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VISCOSITY APPROXIMATION OF A RELAXED ALTERNATING CQ ALGORITHM FOR THE SPLIT EQUALITY PROBLEM

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Abstract. In this paper, we consider a new relaxed alternating CQ algorithm for solving the split equality problem based on the viscosity approximation, and prove that the generated sequence converges strongly to a solution of this problem. The result obtained in this paper extends and improves the corresponding results in the literature.

Keywords. Alternating CQ algorithm; Split equality problem; Split feasibility problem; Viscosity approximation method.

1. Introduction

In 2014, Moudafi [4] introduced the following split equality problem (SEP), which consists of finding two points $x \in C, y \in Q$ satisfying

$$Ax = By, \tag{1.1}$$

where H_1, H_2, and H_3 are real Hilbert spaces, C and Q are convex, closed, and nonempty subsets of spaces H_1 and H_2, respectively, and the operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are both bounded and linear.

In recent years, the SEP has received much attention due to its extensive applications in many applied disciplines. For the recent the algorithmic development and applications for solving the SEP, we refer to [5, 8, 9, 10, 11, 13]. Among these works, Moudafi [4] investigated the following alternating CQ-algorithm (ACQA) for solving the SEP (1.1):

$$\begin{cases} x_{n+1} = P_C(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = P_Q(y_n + \gamma_n B^*(Ax_{n+1} - By_n)), \end{cases} \tag{1.2}$$

with $\gamma_n \in (\varepsilon, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}) - \varepsilon)$, where ε is a sufficiently small positive number, $\{\gamma_n\}$ is a positive increasing sequence. It was proved that the sequence $\{(x_n, y_n)\}$ generated by (1.2) converges to a solution of (1.1) weakly.

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However, implementing algorithm (1.2) requires computing the orthogonal projections P_C and P_Q, which is not easy to implement usually. Therefore, Moudafi (1.2) considered the level sets:

$$C = \{x \in H_1 | c(x) \leq 0\},$$

and

$$Q = \{y \in H_2 | q(x) \leq 0\},$$

where $c : H_1 \to R$ and $q : H_2 \to R$ are two convex subdifferentiable functions in H_1 and H_2, respectively.

In the case of level sets, the associated projections need to be in closed form and they are difficult to compute. To solve this problem, Moudafi [5] suggested the following relaxed alternating CQ-algorithm (RACQA):

$$\begin{cases} x_{n+1} = P_{C_n}(x_n - \gamma A^*(Ax_n - By_n)), \\ y_{n+1} = P_{Q_n}(y_n + \gamma B^*(Ax_{n+1} - By_n)), \end{cases} \tag{1.3}$$

where $\gamma \in (0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$, and $\{C_n\}$ and $\{Q_n\}$ are two sequences of closed convex sets defined as follows

$$C_n = \{x \in H_1 | c(x_n) + \langle \zeta_n, x - x_n \rangle \leq 0\}, \zeta_n \in \partial c(x_n), \tag{1.4}$$

$$Q_n = \{y \in H_2 | q(y_n) + \langle \eta_n, y - y_n \rangle \leq 0\}, \eta_n \in \partial q(y_n), \tag{1.5}$$

where the subdifferential operators ∂c and ∂q of c and q are bounded on bounded sets. Since $\{C_n\}$ and $\{Q_n\}$ are half-spaces, one sees that P_{C_n} and P_{Q_n} are easy to calculate. Thus RACQA (1.3) is implementable. Under appropriate conditions, Moudafi proved that the sequence $\{(x_n, y_n)\}$ generated by the RACQA (1.3) converges to a solution of problem (1.1) weakly.

Recently, Gao and Liu [7] studied the following relaxed alternating CQ algorithm for solving the SEP (1.1):

$$\begin{cases} u_n = x_n - [(1-\tau)(x_n - P_{C_n}x_n) + \tau A^*(Ax_n - By_n)], \\ w_n = \rho_n x_n + (1-\rho_n)u_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)w_n, \\ v_n = y_n - [(1-\tau)(y_n - P_{Q_n}y_n) - \tau B^*(Ax_{n+1} - By_n)], \\ p_n = \rho_n y_n + (1-\rho_n)v_n, \\ y_{n+1} = \alpha_n v + (1-\alpha_n)p_n, \end{cases} \tag{1.6}$$

and they proved a strong convergence result of the algorithm in Hilbert spaces.

In 2000, Moudafi [6] introduced the following iterative formula:

Let H be a Hilbert space, and let $K \subset H$ be a convex and closed set. Suppose that $f : K \to K$ is a contraction and $S : K \to K$ is a nonexpansive mapping. Then the sequence $\{x_n\}$ defined by

$$x_0 \in K, x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)S(x_n), \text{ for all } N \cup \{0\} \text{ and } \alpha_n \in (0,1),$$

under certain conditions, converges strongly to a point $\bar{x} \in Fix(S)$, which is a solution to the variational inequality

$$\langle (I - f)\bar{x}, \bar{x} - x \rangle \leq 0, \text{ for all } x \in Fix(S).$$

Moudafi's generalizations are called the viscosity approximation, which was originally from Attouch [1]. Viscosity approximation methods were widely used in the literature to obtain strong convergence results for various nonlinear problems; see, e.g, [2, 3, 6, 10, 12] and the references therein.

Inspired and motivated by the works mentioned above, using a viscosity approximation method, we consider a new relaxed alternating CQ algorithm for solving the SEP (1.1), and obtain astrong convergence theorem in the framework of Hilbert spaces. The result obtained in this paper extends and improves the associated results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. We borrow $\to$ and $\rightharpoonup$ to denote strong and weak convergence, respectively, and $Fix(U)$ to stand for the set of the fixed points of an operator U, and I to denote the identity operator on H. $\omega_\omega(x_n) = \{x : \exists x_{n_k} \rightharpoonup x\}$ is borrowed to stand for the weak ω-limit set of $\{x_n\}$.

Definition 2.1. Let D be a nonempty subset of H. A mapping $U : D \to H$ is said to be
(i) Lipschitzian if there exists a constant $\beta > 0$ such that

$$\|Ux - Uy\| \leq \beta \|x - y\|, \quad \forall x, y \in D,$$

especially, if $\beta \in (0, 1)$, U is said to be a contraction with constant β;
(ii) nonexpansive if

$$\|Ux - Uy\| \leq \|x - y\|, \quad \forall x, y \in D;$$

(iii) firmly nonexpansive if

$$\|Ux - Uy\|^2 + \|(I - U)x - (I - U)y\|^2 \leq \|x - y\|^2, \quad \forall x, y \in D,$$

which is equivalent to

$$\|Ux - Uy\|^2 \leq \langle x - y, Ux - Uy \rangle, \quad \forall x, y \in D.$$

Definition 2.2. [7] Assume that $f : H \to (-\infty, +\infty]$ is a proper function and $\lambda \in (0, 1)$.
(i) f is convex if

$$f(\lambda x + (1 - \lambda) y) \leq \lambda f(x) + (1 - \lambda) f(y), \quad \forall x, y \in H;$$

(ii) The subdifferential of f is the set-valued operator

$$\partial f : H \to 2^H : x \to \{u \in H | (\forall y \in H) \, \langle y - x, u \rangle + f(x) \leq f(y)\};$$

(iii) f is subdifferential at $x \in H$ if $\partial f(x) \neq \emptyset$, and the elements of $\partial f(x)$ are the subgradients f at x.

Definition 2.3. Let C be a closed, convex, and nonempty subset of a real Hilbert space H. A mapping $F : C \to H$ is said to be
(i) monotone if

$$\langle Fx - Fy, x - y \rangle \geq 0, \forall x, y \in C;$$

(ii) strictly monotone if

$$\langle Fx - Fy, x - y \rangle > 0, \forall x, y \in C, x \neq y;$$

(iii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Fx - Fy, x - y \rangle \geq \eta \|x - y\|^2, \forall x, y \in C.$$

Let C be a subset of H, $x \in H$, and $p \in C$. p is called a projection of x onto C if $\|x - p\| = d_C(x)$. If any point on H can be projected onto C, then the set C is a Chebyshev set. Any point on H is mapped to a unique projection operator on C, denoted by P_C. Let C be a closed, convex, and nonempty subset of H. Then C is a Chebyshev set, for every x and p in H, $p = P_C x \Leftrightarrow p \in C$ *and* $(\forall y \in C)$ $\langle x-p, y-p\rangle \leq 0$. For all $x, y \in H$ and a nonnegative real number η,

$$\langle x,y\rangle = \frac{1}{2}(\|x\|^2+\|y\|^2-\|x-y\|^2),\ \|x+y\|^2 \leq (1+\eta)\|x\|^2+(1+\frac{1}{\eta})\|y\|^2.$$

Lemma 2.4. [7] *Let $\{s_k\}$ be a sequence of nonnegative sequence such that $s_{k+1} \leq c_k + \lambda_k b_k + (1-\lambda_k)s_k$, where $\{\lambda_k\}$ is a sequence in $[0,1]$, $\{b_k\}$, and $\{c_k\}$ are two sequences in R such that*
(i) $\sum_{k=0}^{\infty}\lambda_k = \infty$;
(ii) $\limsup_{k\to\infty} b_k \leq 0$ or $\sum_{k=0}^{\infty}|\lambda_k b_k| < \infty$;
(iii) $c_k \geq 0$ for all k, $\sum\limits_{k=0}^{\infty} c_k < \infty$.
Then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.5. [12] *Let H be a real Hilbert space. Suppose that $F: H \to H$ is k-Lipschitzian and η-strongly monotone over a closed convex set $C \subset H$. Then, the following VIP $\langle v-u^*, F(u^*)\rangle \geq 0$, $\forall v \in C$, has its unique solution $u^* \in C$.*

3. Main Results

In this section, we introduce a new algorithm via the viscosity approximation for solving the SEP, and prove the strong convergence of the algorithm. We shall assume that problem (1.1) is consistent, that is, its solution set, denoted by Ω, is nonempty.

Now we make the following assumptions:
(c_1) $f_1: H_1 \to H_1, f_2: H_2 \to H_2$ are two contractions with constants β_1 and β_2, where $\beta_1, \beta_2 \in (0, \frac{1}{\sqrt{2}})$ and set $\beta = \max\{\beta_1, \beta_2\}$;
(c_2) The operators $A: H_1 \to H_3$, $B: H_2 \to H_3$ are bounded and linear, and A^* and B^* are the adjoint of A and B, respectively;
(c_3) $c = \max\{\|A\|^2, \|B\|^2, 2\}$ and $\eta \in (\frac{\tau c}{1-\tau c}, \frac{1+\tau}{1-\tau})$ with $0 < \tau < \frac{1}{1+c}$.

Algorithm 3.1. Choose an initial guess $(x_0, y_0) \in H_1 \times H_2$ arbitrarily. Update (x_{n+1}, y_{n+1}) by the iteration formula

$$\begin{cases} u_n = x_n - [(1-\tau)(x_n - P_{C_n}x_n) + \tau A^*(Ax_n - By_n)], \\ w_n = \rho_n x_n + (1-\rho_n)u_n, \\ x_{n+1} = \alpha_n f_1(x_n) + (1-\alpha_n)w_n, \\ v_n = y_n - [(1-\tau)(y_n - P_{Q_n}y_n) - \tau B^*(Ax_{n+1} - By_n)], \\ p_n = \rho_n y_n + (1-\rho_n)v_n, \\ y_{n+1} = \alpha_n f_2(y_n) + (1-\alpha_n)p_n, \end{cases} \tag{3.1}$$

where $\alpha_n \subset (0,1)$, satisfying $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{i=0}^{\infty}\alpha_i = \infty$, $\{\alpha_n\}, \{\rho_n\} \subset (0,1)$ are non-increasing sequences, and $0 < \tau < \frac{1}{1+c}$ with $c = \max\{\|A\|^2, \|B\|^2, 2\}$.

Put $H^* = H_1 \times H_2$. Define the inner product of H^* as follows:

$$\langle (x_1,y_1),(x_2,y_2)\rangle = \langle x_1,x_2\rangle + \langle y_1,y_2\rangle, \quad \forall (x_1,y_1),(x_2,y_2) \in H^*.$$

It is easy to see that H^* is also a real Hilbert space and $\|(x,y)\| = (\|x\|^2+\|y\|^2)^{\frac{1}{2}}$, $\forall (x,y) \in H^*$.

Theorem 3.2. *Let H_1, H_2, and H_3 be real Hilbert spaces satisfying all the assumptions of (c_1)-(c_3). Then the sequence $\{(x_n, y_n)\}$ generated by Algorithm 3.1 converges strongly to $(x^*, y^*) \in \Omega$ which is the unique solution of the following variational inequality problem (VIP)*

$$\langle ((I - f_1)x^*, (I - f_2)y^*), (x, y) - (x^*, y^*) \rangle \geq 0, \quad \forall (x, y) \in \Omega. \tag{3.2}$$

Proof. We divide the proof into several steps.
Step 1. We prove that the VIP (3.2) has a unique solution $(x^*, y^*) \in \Omega$.

Obviously, Ω is a closed, convex, and nonempty subset in H^*. Define $F : \Omega \subset H^* \to H^*$ as follows $F(x, y) = ((I - f_1)x, (I - f_2)y)$, $\forall (x, y) \in \Omega$. For any $(x_1, y_1), (x_2, y_2) \in \Omega$, since f_1 and f_2 are two contractions, we have

$$\begin{aligned}
&\langle F(x_1, y_1) - F(x_2, y_2), (x_1, y_1) - (x_2, y_2) \rangle \\
&= \langle (I - f_1)x_1 - (I - f_1)x_2, x_1 - x_2 \rangle + \langle (I - f_2)y_1 - (I - f_2)y_2, y_1 - y_2 \rangle \\
&\geq \|x_1 - x_2\|^2 - \|f_1(x_1) - f_1(x_2)\| \, \|x_1 - x_2\| + \|y_1 - y_2\|^2 \\
&\quad - \|f_2(y_1) - f_2(y_2)\| \, \|y_1 - y_2\| \\
&\geq (1 - \beta)\|(x_1, y_1) - (x_2, y_2)\|^2.
\end{aligned}$$

Thus F is $(1 - \beta)$-strongly monotone, and

$$\begin{aligned}
\|F(x_1, y_1) - F(x_2, y_2)\|^2 &= \|(I - f_1)x_1 - (I - f_1)x_2\|^2 + \|(I - f_2)y_1 - (I - f_2)y_2)\|^2 \\
&\leq 2(1 + {\beta_1}^2)\|x_1 - x_2\|^2 + 2(1 + {\beta_2}^2)\|y_1 - y_2\|^2 \\
&\leq 2(1 + \beta^2)\|(x_1, y_1) - (x_2, y_2)\|^2,
\end{aligned}$$

which indicates that F is $2(1 + \beta^2)$-Lipschitzian. Therefore, it follows from Lemma 2.5 that the VIP (3.2) has a unique solution $(x^*, y^*) \in \Omega$.
Step 2. We prove that sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

By (3.1), (c_1), and the definition of contractions, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq & \alpha_n \|f_1(x_n) - x^*\|^2 + (1 - \alpha_n)\|w_n - x^*\|^2 \\
\leq & \alpha_n (\|f_1(x_n) - f_1(x^*)|| + ||f_1(x^*) - x^*\|)^2 + (1 - \alpha_n) \\
& \times [\rho_n \|x_n - x^*\|^2 + (1 - \rho_n) \|u_n - x^*\|^2] \\
\leq & \left[\rho_n - \alpha_n(\rho_n - 2\beta^2)\right] \|x_n - x^*\|^2 + 2\alpha_n \|f_1(x^*) - x^*\|^2 \\
& + (1 - \alpha_n)(1 - \rho_n) \|u_n - x^*\|^2,
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1} - y^*\|^2 \leq & 2\alpha_n ({\beta_2}^2 \|y_n - y^*\|^2 + \|f_2(y^*) - y^*\|^2) + (1 - \alpha_n) \\
& \times [\rho_n \|y_n - y^*\|^2 + (1 - \rho_n) \|v_n - y^*\|^2] \\
= & \left[\rho_n - \alpha_n(\rho_n - 2\beta^2)\right] \|y_n - y^*\|^2 + 2\alpha_n \|f_2(y^*) - y^*\|^2 \\
& + (1 - \alpha_n)(1 - \rho_n) \|v_n - y^*\|^2.
\end{aligned}$$

Adding the above two formulas, we arrive at

$$\begin{aligned}\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2 \leq &[\rho_n-\alpha_n(\rho_n-2\beta^2)](\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\ &+2\alpha_n(\|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2)\\ &+(1-\alpha_n)(1-\rho_n)(\|u_n-x^*\|^2+\|v_n-y^*\|^2). \end{aligned} \tag{3.3}$$

It follows from (3.1) and (c_3) that

$$\begin{aligned}&\|u_n-x^*\|^2\\ &=\|x_n-x^*\|^2-2(1-\tau)\langle x_n-P_{C_n}x_n+P_{C_n}x_n-x^*,x_n-P_{C_n}x_n\rangle\\ &\quad -2\tau\langle x_n-x^*,A^*(Ax_n-By_n)\rangle+||(1-\tau)(x_n-P_{C_n}x_n)+\tau A^*(Ax_n-By_n)\|^2\\ &\leq \|x_n-x^*\|^2-2(1-\tau)\|x_n-P_{C_n}x_n\|^2-2\tau\langle Ax_n-Ax^*,Ax_n-By_n\rangle\\ &\quad +(1-\tau)^2(1+\eta)\|x_n-P_{C_n}x_n\|^2+\tau^2(1+\frac{1}{\eta})\|A\|^2\|Ax_n-By_n\|^2\\ &\leq \|x_n-x^*\|^2-2(1-\tau)\|x_n-P_{C_n}x_n\|^2-\tau(\|Ax_n-Ax^*\|^2\\ &\quad +\|Ax_n-By_n\|^2-\|By_n-By^*\|^2)+(1-\tau)^2(1+\eta)\|x_n-P_{C_n}x_n\|^2\\ &\quad +\tau^2(1+\frac{1}{\eta})c\|Ax_n-By_n\|^2\\ &\leq \|x_n-x^*\|^2-(1-\tau)[2-(1+\eta)(1-\tau)]\|x_n-P_{C_n}x_n\|^2\\ &\quad -\tau[1-(1+\frac{1}{\eta})\tau c]\|Ax_n-By_n\|^2-\tau\|Ax_n-Ax^*\|^2+\tau\|By_n-By^*\|^2\\ &\leq \|x_n-x^*\|^2+\tau\|B\|^2\|y_n-y^*\|^2. \end{aligned} \tag{3.4}$$

Similarly, we obtain

$$\begin{aligned}&\|v_n-y^*\|^2\\ &=\|y_n-y^*\|^2-2(1-\tau)\langle y_n-y^*,y_n-P_{Q_n}y_n\rangle+2\tau\langle y_n-y^*,B^*(Ax_{n+1}-By_n)\rangle\\ &\quad +\|(1-\tau)(y_n-P_{Q_n}y_n)-\tau B^*(Ax_{n+1}-By_n)\|^2\\ &\leq\|y_n-y^*\|^2-2(1-\tau)\|y_n-P_{Q_n}y_n\|^2+2\tau\langle By_n-By^*,Ax_{n+1}-By_n\rangle\\ &\quad +(1-\tau)^2(1+\eta)\|y_n-P_{Q_n}y_n\|^2+\tau^2(1+\frac{1}{\eta})\|B\|^2\|Ax_{n+1}-By_n\|^2\\ &\leq\|y_n-y^*\|^2-2(1-\tau)\|y_n-P_{Q_n}y_n\|^2-\tau(\|Ax_{n+1}-By_n\|^2+\|By_n-By^*\|^2\\ &\quad -\|Ax_{n+1}-Ax^*\|^2)+(1-\tau)^2(1+\eta)\|y_n-P_{Q_n}y_n\|^2\\ &\quad +\tau^2(1+\frac{1}{\eta})c\|Ax_{n+1}-By_n\|^2\\ &=\|y_n-y^*\|^2-(1-\tau)[2-(1+\eta)(1-\tau)]\|y_n-P_{Q_n}y_n\|^2-\tau[1-(1+\frac{1}{\eta})\tau c]\\ &\quad \times\|Ax_{n+1}-By_n\|^2-\tau\|By_n-By^*\|^2+\tau\|Ax_{n+1}-Ax^*\|^2\\ &\leq\|y_n-x^*\|^2+\tau\|A\|^2\|x_{n+1}-x^*\|^2. \end{aligned} \tag{3.5}$$

Let

$$a_n = (1-\tau)\,[2-(1+\eta)(1-\tau)]\,\|x_n - P_{C_n}x_n\|^2 + \tau[1-(1+\frac{1}{\eta})\tau c]\|Ax_n - By_n\|^2,$$

and

$$b_n = (1-\tau)\,[2-(1+\eta)(1-\tau)]\,\|y_n - P_{Q_n}y_n\|^2 + \tau[1-(1+\frac{1}{\eta})\tau c]\|Ax_{n+1} - By_n\|^2.$$

From equations (3.3)-(3.5), we have

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2+\|y_{n+1}-y^*\|^2\\
\leq&[\rho_n-\alpha_n(\rho_n-2\beta^2)](\|x_n-x^*\|^2+\|y_n-y^*\|^2)+2\alpha_n(\|f_1(x^*)-x^*\|^2\\
&+\|f_2(y^*)-y^*\|^2)+(1-\alpha_n)(1-\rho_n)[(\|x_n-x^*\|^2+\|y_n-y^*\|^2)\\
&-(1-\tau)[2-(1+\eta)(1-\tau)](\|x_n-P_{C_n}x_n\|^2+\|y_n-P_{Q_n}y_n\|^2)\\
&-\tau[1-(1+\frac{1}{\eta})\tau c](\|Ax_n-By_n\|^2+\|Ax_{n+1}-By_n\|^2)\\
&-\tau\|Ax_n-Ax^*\|^2+\tau\|Ax_{n+1}-Ax^*\|^2]\\
=&[1-\alpha_n(1-2\beta^2)](\|x_n-x^*\|^2+\|y_n-y^*\|^2-\frac{(1-\alpha_n)(1-\rho_n)}{1-\alpha_n(1-2\beta^2)}\\
&\times\tau\|Ax_n-Ax^*\|^2)+2\alpha_n(\|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2)\\
&-(1-\alpha_n)(1-\rho_n)\{(1-\tau)[2-(1+\eta)(1-\tau)](\|x_n-P_{C_n}x_n\|^2\\
&+\|y_n-P_{Q_n}y_n\|^2)+\tau[1-(1+\frac{1}{\eta})\tau c](\|Ax_n-By_n\|^2+\|Ax_{n+1}-By_n\|^2)\}\\
&+\tau(1-\alpha_n)(1-\rho_n)\|Ax_{n+1}-Ax^*\|^2\\
\leq&[1-\alpha_n(1-2\beta^2)](\|x_n-x^*\|^2+\|y_n-y^*\|^2-(1-\alpha_n)(1-\rho_n)\\
&\times\tau\|Ax_n-Ax^*\|^2)+2\alpha_n(\|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2)\\
&-(1-\alpha_n)(1-\rho_n)(a_n+b_n)+\tau(1-\alpha_{n+1})(1-\rho_{n+1})\|Ax_{n+1}-Ax^*\|^2. \qquad (3.6)
\end{aligned}$$

Let

$$\Gamma_n(x^*,y^*) = \|x_n-x^*\|^2+\|y_n-y^*\|^2-(1-\alpha_n)(1-\rho_n)\,\tau\|Ax_n-Ax^*\|^2,$$

which yields

$$\begin{aligned}
\Gamma_n(x^*,y^*) &\geq [1-\tau(1-\alpha_n)(1-\rho_n)\|A\|^2]\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\geq (1-\tau\|A\|^2)\|x_n-x^*\|^2+\|y_n-y^*\|^2\\
&\geq 0. \qquad (3.7)
\end{aligned}$$

It follows from (3.6) that

$$\begin{aligned}\Gamma_{n+1}(x^*,y^*) \leq &(1-\alpha_n(1-2\beta^2))\Gamma_n(x^*,y^*)+2\alpha_n(\|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2)\\ &-(1-\alpha_n)(1-\rho_n)(a_n+b_n)\\ =&(1-\alpha_n(1-2\beta^2))\Gamma_n(x^*,y^*)+\alpha_n(1-2\beta^2)[\frac{2}{1-2\beta^2}(\|f_1(x^*)-x^*\|^2\\ &+\|f_2(y^*)-y^*\|^2)]-(1-\alpha_n)(1-\rho_n)(a_n+b_n)\\ \leq&(1-\alpha_n(1-2\beta^2))\Gamma_n(x^*,y^*)+\alpha_n(1-2\beta^2)[\frac{2}{1-2\beta^2}(\|f_1(x^*)-x^*\|^2\\ &+\|f_2(y^*)-y^*\|^2)].\end{aligned}$$

Thus

$$\begin{aligned}\Gamma_{n+1}(x^*,y^*) \leq&\max\{\Gamma_n(x^*,y^*),\frac{2}{1-2\beta^2}(\|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2)\}\\ \leq&\max\{\Gamma_1(x^*,y^*),\frac{2}{1-2\beta^2}(\|f_1(x^*)-x^*\|^2+\|f_2(y^*)-y^*\|^2)\},\end{aligned}$$

that is, $\{\Gamma_{n+1}(x^*,y^*)\}$ is bounded. So it follows from (3.7) that $\{x_n\}$ and $\{y_n\}$ are also bounded. Furthermore, due to (3.4) and (3.5), $\{u_n\}$ and $\{v_n\}$ are both bounded.
Step 3. Prove

$$\Gamma_{n+1}(x^*,y^*)\leq\left[1-\alpha_n(1-(1-\alpha_n)\beta^2)\right]\Gamma_n(x^*,y^*)+\alpha_n(1-(1-\alpha_n)\beta^2)\delta_n,$$

where

$$\begin{aligned}\delta_n=&\frac{2(1-\alpha_n)}{1-(1-\alpha_n)\beta^2}(\langle f_1(x^*)-x^*,w_n-x^*\rangle+\langle f_2(y^*)-y^*,p_n-y^*\rangle)\\ &+\frac{\alpha_n(M_1+M_2)}{1-(1-\alpha_n)\beta^2}-\frac{(1-\alpha_n)(1-\rho_n)(a_n+b_n)}{\alpha_n(1-(1-\alpha_n)\beta^2)}.\end{aligned}\tag{3.8}$$

By (3.1), we have

$$\begin{aligned}&\|x_{n+1}-x^*\|^2\\ &=(1-\alpha_n)^2\|w_n-x^*\|^2+2\alpha_n(1-\alpha_n)\langle f_1(x_n)-x^*,w_n-x^*\rangle+\alpha_n^2\|f_1(x_n)-x^*\|^2\\ &=(1-\alpha_n)^2\|w_n-x^*\|^2+2\alpha_n(1-\alpha_n)(\langle f_1(x_n)-f_1(x^*),w_n-x^*\rangle\\ &\quad+\langle f_1(x^*)-x^*,w_n-x^*\rangle)+\alpha_n^2\|f_1(x_n)-x^*\|^2\\ &\leq(1-\alpha_n)^2\|w_n-x^*\|^2+\alpha_n(1-\alpha_n)(\|f_1(x_n)-f_1(x^*)\|^2+\|w_n-x^*\|^2)\\ &\quad+2\alpha_n(1-\alpha_n)\langle f_1(x^*)-x^*,w_n-x^*\rangle+\alpha_n^2\|f_1(x_n)-x^*\|^2\\ &\leq(1-\alpha_n)\|w_n-x^*\|^2+\alpha_n(1-\alpha_n)\beta^2\|x_n-x^*\|^2+2\alpha_n(1-\alpha_n)\\ &\quad\times\langle f_1(x^*)-x^*,w_n-x^*\rangle+\alpha_n^2M_1\\ &\leq(1-\alpha_n)[\rho_n\|x_n-x^*\|^2+(1-\rho_n)\|u_n-x^*\|^2]+\alpha_n(1-\alpha_n)\beta^2\|x_n-x^*\|^2\\ &\quad+2\alpha_n(1-\alpha_n)\langle f_1(x^*)-x^*,w_n-x^*\rangle+\alpha_n^2M_1,\end{aligned}$$

where $M_1 = \sup_{n\geq 0}\{\|f_1(x_n) - x^*\|^2\}$. Substituting (3.4) into the inequality above yields

$$\begin{aligned}\|x_{n+1}-x^*\|^2 \leq &(1-\alpha_n)[\rho_n\|x_n-x^*\|^2 + (1-\rho_n)(\|x_n-x^*\|^2 - a_n - \tau\|Ax_n - Ax^*\|^2\\ &+\tau\|By_n - By^*\|^2)] + \alpha_n(1-\alpha_n)\beta^2\|x_n-x^*\|^2 + 2\alpha_n(1-\alpha_n)\\ &\times\langle f_1(x^*)-x^*, w_n - x^*\rangle + \alpha_n^2 M_1\\ \leq &[1-\alpha_n\left(1-(1-\alpha_n)\beta^2\right)]\|x_n-x^*\|^2 - (1-\alpha_n)(1-\rho_n)a_n\\ &-(1-\alpha_n)(1-\rho_n)\tau(\|Ax_n-Ax^*\|^2 - \|By_n - By^*\|^2)\\ &+2\alpha_n(1-\alpha_n)\langle f_1(x^*)-x^*, w_n-x^*\rangle + \alpha_n^2 M_1.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\|y_{n+1}-y^*\|^2 \leq &[1-\alpha_n\left(1-(1-\alpha_n)\beta^2\right)]\|y_n-y^*\|^2 - (1-\alpha_n)(1-\rho_n)b_n\\ &+(1-\alpha_n)(1-\rho_n)\tau(\|Ax_{n+1}-Ax^*\|^2 - \|By_n-By^*\|^2)\\ &+2\alpha_n(1-\alpha_n)\langle f_2(y^*)-y^*, p_n - y^*\rangle + {\alpha_n}^2 M_2,\end{aligned}$$

where $M_2 = \sup_{n\geq 0}\{\|f_2(y_n) - y^*\|^2\}$. Observe that both $\{\alpha_n\}$ and $\{\rho_n\}$ are non-increasing in $(0,1)$. It follows that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2 + \|y_{n+1}-y^*\|^2\\ \leq &[1-\alpha_n\left(1-(1-\alpha_n)\beta^2\right)](\|x_n-x^*\|^2 + \|y_n-y^*\|^2\\ &-\frac{(1-\alpha_n)(1-\rho_n)}{1-\alpha_n(1-(1-\alpha_n)\beta^2)}\tau\|Ax_n-Ax^*\|^2) - (1-\alpha_n)(1-\rho_n)(a_n+b_n)\\ &+2\alpha_n(1-\alpha_n)(\langle f_1(x^*)-x^*, w_n-x^*\rangle + \langle f_2(y^*)-y^*, p_n-y^*\rangle)\\ &+{\alpha_n}^2(M_1+M_2) + (1-\alpha_n)(1-\rho_n)\tau\|Ax_{n+1}-Ax^*\|^2\\ \leq &[1-\alpha_n(1-(1-\alpha_n)\beta^2)](\|x_n-x^*\|^2 + \|y_n-y^*\|^2 - (1-\alpha_n)\\ &\times(1-\rho_n)\,\tau\|Ax_n-Ax^*\|^2) + \alpha_n(1-(1-\alpha_n)\beta^2)[\frac{2(1-\alpha_n)}{1-(1-\alpha_n)\beta^2}\\ &\times(\langle f_1(x^*)-x^*, w_n-x^*\rangle + \langle f_2(y^*)-y^*, p_n-y^*\rangle) + \frac{\alpha_n(M_1+M_2)}{1-(1-\alpha_n)\beta^2}\\ &-\frac{(1-\alpha_n)(1-\rho_n)(a_n+b_n)}{\alpha_n(1-(1-\alpha_n)\beta^2)}] + (1-\alpha_{n+1})(1-\rho_{n+1})\tau\|Ax_{n+1}-Ax^*\|^2,\end{aligned}$$

which implies

$$\Gamma_{n+1}(x^*,y^*) \leq \left[1-\alpha_n(1-(1-\alpha_n)\beta^2)\right]\Gamma_n(x^*,y^*) + \alpha_n(1-(1-\alpha_n)\beta^2)\delta_n. \tag{3.9}$$

Step 4. We prove that $\{(x_n,y_n)\}$ converges strongly to $(\hat{x},\hat{y})$.

Note that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$ are bounded. In view of (3.1) and (3.8), we see that

$$\begin{aligned}\delta_n \leq &\frac{2(1-\alpha_n)}{1-(1-\alpha_n)\beta^2}(\langle f_1(x^*)-x^*, w_n-x^*\rangle + \langle f_2(y^*)-y^*, p_n-y^*\rangle) + \frac{\alpha_n(M_1+M_2)}{1-(1-\alpha_n)\beta^2}\\ \leq &\frac{2(1-\alpha_n)}{1-(1-\alpha_n)\beta^2}[\|f_1(x^*)-x^*\|(\rho_n\|x_n-x^*\| + (1-\rho_n)\|u_n-x^*\|)\\ &+\|f_2(y^*)-y^*\|(\rho_n\|y_n-y^*\| + (1-\rho_n)\|v_n-y^*\|)].\end{aligned}$$

Therefore, $\delta_n < \infty$, that is, $\limsup_{n\to\infty}\delta_n < \infty$.

Next, we prove $\limsup_{n\to\infty}\delta_n \geq -1$ to guarantee that $\{\delta_n\}$ is bounded. We assume that $\limsup_{n\to\infty}\delta_n < -1$. Then, $\forall n > n_0$, $\exists n_0 > 0$, s.t. $\delta_n < -1$. According to inequality (3.9), for $\forall n > n_0$,

$$\begin{aligned}\Gamma_{n+1}(x^*,y^*) &\leq \left[1-\alpha_n\left(1-(1-\alpha_n)\beta^2\right)\right]\Gamma_n(x^*,y^*)-\alpha_n\left(1-(1-\alpha_n)\beta^2\right)\\ &\leq \Gamma_n(x^*,y^*)-\alpha_n\left(1-(1-\alpha_n)\beta^2\right).\end{aligned}$$

It follows that $\Gamma_{n+1}(x^*,y^*) \leq \Gamma_{n_0}(x^*,y^*) - \sum_{i=n_0}^{n}\alpha_i\left(1-(1-\alpha_i)\beta^2\right)$. Since $\sum_{i=n_0}^{\infty}\alpha_i = \infty$, then $\sum_{i=n_0}^{\infty}\alpha_i\left(1-(1-\alpha_i)\beta^2\right) = \infty$. There exists $N > n_0$ s.t. $\sum_{i=n_0}^{N}\alpha_i\left(1-(1-\alpha_i)\beta^2\right) > \Gamma_{n_0}(x^*,y^*)$. Furthermore, $\Gamma_{N+1}(x^*,y^*) \leq \Gamma_{n_0}(x^*,y^*) - \sum\limits_{i=n_0}^{N}\alpha_i\left(1-(1-\alpha_i)\beta^2\right) < 0$, which contradicts the fact that $\{\Gamma_n(x^*,y^*)\}$ is a non-negative real sequence. Hence $\limsup_{n\to\infty}\delta_n \geq -1$, and then $\limsup_{n\to\infty}\delta_n$ is finite. Since $\limsup_{n\to\infty}\delta_n$ exists, we can take its subsequence $\{n_k\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\delta_n = \lim_{k\to\infty}\delta_{n_k}\\ &= \lim_{k\to\infty}\frac{2(1-\alpha_{n_k})}{1-(1-\alpha_{n_k})\beta^2}\left(\langle f_1(x^*)-x^*, w_{n_k}-x^*\rangle + \langle f_2(y^*)-y^*, p_{n_k}-y^*\rangle\right)\\ &\quad + \frac{\alpha_{n_k}(M_1+M_2)}{1-(1-\alpha_{n_k})\beta^2} - \frac{(1-\alpha_{n_k})(1-\rho_{n_k})(a_{n_k}+b_{n_k})}{\alpha_{n_k}(1-(1-\alpha_{n_k})\beta^2)}\\ &\leq \lim_{k\to\infty}\frac{2}{1-\beta^2}\left(\langle f_1(x^*)-x^*, w_{n_k}-x^*\rangle + \langle f_2(y^*)-y^*, p_{n_k}-y^*\rangle\right).\end{aligned} \tag{3.10}$$

It follows from (3.1) that

$$\|w_n\| \leq \rho_n\|x_n\| + (1-\rho_n)\|u_n\|,\ \|p_n\| \leq \rho_n\|y_n\| + (1-\rho_n)\|v_n\|,$$

which further yield that $\{w_n\},\{p_n\}$ are bounded. Thus $\{w_n\},\{p_n\}$ have weakly convergent subsequences. Without loss of generality, we assume $w_{n_k} \rightharpoonup \omega^*$, $p_{n_k} \rightharpoonup \omega^*$, which together with (3.10) implies that

$$\begin{aligned}&\lim_{k\to\infty}\frac{(1-\alpha_{n_k})(1-\rho_{n_k})(a_{n_k}+b_{n_k})}{\alpha_{n_k}(1-(1-\alpha_{n_k})\beta^2)}\\ &= \lim_{k\to\infty}\frac{(1-\alpha_{n_k})(1-\rho_{n_k})}{\alpha_{n_k}(1-(1-\alpha_{n_k})\beta^2)}\{(1-\tau)[2-(1+\eta)(1-\tau)](\left\|x_{n_k}-P_{C_{n_k}}x_{n_k}\right\|^2\\ &\quad + \left\|y_{n_k}-P_{Q_{n_k}}y_{n_k}\right\|^2) + \tau[1-(1+\frac{1}{\eta})\tau c](\|Ax_{n_k}-By_{n_k}\|^2 + \left\|Ax_{n_k+1}-By_{n_k}\right\|^2)\}\end{aligned}$$

exist. Since $\{\rho_{n_k}\}$ is bounded and non-increasing and $\frac{1-\alpha_{n_k}}{\alpha_{n_k}} \to \infty$, we have

$$\begin{aligned}&\lim_{k\to\infty}(1-\tau)[2-(1+\eta)(1-\tau)](\left\|x_{n_k}-P_{C_{n_k}}x_{n_k}\right\|^2 + \left\|y_{n_k}-P_{Q_{n_k}}y_{n_k}\right\|^2)\\ &+\tau[1-(1+\frac{1}{\eta})\tau c](\|Ax_{n_k}-By_{n_k}\|^2 + \left\|Ax_{n_k+1}-By_{n_k}\right\|^2) = 0.\end{aligned} \tag{3.11}$$

By (c_3), we obtain $(1-\tau)[2-(1+\eta)(1-\tau)]>0, \tau[1-(1+\frac{1}{\eta})\tau c]>0$, which together with (3.11) implies that

$$\lim_{k\to\infty}\left\|x_{n_k}-P_{C_{n_k}}x_{n_k}\right\|=\lim_{k\to\infty}\left\|y_{n_k}-P_{Q_{n_k}}y_{n_k}\right\|=0, \tag{3.12}$$

and

$$\lim_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|=\lim_{k\to\infty}\left\|Ax_{n_k+1}-By_{n_k}\right\|=0. \tag{3.13}$$

From (3.12) and (3.13), we have

$$\begin{aligned}\lim_{k\to\infty}\|u_{n_k}-x_{n_k}\| &= \lim_{k\to\infty}\left\|(1-\tau)(x_{n_k}-P_{C_{n_k}}x_{n_k})+\tau A^*(Ax_{n_k}-By_{n_k})\right\| \\ &\le \lim_{k\to\infty}[(1-\tau)\left\|x_{n_k}-P_{C_{n_k}}x_{n_k}\right\|+\tau\|A\|\,\|Ax_{n_k}-By_{n_k}\|]=0. \end{aligned} \tag{3.14}$$

Furthermore,

$$\lim_{k\to\infty}\|w_{n_k}-x_{n_k}\|=\lim_{k\to\infty}(1-\rho_{n_k})\|u_{n_k}-x_{n_k}\|=0. \tag{3.15}$$

Similarly, we obtain

$$\lim_{k\to\infty}\|v_{n_k}-y_{n_k}\|\le\lim_{k\to\infty}(1-\tau)\left\|y_{n_k}-P_{Q_{n_k}}y_{n_k}\right\|+\tau\|B\|\left\|Ax_{n_k+1}-By_{n_k}\right\|=0, \tag{3.16}$$

and

$$\lim_{k\to\infty}\|p_{n_k}-y_{n_k}\|\le\lim_{k\to\infty}(1-\rho_n)\|v_{n_k}-y_{n_k}\|=0. \tag{3.17}$$

Next, we prove that any weak cluster point $(\hat{x},\hat{y})$ of $\{(x_n,y_n)\}$ is the solution of the SEP (1.1), that is, $(\hat{x},\hat{y})\in\Omega$. Since $\omega_\omega\{(x_n,y_n)\}\neq\emptyset$, then $\exists(x_{n_k},y_{n_k})\rightharpoonup(\hat{x},\hat{y})$. Thus $x_{n_k}\rightharpoonup\hat{x}$, $y_{n_k}\rightharpoonup\hat{y}$. Observe that there exists the constant $\delta_1>0$, for $\forall n\ge 0$, $\|\zeta_n\|\le\delta_1$, where $\zeta_n\in\partial c(x_n)$. Hence,

$$\frac{u_{n_k}-\tau x_{n_k}+\tau A^*(Ax_{n_k}-By_{n_k})}{1-\tau}=P_{C_{n_k}}x_{n_k}\in C_{n_k}.$$

Thus $\exists(x_{n_k},y_{n_k})\rightharpoonup(\hat{x},\hat{y})$, and then $x_{n_k}\rightharpoonup\hat{x}, y_{n_k}\rightharpoonup\hat{y}$. From (1.4), we have

$$c(x_{n_k})+\left\langle\zeta_{n_k},\frac{u_{n_k}-x_{n_k}+\tau A^*(Ax_{n_k}-By_{n_k})}{1-\tau}\right\rangle\le 0.$$

Combining (3.13) and (3.14), we arrive at

$$\begin{aligned}c(x_{n_k}) &\le \left\langle\zeta_{n_k},\frac{x_{n_k}-u_{n_k}-\tau A^*(Ax_{n_k}-By_{n_k})}{1-\tau}\right\rangle \\ &\le \frac{\delta_1}{1-\tau}[\|x_{n_k}-u_{n_k}\|+\tau\|A\|\,\|Ax_{n_k}-By_{n_k}\|]\to 0\end{aligned}$$

as $k\to\infty$. From the weak lower semicontinuity of c, we obtain $c(\hat{x})\le\liminf_{k\to\infty}c(x_{n_k})\le 0$. Thus $\hat{x}\in C$.

Similarly, because ∂q is bounded on bounded sets, there is a constant $\delta_2>0$, for $\forall n_k\ge 0$, such that $\|\eta_{n_k}\|\le\delta_2$, where $\eta_{n_k}\in\partial q(y_{n_k})$. Note that

$$\frac{v_{n_k}-\tau y_{n_k}-\tau B^*\left(Ax_{n_k+1}-By_{n_k}\right)}{1-\tau}=P_{Q_{n_k}}y_{n_k}\in Q_{n_k}.$$

It follows from (1.5) that

$$q(y_{n_k})+\left\langle \eta_{n_k}, \frac{v_{n_k}-y_{n_k}-\tau B^*\left(Ax_{n_k+1}-By_{n_k}\right)}{1-\tau}\right\rangle \leq 0.$$

From (3.13) and (3.16), we have

$$\begin{aligned} q(y_{n_k}) &\leq \left\langle \eta_{n_k}, \frac{y_{n_k}-v_{n_k}+\tau B^*\left(Ax_{n_k+1}-By_{n_k}\right)}{1-\tau}\right\rangle \\ &\leq \frac{\delta_2}{1-\tau}\left[\|y_{n_k}-v_{n_k}\|+\tau\|B\|\left\|Ax_{n_k+1}-By_{n_k}\right\|\right] \to 0 \end{aligned}$$

as $k\to\infty$. The weak lower semicontinuity of q implies that $q(\hat{y})\leq \liminf_{k\to\infty} q(y_{n_k})\leq 0$. Thus $\hat{y}\in Q$.

Furthermore, $Ax_n - By_n \rightharpoonup A\hat{x}-B\hat{y}$ and the weak lower semicontinuity of the squared norm imply $\|A\hat{x}-B\hat{y}\|^2 \leq \liminf_{k\to\infty}\|Ax_{n_k}-By_{n_k}\|^2 = 0$, and then $(\hat{x},\hat{y})\in\Omega$. By (3.15) and (3.17), we have $w_{n_k}\rightharpoonup \hat{x}, p_{n_k}\rightharpoonup \hat{y}$. From (3.2) and (3.10), we obtain

$$\begin{aligned} \limsup_{n\to\infty}\delta_n &\leq \lim_{k\to\infty}\frac{2}{1-\beta^2}\left(\langle f_1(x^*)-x^*, w_{n_k}-x^*\rangle+\langle f_2(y^*)-y^*, p_{n_k}-y^*\rangle\right) \\ &= \frac{2}{1-\beta^2}\left(\langle f_1(x^*)-x^*, \hat{x}-x^*\rangle+\langle f_2(y^*)-y^*, \hat{y}-y^*\rangle\right) \\ &= \frac{2}{1-\beta^2}\left[-\langle((I-f_1)x^*,(I-f_2)y^*),(\hat{x},\hat{y})-(x^*,y^*)\rangle\right] \\ &\leq 0. \end{aligned}$$

It follows from Lemma 2.4 and (3.9) that $\lim_{n\to\infty}\Gamma_n(x^*,y^*)=0$. It follows from (3.7) that $\lim_{n\to\infty}\|x_n-x^*\|=\lim_{n\to\infty}\|y_n-y^*\|=0$. Therefore $(x_n,y_n)\to(x^*,y^*)\in\Omega$, that is, $\{(x_n,y_n)\}$ is strongly convergent to (x^*,y^*), which is the unique solution to the following VIP

$$\langle((I-f_1)x^*,(I-f_2)y^*),(\hat{x},\hat{y})-(x^*,y^*)\rangle\geq 0, \quad \forall (x,y)\in\Omega,$$

which completes the proof. □

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