K = \bigcup_{n=0}^{\infty} \overline{\text{co}}(B \cup F(K_n)) \subseteq \overline{\text{co}}(B \cup F(K)).$$

On the other hand, K is convex since K_n is convex for $n = 0, 1, 2, \cdots$. As K contains B and $\bigcup_{n=0}^{\infty} F(K_n) = F(K)$, $\overline{\text{co}}(B \cup F(K)) \subseteq K$.

Furthermore, one can prove that K is weakly compact. Since K_n and K are closed convex, one has $K_n = \overline{K_n^w}$ and $K = \overline{K^w}$. Condition (3) implies that K_0 is weakly compact. Since F takes relatively weakly compact sets into relatively weakly compact sets, $F(K_n)$ is relatively weakly compact. This together with conditon (3) guarantees that K_n is weakly compact. By condition (2), there exists a countable subset C_n of K_n with $\overline{C_n^w} = \overline{K_n^w} = K_n$. Put $C = \bigcup_{n=0}^{\infty} C_n$. Since

$$K = \bigcup_{n=0}^{\infty} K_n = \bigcup_{n=0}^{\infty} \overline{C_n^w} = \overline{\{\bigcup_{n=0}^{\infty} C_n\}^w} = \overline{C^w},$$

one has that K is weakly compact by condition (1).

Define a map $G : K \multimap K$ by $G(x) = F(x) \cap K$. Since $F(K) \subset K$, $G(x) \neq \emptyset$ for all $x \in K$. Note that $G : K \multimap \mathscr{K}(K)$ and G has a weakly sequentially closed graph. Now Proposition 1.1 guarantees the existence of a point $x \in G(x) \subset F(x)$. □

From Theorem 2.1, we have the new Mönch type result.

Corollary 2.2. *Assume that E is a metrizable locally convex linear topological space. Assume that Q is a convex, closed, and convex subset of E and $x_0 \in Q$. Assume that $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph and F takes relatively weakly compact sets into relatively weakly compact sets. If*

(1) *for a subset A of Q and a countable subset C of A, $A = \overline{\mathrm{co}}(\{x_0\} \cup F(A))$ with $\overline{C^w} = A$ implies A is weakly compact;*
(2) *for any relatively weakly compact subset A of E, there exists a countable set $S \subset A$ with $\overline{S^w} = \overline{A^w}$; and*
(3) *if A is a weakly compact subset of E, then $\overline{\mathrm{co}}(A)$ is weakly compact,*

then F has a fixed point.

The same result can be obtained by modifying condition (1) of Theorem 2.1.

Theorem 2.3. *Assume that E is a metrizable locally convex linear topological space. Assume that Q is a convex, closed, and nonempty subset of E, and assume that B is a relatively weakly compact subset of Q. Assume that $F : Q \multimap \mathscr{K}(Q)$ is a weakly sequentially closed graph and F takes relatively weakly compact sets into relatively weakly compact sets. If*

(1) *for a subset A of Q and a countable subset C of A, $A = \mathrm{co}(B \cup F(A))$ with $\overline{C^w} = \overline{A^w}$ implies $\overline{A^w}$ is weakly compact;*
(2) *for any relatively weakly compact subset A of E, there exists a countable set $S \subset A$ with $\overline{S^w} = \overline{A^w}$; and*
(3) *if A is a weakly compact subset of E, then $\overline{\mathrm{co}}(A)$ is weakly compact,*

then F has a fixed point.

Proof. Put $K_0 = \mathrm{co}(B)$, $K_{n+1} = \mathrm{co}(B \cup F(K_n))$ for $n = 0, 1, 2, \cdots$, and $K = \bigcup_{n=0}^{\infty} K_n$. The same argument in the proof of Theorem 2.1 yields $K = \mathrm{co}(B \cup F(K))$ and $\overline{K} = \overline{K^w}$ is weakly compact. Define a map $G : \overline{K^w} \multimap \overline{K^w}$ by $G(x) = F(x) \cap \overline{K^w}$. Now we prove $G(x) \neq \emptyset$ for all $x \in \overline{K^w}$. Since $\overline{K^w}$ is weakly closed, there exists a weak convergence sequence $\{x_n\}$ in K such that $x_n \rightharpoonup x$. Since $F(K) \subset K$, one has $G(x_n) \neq \emptyset$. One takes $y_n \in G(x_n) = F(x_n)$ for each n. Since $\overline{K^w}$ is weakly compact, $y_n \rightharpoonup y$ for some $y \in \overline{K^w}$ without loss of generality. Since F has a weakly sequentially closed graph, $y \in F(x)$. Thus $y \in F(x) \cap \overline{K^w} = G(x)$. Note that $G(x) \in \mathscr{K}(\overline{K^w})$ for all $x \in \overline{K^w}$ and G has a weakly sequentially closed graph. Proposition 1.1 guarantees a point $x \in G(x) \subset F(x)$. This completes the proof. □

Remark 2.4. 1. Theorem 2.3 generalizes the Theorem 2.2 in [9].

2. O'Regan presented the following properties in [9]:

(a) If K is a weakly compact subset of E, and K with the relative weak topology is metrizable, then (2) holds. So if E is a Banach space whose dual E^* is separable, then E satisfies (2).

(b) If a Krein-Šmulian type theorem holds, then condition (3) is satisfied. For example, E could be a Banach space or a quasicomplete locally convex linear topological space.

3. Fixed Point Theorems on a Banach Space

Theorems in this section do not need to assume the condition of Theorem 2.1 and Theorem 2.3 that F takes relatively weakly compact sets into relatively weakly compact sets.

Theorem 3.1. *Assume that E is a Banach space. Assume that Q is a convex, closed, and nonempty subset of E. In addition, assume that B is a relatively weakly compact subset of Q and $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph. If*

(1) *for a subset A of Q and a countable subset C of A, $A = \overline{\text{co}}(B \cup F(A))$ and $C \subset \overline{\text{co}}(B \cup F(C))$ implies $\overline{C^w}$ is weakly compact,*

then F has a fixed point.

Proof. Let $\mathscr{H}$ be the family of all subsets D of Q with $\overline{\text{co}}(B \cup F(D)) \subset D$. Note $\mathscr{H} \neq \emptyset$ since $Q \in \mathscr{H}$. Let $K = \cap_{D \in \mathscr{H}} D$ and $K_1 = \overline{\text{co}}(B \cup F(K))$.

We now prove $K = K_1$. For any $D \in \mathscr{H}$, we have that

$$K_1 = \overline{\text{co}}(B \cup F(K)) \subset \overline{\text{co}}(B \cup F(D)) \subset D,$$

so $K_1 \subset K$ and $\overline{\text{co}}(B \cup F(K_1)) \subset \overline{\text{co}}(B \cup F(K)) = K_1$. As a result $K_1 \in \mathscr{H}$, so $K \subset K_1$ and

$$K = \overline{\text{co}}(B \cup F(K)). \tag{3.1}$$

If $K (= \overline{K} = \overline{K^w})$ is not weakly compact, then one sees from the Eberlein-Šmulian theorem that there exists a sequence $y_1, y_2, \cdots$ in K without a weakly convergent subsequence.

Let $C_1 = \{y_1, y_2, \cdots\}$. Next, we find a countable set $C \subset K$ such that $C \subset \overline{\text{co}}(B \cup F(C))$. First, we construct a countable set $C_2 \subset K$ with $C_1 \subset C_2$ and $C_1 \subset \overline{\text{co}}(B \cup F(C_2))$. Since $K = \overline{\text{co}}(B \cup F(K))$ and $C_1 \subset K$, each y_n is the limit of a sequence of finite convex combination of points from $B \cup F(K)$. Hence, there exists a countable set $Q_0 \subset B \cup F(K)$ with $y_n \in \overline{\text{co}}(Q_0)$ for each n, i.e, $C_1 \subset \overline{\text{co}}(Q_0)$. In particular, there exists a countable set $A_2 \subset K$ with $C_1 \subset \overline{\text{co}}(B \cup F(A_2))$. Let $C_2 = C_1 \cup A_2$. Then $C_1 \subset \overline{\text{co}}(B \cup F(C_2))$. Proceed as above, we obtain countable sets $C_3, C_4, \cdots$ with $C_n \subset K$, $C_n \subset C_{n+1}$, and $C_n \subset \overline{\text{co}}(B \cup F(C_{n+1}))$ for $n = 1, 2, \cdots$. Let $C = \cup_{n=1}^{\infty} C_n$. For each $x \in C$, we have $x \in C_n$ for some $n \in \{1, 2, \cdots\}$, which implies

$$x \in \overline{\text{co}}(B \cup F(C_{n+1})) \subset \overline{\text{co}}(B \cup F(C)).$$

Thus $C \subset \overline{\text{co}}(B \cup F(C))$. Now, (1) guarantees that $\overline{C^w}$ is weakly compact. This is a contradiction since $\overline{C^w}$ contains the sequence $\{y_n\}$ (note $C_1 \subset C$ and $\overline{C^w} \subset \overline{K^w} = K$), which has no weakly convergent subsequence. From (3.1), one has that $F : K \multimap \mathscr{K}(K)$ has a weakly sequentially closed graph. So, F has a fixed point by Proposition 1.1. $\square$

Theorem 3.1 generalizes the Theorem 2.1 in [10].

Theorem 3.2. *Assume that E is a Banach space, and assume that Q is convex, closed, and nonempty subset of E. Assume that B is a relatively weakly compact subset of Q, and assume that $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph. If*

(1) *for a subset A of Q and a countable subset C of A, $A = \text{co}(B \cup F(A))$ and $C \subset \overline{\text{co}}(B \cup F(C))$ implies $\overline{C^w}$ is weakly compact,*

then F has a fixed point.

Proof. Let $\mathscr{H}$ be the family of all subsets D of Q with $\text{co}(B\cup F(D))\subset D$, and let $K=\cap_{D\in\mathscr{H}}D$. A slight adjustment of the argument in the proof of Theorem 3.1 yields that $K=\text{co}(B\cup F(K))$ and $\overline{K}=\overline{K^w}$ is weakly compact. Define a map $G:\overline{K^w}\multimap\overline{K^w}$ by $G(x)=F(x)\cap\overline{K^w}$. Then by the same argument of Theorem 2.3, we can conclude that $G(x)\neq\emptyset$ for all $x\in\overline{K^w}$ and there exists a point $x\in G(x)\subset F(x)$. □

Theorem 3.2 generalizes the Theorem 2.7 in [10].

Theorem 3.3. *Assume that E is a Banach space and Q a convex, closed, and nonempty subset of E. Assume that B is a relatively weakly compact subset of Q and $F:Q\multimap\mathscr{K}(Q)$ has a weakly sequentially closed graph. If, for a subset A of Q with $A=\overline{\text{co}}(B\cup F(A))$, the followings hold:*

(1) *for any countable subset N of A, there exists a countable set $P\subset A$ with $\overline{\text{co}}(B\cup F(N))\subset\overline{P^w}$; and*
(2) *for a countable subset C of A, $\overline{C^w}=\overline{\text{co}}(B\cup F(C))$ implies $\overline{C^w}$ is weakly compact,*

then F has a fixed point.

Proof. Let K be as in the proof of Theorem 3.1. Then $K=\overline{\text{co}}(B\cup F(K))$. We now claim

$$K(=\overline{K}=\overline{K^w})\text{ is weakly compact.}$$

Suppose that the claim is false. As in the proof of Theorem 3.1, there exist a sequence $y_1,y_2,\cdots$ in K without a weakly convergent subsequence and a countable set $A_2\subset K$ with

$$C_1=\{y_1,y_2,\cdots\}\subset\overline{\text{co}}(B\cup F(A_2)).$$

From (1), there exists a countable set $P_2\subset K$ with $\overline{\text{co}}(B\cup F(C_1))\subset\overline{P_2^w}$. Let $C_2=C_1\cup A_2\cup P_2$. Then

$$C_1\subset\overline{\text{co}}(B\cup F(C_2))\text{ and }\overline{\text{co}}(B\cup F(C_1))\subset\overline{C_2^w}.$$

By induction, we obtain countable sets $C_3,C_4,\cdots$ with $C_n\subset K, C_n\subset C_{n+1}$, $C_n\subset\overline{\text{co}}(B\cup F(C_{n+1}))$, and $\overline{\text{co}}(B\cup F(C_n))\subset\overline{C_{n+1}^w}$ for $n=1,2,\cdots$. Let $C=\cup_{n=1}^{\infty}C_n$. Then $C\subset\overline{\text{co}}(B\cup F(C))$.

Now we prove $\overline{\text{co}}(B\cup F(C))=\overline{C^w}$. Note that $\overline{\text{co}}(B\cup F(C))$ is weakly closed and $\overline{C^w}\subset\overline{\text{co}}(B\cup F(C))$. If $x\in\text{co}(B\cup F(C))$, then x is a finite convex combination of some points in $B\cup F(C)$. Since $C=\cup_{n=1}^{\infty}C_n$, then there exists an n such that $x\in\text{co}(B\cup F(C_n))\subset\overline{C_{n+1}^w}$. Therefore,

$$\text{co}(B\cup F(C))\subset\cup_{n=1}^{\infty}\overline{C_{n+1}^w}=\overline{C^w}.$$

Hence,

$$\overline{\text{co}}(B\cup F(C))=\overline{\text{co}(B\cup F(C))}^w\subset\overline{C^w},$$

and $\overline{\text{co}}(B\cup F(C))=\overline{C^w}$. Now (2) guarantees that $\overline{C^w}$ is weakly compact. This is a contradiction since $\overline{C^w}$ contains the sequence $\{y_n\}$, which has no weakly convergent subsequence. Thus K is weakly compact. Since $F:K\multimap\mathscr{K}(K)$ has a weakly sequentially closed graph, F has a fixed point by Proposition 1.1. □

Remark 3.4. 1. Theorem 3.3 generalizes the Theorem 2.5 in [10].

2. With some minor adjustment to the Remark 2.6 of [10], it can be demonstrated that the condition (1) in Theorem 3.3 is satisfied by a single valued map F and a countable set B. Since $N\subset A$, $\overline{\text{co}}(B\cup F(N))\subset\overline{\text{co}}(B\cup F(A))=A$. As $B\cup F(N)$ is countable, one has that $\overline{\text{co}}(B\cup F(N))$ is separable and therefore weakly separable. Thus there exists a countable set $P\subset A$ such that $P\subset\overline{\text{co}}(B\cup F(N))\subset\overline{P^w}$.

Theorem 3.5. *Assume that E is a Banach space, and assume that Q is convex, closed, and nonempty subset of E. Assume that B is a relatively weakly compact subset of Q, and assume that $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph. If, for a subset A of Q with $A = \mathrm{co}(B \cup F(A))$, the followings hold:*

(1) *for any countable subset N of A, there exists a countable set $P \subset A$ with $\overline{\mathrm{co}}(B \cup F(N)) \subset \overline{P^w}$; and*
(2) *for a countable subset C of A, $\overline{C^w} = \overline{\mathrm{co}}(B \cup F(C))$ implies $\overline{C^w}$ is weakly compact,*

then F has a fixed point.

Proof. Let $\mathscr{H}$ and K be as in Theorem 3.2. Then a slight adjustment of the argument in the proof of Theorem 3.3 yields that $\overline{K^w}$ is weakly compact, and the map $G : \overline{K^w} \multimap \overline{K^w}$ defined by $G(x) = F(x) \cap \overline{K^w}$ has a fixed point $x \in G(x) \subset F(x)$. □

We remark here that Theorem 3.5 generalizes the Theorem 2.9 in [10].

4. Weakly Condensing Maps

Definition 4.1. Let E be a Banach space and $\mathscr{B}(E)$ denote the family of nonempty bounded subsets of E. A function $\omega : \mathscr{B}(E) \to \mathbb{R}_+$ is said to be a measure of weak non-compactness (MWNC) on E if it satisfies the following properties for any $\Omega_1, \Omega_2 \in \mathscr{B}(E)$:

(1) $\omega(\overline{\mathrm{co}}(\Omega_1)) = \omega(\Omega_1)$;
(2) $\Omega_1 \subset \Omega_2$ implies $\omega(\Omega_1) \le \omega(\Omega_2)$;
(3) $\omega(\Omega_1 \cup \{a\}) = \omega(\Omega_1)$ for all $a \in E$;
(4) $\omega(\Omega_1) = 0$ if and only if Ω_1 is weakly relatively compact in E.

Definition 4.2. For $Q \subset E$, a map $F : Q \multimap E$ is called ω-condensing if $\omega(F(A)) < \omega(A)$ for any bounded subsets A of Q with $\omega(A) > 0$, and countably ω-condensing if $\omega(F(A)) < \omega(A)$ for any countably bounded subsets A of Q with $\omega(A) > 0$.

For details, we refer to [2, 3]. We can also obtain a fixed point theorem for ω-condensing maps from the following theorem.

Theorem 4.3. *Assume that E is a Banach space, and assume that Q is a convex, closed and nonempty countably bounded subset of E. Assume that $a \in Q$ and $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph. Suppose that, for every countably bounded subset A of Q with $A = \overline{\mathrm{co}}(\{a\} \cup F(A))$, A is weakly compact. Then F has a fixed point.*

Proof. Let $\mathscr{H}$ be the family of all countably bounded subsets D of Q with $\overline{\mathrm{co}}(\{a\} \cup F(D)) \subset D$. Note $\mathscr{H} \neq \emptyset$ due to $Q \in \mathscr{H}$. Let $K = \cap_{D \in \mathscr{H}} D$. By the similar argument of the proof of Theorem 3.1, one has $K = \overline{\mathrm{co}}(\{a\} \cup F(K))$. This implies that K is weakly compact, and $F : K \multimap \mathscr{K}(K)$ has a weakly sequentially closed graph. Thus F has a fixed point by Proposition 1.1. □

Theorem 4.4. *Assume that E is a Banach space and Q is a convex, closed, and nonempty countably bounded subset of E. If a countably ω-condensing map $F : Q \multimap \mathscr{K}(Q)$ has a weakly sequentially closed graph, then F has a fixed point.*

Proof. Suppose that there exists a countably bounded subset A of Q with $A = \overline{\mathrm{co}}(\{a\} \cup F(A))$, but A is not weakly compact, where $a \in Q$. Then $\omega(A) > 0$ and $\omega(A) = \omega(\overline{\mathrm{co}}(\{a\} \cup F(A))) = \omega(F(A)) < \omega(A)$ since F is countably ω-condensing. This is a contradiction. In view of Theorem 4.3, one obtains that F has a fixed point. □

Remark 4.5. 1. Theorem 3.1 [3] has slightly different conditions from Theorem 4.4. Indeed, F is not weakly compact convex valued, but weakly sequentially continuous.

2. Theorem 2.3 [2] is for a ω-condensing map with convex values.

3. The two theorems mentioned above only assume that the multimap F has a bounded range, but the proofs of them use that Q is in $\mathscr{H} = \{D \subset Q : a \in D, D$ is a bounded convex subset of $Q\}$.

Funding

This paper was supported by the Sehan University Research Fund in 2022.

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