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INERTIAL EXTRAPOLATION METHOD WITH REGULARIZATION FOR SOLVING MONOTONE BILEVEL VARIATION INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. The purpose of this paper is to introduce a generalized inertial extrapolation iterative method with regularization for approximating a solution of monotone and Lipschitz variational inequality and fixed point problems. In real Hilbert spaces, the strong convergence of the iterative method is obtained under certain conditions imposed on regularization parameters. Some numerical experiments are provided to show the efficiency and applicability of the proposed method.

Keywords. Bilevel variational inequality; Fixed point; Inertial iterative scheme; Nonexpansive mapping.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$. Let C be a nonempty, closed and convex subset of H, and let $A : H \to H$ be a nonlinear operator. The classical Variational Inequality Problem (VIP), which was independently introduced by Stampacchia [30] and Fichera [12, 13] for modeling problems arising from mechanics and for solving the Signorini problem, is formulated as finding $x \in C$ such that $\langle Ax, y-x\rangle \geq 0$, $\forall y \in C$. It is known that many problems in economics, mathematical sciences, and mathematical physics can be formulated as the VIP. We denoted the solution set of the VIP by $VI(A,C)$. In [7], Censor *et al.* considreed the following Split Variational Inequality Problem (SVIP), which is to find $x^* \in C$ that solves $\langle A_1x^*, x-x^*\rangle \geq 0$, $\forall x \in C$ such that $y^* = Tx^* \in Q$ solves $\langle A_2y^*, y-y^*\rangle \geq 0$, $\forall y \in Q$, where C and Q are nonempty, closed, and convex subsets of real Hilbert spaces H_1

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and H_2, respectively, $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$ are two operators, and $T : H_1 \to H_2$ is a bounded linear operator. When $A_1 = A_2 = 0$, the SVIP reduces to the Split Feasibility Problem (SFP). That is to find $x^* \in C$ such that $y^* = Tx^* \in Q$. The SFP, which was introduced by Censor and Elfving [6] in the framework of finite-dimensional Hilbert spaces, finds various applications in many real-life problems, such as image recovery, signal processing, control theory, data compression, computer tomography and so on; see, e.g., [4, 8] and the references therein. Therefore, a lot of researchers in this direction extensively studied this problem. For instance, Ceng *et al.* [5] proposed the following iterative method for solving the SFP:

$$\begin{cases} x_0 = x \in C, \\ y_n = (1-\beta_n)x_n + \beta_n P_C(x_n - \lambda \triangledown f_{\alpha_n}(x_n)), \\ x_{n+1} = \gamma_n x_n + (1-\gamma_n) S P_C(y_n - \lambda \triangledown f_{\alpha_n}(y_n)), \end{cases}$$

where $\triangledown f_{\alpha_n} = \alpha_n I + T^*(I - P_Q)T$, $S : C \to C$ is a nonexpansive mapping, and the sequences of parameters $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are in $(0,1)$. The above iterative algorithm is a combination of the regularization method and extragradient method due to Nadezhkina and Takahashi [26]. Under some mild assumptions, they established that the sequence generated by the iterative method converges weakly to a common solution of the SFP and fixed point problem for nonexpansive mapping. In 2020, Chuasuk and Kaewcharoen [9] proposed the following iterative scheme:

$$\begin{cases} x_0 \in H_1, \\ y_n = P_C(x_n - \lambda_n(T^*(I - SP_Q))T + \alpha_n I)x_n), \\ z_n = P_C(x_n - \lambda_n(T^*(I - SP_Q))T + \alpha_n I)y_n), \\ w_n = (1-\sigma_n)z_n + \sigma_n U z_n, \\ s_n = (1-\beta_n)z_n + \beta_n U w_n, \\ x_{n+1} = (1-\gamma_n)z_n + \gamma_n U s_n, \end{cases}$$

where $S : Q \to Q$ is a nonexpansive mapping, $U : C \to C$ is a pseudo-contractive and L-Lipschitzian continuous mapping, and the sequences of parameters $\{\sigma_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are in $(0,1)$. Under some mild assumptions, they established that the sequence generated by the iterative method converges weakly to a common solution of the SFP and the fixed point problem of a nonexpansive mapping and a pseudo-contractive mapping. The above iterative scheme is the combination of an extragradient method with the regularization due to a generalized Ishikawa iterative scheme. Regularization methods have been employed in a number of optimization problems. Let $f : H_1 \to \mathbb{R}$ be a continuous differentiable function. Then the minimization problem $\min_{x \in C} f(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2$ is ill-posed (see [35]). To address this problem, Xu [35] considered the following Tikhonov regularized problem: $\min_{x \in C} f_\alpha(x) := \frac{1}{2}\|Tx - P_Q Tx\|^2 + \frac{1}{2}\alpha\|x\|$, where $\alpha > 0$ is the regularization parameter.

The traditional Tikhonov regularization methods are usually used to solve ill-posed optimization problems. One of the advantages of the regularization methods are their possible strong convergence to the minimum-norm solutions of optimization problems; see, e.g., [5, 15, 20, 35] and the references therein.

In [18], Hieu and Quy introduced a regularization extragradient method, which is described as follows:

$$\begin{cases} x_0, y_0 \in C, \\ x_{n+1} = P_C(x_n - \lambda_n(Ay_n + \alpha_n x_n)), \\ y_{n+1} = P_C(x_{n+1} - \lambda_{n+1}(Ay_n + \alpha_{n+1} x_{n+1})), \end{cases}$$

where A is monotone and Lipschitz continuous on C with $L > 0$, $\{\lambda_n\} \subset [a,b] \subset (0, \frac{\sqrt{2}-1}{L})$, and α_n satisfies certain conditions. A strong convergence theorem was established. In addition, Hieu, Quy, and Duong [19] introduced the following double projection method with regularization. It reads

$$\begin{cases} x_0 \in C, \\ y_n = P_C(x_n - \lambda_n(Ax_n + \alpha_n x_n)), \\ x_{n+1} = P_C(x_n - \lambda_n(Ay_n + \alpha_n x_n)), \end{cases}$$

where A is monotone and Lipschitz continuous on C with $L > 0$, $\{\lambda_n\} \subset [a,b] \subset (0, \frac{1}{L})$, and α_n satisfies certain conditions. They obtained a strong convergence theorem of solutions in Hilbert spaces. In 2008, Mainge [23] introduced and studied a variational inequality problem of the form:

$$\text{Find } x^* \in VI(A,C) \text{ such that } \langle Fx^*, x - x^* \rangle \geq 0,\ \forall\, x \in VI(A,C), \tag{1.1}$$

where $F : H \to H$ is L-Lispschitz continuous and γ-strongly monotone. He proposed a hybrid extragradient-viscosity method described and obtained a strong convergence theorem of solutions in Hilbert spaces. In [17], Hieu, Dang, and Anh introduced a regularization-projection methods for solving problem (1.1) as follows

$$\begin{cases} u_0 \in H, \\ v_n = P_C(u_n - \lambda_n(Au_n + \alpha_n u_n)), \\ T_n = \{z \in H : \langle u_n - \lambda_n(Au_n + \alpha_n Fu_n) - v_n, z - v_n \rangle \leq 0\}, \\ u_{n+1} = P_{T_n}(u_n - \lambda_n(Av_n + \alpha_n u_n)), \\ \text{update } \lambda_{n+1} :\ \text{if } \lambda_n \|Au_n - Av_n\| \leq \mu \|u_n - v_n\|, \text{ then } \lambda_{n+1} = \lambda_n, \\ \text{else } \lambda_{n+1} = \frac{\mu \|u_n - v_n\|}{\|Au_n - Av_n\|}, \end{cases}$$

where $\lambda_0 \in (0,\infty), \mu \in (0,1)$, and $\{\alpha_n\} \subset (0,\infty)$. It was established that $\{u_n\}$ converges strongly to the solution of problem (1.1). They further established that the main idea of the regularization method for handling a monotone VIP is to add a strongly monotone operator depending on the so-called regularization parameter to the monotone cost operator for obtaining a strongly monotone VIP. The regularized problem has a unique solution continuously depending on the regularization parameter. They associated the VIP with the following regularized variational inequality problem (RVIP): Find $x \in C$ such that $\langle Ax + \alpha Fx, y - x \rangle \geq 0, \forall y \in C$, where $\alpha > 0$ is a real parameter, and $F : H \to H$ is L-Lispschitz continuous and γ-strongly monotone. Since A is monotone and Lipschitz continuous, $A + \alpha F$ is strongly monotone and Lipschitz continuous. Thus, the RVIP is uniquely solvable for each $\alpha > 0$, and this unique solution is denoted by p_α. They further studied the relationship between the regularization solution p_α of the RVIP and the unique solution p^* of problem (1.1). We shall give this in Section 2. For details about the

$RVIP$, we refer to, e.g., [17, 18, 19]. An interesting generalization of (1.1) is defined as follows:

$$\text{Find } p^* \in VI(A,C)\cap F(S) \text{ such that } \langle Fp^*, x-p^*\rangle \geq 0,\ \forall\, x\in VI(A,C), \tag{1.2}$$

where $S: H\to H, F: H\to H$ is L-Lispschitz continuous and γ-strongly monotone.

It is of interest construct a viscosity type iterative method with the regularization for problem (1.2). On the other hand, the inertial extrapolation method has been proven to be an effective way to accelerate the rate of convergence of iterative algorithms. The technique is based on a discrete version of a second order dissipative dynamical system [2, 3]. The inertial type algorithms use its two previous iterates to obtain its next iterate [1, 24, 25]. For details on inertia extrapolation, we refer to [10, 11, 22, 27, 31] and the references therein. Another interesting question is to further enhance the effectiveness of the inertial term. Motivated by the recent interest in this direction of this research, our purpose is to introduce the following problem:

$$\text{Find } p^* \in RVIP(A,C)\cap F(S) \text{ such that } \langle Fp^*, x-p^*\rangle \geq 0,\ \forall\, x\in RVIP(A,C), \tag{1.3}$$

where $S: H\to H, F: H\to H$ is L-Lispschitz continuous and γ-strongly monotone. In addition, we introduce a new generalized inertial viscosity extrapolation method with the regularization technique for solving problem (1.2) when the underlying operator A is monotone and Lipschitz continuous, and F is L-Lispschitz continuous and γ-strongly monotone. Our method uses the stepsizes that are generated at each iteration by some simple computations, which allows it to be easily implemented without the prior knowledge of the operator norm or the coefficient of an underlying operator. Furthermore, we prove that the proposed method converges strongly to a solution of problem (1.2) in real Hilbert spaces. Moreover, numerical experiment are presented to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces. Our highlights are the regularization approach, the generalized inertial introduced, and the new proof for the strong convergence. The rest of this paper is organized as follows: In Section 2, we recall some useful definitions and results that are relevant for our study. In Section 3, we present our proposed method and highlight some of its useful features. In Section 4, we establish strong convergence of our method and. In Section 5 we present some numerical experiments to show the efficiency and applicability of our method in the framework of infinite dimensional Hilbert spaces. In Section 6, the last section, we give the concluding remark.

2. Preliminaries

In this section, we begin by recalling some known and useful results, which are needed in the sequel.

Let H be a real Hilbert space. The set of the fixed points of a nonlinear mapping $T: H\to H$ will be denoted by $F(T)$, that is, $F(T)=\{x\in H: Tx=x\}$. We denote strong and weak convergence by "$\to$" and "$\rightharpoonup$", respectively. For any $x,y\in H$ and $\alpha\in[0,1]$, it is well-known that $\langle x,y\rangle = \frac{1}{2}(\|x\|^2+\|y\|^2-\|x-y\|^2)$, $\|x-y\|^2\leq \|x\|^2+2\langle y, x-y\rangle$, and $\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2$.

Let $T: H\to H$ be a nonlinear mapping. T is said to be

(a) L-Lipschitz continuous if there exists $L>0$ such that $\|Tx-Ty\|\leq L\|x-y\|$, for all $x,y\in H$. If $L=1$, then T is called a nonexpansive mapping;

(b) monotone if $\langle Tx-Ty, x-y\rangle \geq 0$, $\forall x,y\in H$;

(c) γ-strongly monotone if there exists $\alpha>0$ such that $\langle Tx-Ty, x-y\rangle\geq \gamma\|x-y\|^2$, $\forall x,y\in H$.

It is known that the fixed point set of nonexpansive mappings is closed and convex. For a nonexpansive mapping T, it satisfies the following inequality $2\langle (x-Tx)-(y-Ty), Ty-Tx\rangle \leq \|(Tx-x)-(Ty-y)\|^2, \forall x,y \in H$. furthermore, for all $x \in H$ and $x^* \in F(T)$, $2\langle x-Tx, x^*-Tx\rangle \leq \|Tx-x\|^2, \forall x,y \in H$. Let C be a nonempty, closed, and convex subset of H. For any $u \in H$, there exists a unique point $P_C u \in C$ such that $\|u-P_C u\| \leq \|u-y\|$, $\forall y \in C$. P_C is called the metric projection of H onto C. It is well-known that P_C satisfies $\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$, for all $x,y \in H$. Furthermore, P_C is characterized by the property $\|x-y\|^2 \geq \|x-P_C x\|^2 + \|y-P_C x\|^2$ and $\langle x-P_C x, y-P_C x\rangle \leq 0$, for all $x \in H$ and $y \in C$. In addition, P_C is firmly nonexpansive, that is, $\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$.

Recall from [16] that a mapping $T: C \to C$ is said to be demiclosed at 0 if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $Tx = x$. It is known that nonexpansive mappings are demiclosed at 0.

Lemma 2.1. [28] *Let $\{a_n\}$ be a sequence of positive real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0,1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, and let $\{d_n\}$ be a sequence of real numbers. Suppose that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n d_n$, $n \geq 1$. If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for all subsequences $\{a_{n_k}\}$ of $\{a_n\}$ satisfying the condition $\liminf_{k\to\infty}\{a_{n_k+1} - a_{n_k}\} \geq 0$, then, $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2. [17] *Let L and γ be the Lipschitz constant and the modulus of strong monotonicity of a operator F. Then*

(1) $\|p_\alpha\| \leq \|p^*\| + \frac{\|Fp^*\|}{\gamma}$.
(2) $\|p_\alpha - p_\beta\| \leq \frac{\|\alpha-\beta\|}{\alpha} M$ *for all* $\alpha, \beta > 0$, *where* $M = \frac{1}{\gamma}[2L\|p^*\| + (1+\frac{L}{\gamma})\|Fp^*\|]$.
(3) $\lim\limits_{\alpha\to 0} \|p_\alpha - p^*\| = 0$.

3. The Algorithm

In this section, we present our method and highlight some of its important features. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.1. Suppose that the following conditions hold:
Condition A.

(1) H is a Hilbert space, and C is a nonempty, closed, and convex subset of H.
(2) $\{S_n\}$ is a sequence of nonexpansive mappings on H.
(3) $A: H \to H$ is monotone and L_1- Lipschitz continuous operator, and $F: H \to H$ is γ-strongly monotone and L_2-Lipschitz continuous operator, where $L_1, L_2 > 0$, and $\gamma > 0$.
(4) $S: H \to H$ is a nonexpansive mapping, and $f: H \to H$ is a contraction mapping with coefficient $k \in (0,1)$.
(5) The solution set $\Gamma = \{p^* \in VI(A,C) \cap F(S)$ such that $\langle Fp^*, x-p^*\rangle \geq 0,\ \forall x \in VI(A,C)\}$ is nonempty.
(6) The solution set $\Omega = \{p \in RVIP \cap F(S)$ such that $\langle Fp, x-p\rangle \geq 0,\ \forall\ x \in RVIP\}$ is nonempty.

Condition B.

(1) $\beta_n \subset (0,1)$, $\lim\limits_{n\to\infty} \beta_n = 0$, and $\sum_{n=0}^{\infty} \beta_n = \infty$.
(2) $\alpha_n \in (0,1)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

(3) $\{\delta_n\} \subset (0,\delta_0) \in (0,1), \{\gamma_n\}, \{\eta_n\} \subset (0,1)$ such that $\beta_n + \delta_n + \eta_n = 1$, $\lambda_0 > 0, \mu \in (0,1)$, and $\sum_{n=1}^{\infty} \zeta_n < \infty$.

We present the algorithm.

Algorithm 3.2. Give $x_0, x_1 \in H$, $L_2 \in (0,2)$, and $\theta_n \in (0,1)$, and let the parameters λ_0, μ and sequences $\gamma_n, \beta_n, \eta_n$, and δ_n satisfy the conditions above,

Step 1: Given the iterates x_{n-1} and x_n for all $n \in \mathbb{N}$, choose θ_n such that $0 \leq \theta_n \leq \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\theta, \frac{\varepsilon_n}{\|x_n - x_{n-1}\|}\right\}, & \text{if } x_n \neq x_{n-1} \\ \theta, & \text{otherwise,} \end{cases} \tag{3.1}$$

where θ is a positive constant, and $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = \circ(\beta_n)$.

Step 2: Set $w_n = x_n + \theta_n(S_n x_n - S_n x_{n-1})$, compute $z_n = P_C(w_n - \lambda_n(Aw_n + \alpha_n F w_n))$ and $u_n = \gamma_n w_n + (1-\gamma_n) q_n$, where $q_n = P_{T_n}(w_n - \lambda_n(Az_n + \alpha_n F w_n)), T_n = \{w \in H : \langle w_n - \lambda_n(Aw_n + \alpha_n F w_n) - z_n, w - z_n\rangle \leq 0\}$, and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\mu(\|w_n - z_n\|^2 + \|q_n - z_n\|^2)}{2\langle Aw_n - Az_n, q_n - z_n\rangle},\ \lambda_n + \zeta_n\right\}, & \text{if } \langle Aw_n - Az_n, q_n - z_n\rangle > 0, \\ \lambda_n + \zeta_n, & \text{otherwise.} \end{cases}$$

Step 3. Compute $x_{n+1} = \beta_n f(x_n) + \eta_n x_n + \delta_n S u_n$.

Remark 3.3. (1) $C \subset T_n$ for all $n \in \mathbb{N}$. Indeed from the definition of z_n and the characteristic of the metric projection, we have that $\langle w_n - \lambda_n(Aw_n + \alpha_n F w_n) - z_n, w - z_n\rangle \leq 0$ for all $w \in C$. Thus, this together with the definition of T_n implies that $C \subset T_n$ for all $n \in \mathbb{N}$.

(2) Stepsize $\{\lambda_n\}$ is self-adaptive and save computational time unlike the linesearch method that requires loop computations at each iteration, and thus increases computational time.

(3) We do not use the traditional method in [14, 29, 32, 33, 34]. The techniques and ideas employed in our strong convergence analysis are new.

(4) In Algorithm 3.2, it is easy to compute step 1 since the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. It is also easy to see from (3.1) that $\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$. Since $\{\varepsilon_n\}$ is a positive sequence such that $\varepsilon_n = \circ(\beta_n)$, which means that $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$. Hence, $\theta_n\|x_n - x_{n-1}\| \leq \varepsilon_n$ for all $n \in \mathbb{N}$, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$ implies that $\lim\limits_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\beta_n} = 0$.

(5) The sequences of nonexpansive mapping $\{S_n\}$ is helpful for the convergence rate; see Section 5 for the comparison of our proposed iterative algorithm with the sequence $\{S_n\}$ and without the sequence $\{S_n\}$.

(6) The relationship between the regularization solution p_α of the problem (1.3) and the unique solution p^* of the problem (1.2) is the same with Lemma 2.2.

4. Convergence Analysis

The following two lemmas are essential for our convergence theorem.

Lemma 4.1. *Let $\{\lambda_n\}$ be the sequence generated by Algorithm (3.2). Then* $\lim_{n\to\infty} \lambda_n = \lambda$ *and* $\lambda \in [\min\{\lambda_1, \frac{\mu}{L_1}\}, \lambda_1 + \zeta]$.

Proof. From [21, Lemma 3.1], one can obtain the desired conclusion immediately. So, we omit the proof here. $\square$

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2. Then, under Assumption 3.1, $\{x_n\}$ is bounded.*

Proof. Let $p \in \Omega$. Since $\lim\limits_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, there exists $N_1 > 0$ such that $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq N_1$, for all $n \in \mathbb{N}$. Note that $\|w_n - p\| \leq \|x_n - p\| + \theta_n\|S_n x_n - S_n x_{n-1}\| \leq \|x_n - p\| + \beta_n N_1$. Letting $q_n = P_{T_n}(w_n - \lambda_n(Az_n + \alpha_n F w_n))$, we have

$$\begin{aligned}
\|q_n - p\|^2 &\leq \|w_n - \lambda_n(Az_n + \alpha_n F w_n) - p\|^2 - \|w_n - \lambda_n(Az_n + \alpha_n F w_n) - q_n\|^2 \\
&= \|(w_n - p) - \lambda_n(Az_n + \alpha_n F w_n)\|^2 - \|(w_n - q_n) - \lambda_n(Az_n + \alpha_n F w_n)\|^2 \\
&= \|w_n - p\|^2 - \|w_n - q_n\|^2 - 2\lambda_n\langle q_n - p, Az_n + \alpha_n F w_n\rangle \\
&= \|w_n - p\|^2 - \|w_n - q_n\|^2 + 2\langle w_n - z_n, z_n - q_n\rangle + 2\lambda_n\langle Az_n + \alpha_n F w_n, p - z_n\rangle \\
&\quad + 2\lambda_n\langle Az_n - Aw_n, z_n - q_n\rangle + 2\langle w_n - \lambda_n(Aw_n + \alpha_n F w_n) - z_n, q_n - z_n\rangle. \qquad (4.1)
\end{aligned}$$

Since $q_n \in T_n$, we have from the definition of T_n that $\langle w_n - \lambda_n(Aw_n + \alpha_n F w_n) - z_n, q_n - z_n\rangle \leq 0$ and $2\langle w_n - z_n, z_n - q_n\rangle = \|w_n - q_n\|^2 - \|w_n - z_n\|^2 - \|z_n - q_n\|^2$. It follows from (4.1) that

$$\begin{aligned}
\|q_n - p\|^2 \leq\; & 2\lambda_n\langle Az_n - Aw_n, z_n - q_n\rangle + \|w_n - p\|^2 - \|z_n - q_n\|^2 - \|w_n - z_n\|^2 \\
& + 2\lambda_n\langle Az_n + \alpha_n F w_n, p - z_n\rangle.
\end{aligned} \qquad (4.2)$$

Now, using the monotonicity of A, we have that $\langle Az_n - Ap, p - z_n\rangle \leq 0$. Thus,

$$2\lambda_n\langle Az_n + \alpha_n F w_n, p - z_n\rangle \leq 2\lambda_n\langle Ap + \alpha_n F p, p - z_n\rangle + 2\lambda_n\alpha_n\langle F w_n - F p, p - z_n\rangle.$$

Since p is a solution of $RVIP$ and $z_n \in C$, we have that $\langle Ap + \alpha_n F p, z_n - p\rangle \geq 0$, which implies $\langle Ap + \alpha_n F p, p - z_n\rangle \leq 0$ and $2\lambda_n\langle Az_n + \alpha_n F w_n, p - z_n\rangle \leq 2\lambda_n\alpha_n\langle F w_n - F p, p - z_n\rangle$, Thus, from the γ-strongly monotonicity of F, we have that

$$\begin{aligned}
2\lambda_n\langle Az_n + \alpha_n F w_n, p - z_n\rangle &\leq 2\lambda_n\alpha_n\langle F w_n - F p, p - w_n\rangle + 2\lambda_n\alpha_n\langle F w_n - F p, w_n - z_n\rangle \\
&\leq -2\lambda_n\alpha_n\gamma\|p - w_n\|^2 + 2\lambda_n\alpha_n\langle F w_n - F p, w_n - z_n\rangle.
\end{aligned}$$

Thus, (4.2) becomes

$$\begin{aligned}
\|q_n - p\|^2 &\leq (1 - 2\lambda_n\alpha_n)\|w_n - p\|^2 - \|z_n - q_n\|^2 - \|w_n - z_n\|^2 + 2\lambda_n\langle Az_n - Aw_n, z_n - q_n\rangle \\
&\quad + 2\lambda_n\langle F w_n - F p, w_n - z_n\rangle \\
&\leq (1 - 2\lambda_n\alpha_n)\|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - w_n\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - q_n\|^2 \\
&\quad + 2\lambda_n\alpha_n L_2\|w_n - p\|\|w_n - z_n\| \\
&\leq (1 - \lambda_n\alpha_n(2 - L_2))\|w_n - p\|^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}} - \lambda_n\alpha_n L_2\right)\|z_n - w_n\|^2 \\
&\quad - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\|z_n - q_n\|^2.
\end{aligned}$$

In view of $\lim\limits_{n\to\infty}(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}) = 1 - \mu > 0$, there exists $N \geq 0$ such that, for $n \geq N$, $1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > 0$. Thus, it follows that, for all $n \geq N$,

$$\|q_n - p\|^2 = \|w_n - p\|^2 \Rightarrow \|q_n - p\| \leq \|w_n - p\|. \qquad (4.3)$$

Thus, $\|u_n - p\| \le \gamma_n\|w_n - p\| + (1-\gamma_n)\|q_n - p\| \le \gamma_n\|w_n - p\| + (1-\gamma_n)\|w_n - p\| = \|w_n - p\|$. Hence,

$$\begin{aligned}
\|x_{n+1} - p\| &\le \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + \eta_n\|x_n - p\| + \delta_n\|Su_n - p\| \\
&\le \beta_n k\|x_n - p\| + \alpha_n\|f(p) - p\| + \eta_n\|x_n - p\| + \delta_n\|u_n - p\| \\
&\le \beta_n k\|x_n - p\| + \alpha_n\|f(p) - p\| + \eta_n\|x_n - p\| + \delta_n\|x_n - p\| + \delta_n\beta_n N_1 \\
&\le (1 - \beta_n(1-k))\|x_n - p\| + \beta_n(1-k)\left[\frac{\delta_n N_1 + \|f(p) - p\|}{(1-k)}\right].
\end{aligned}$$

This implies $\|x_{n+1} - p\| \le \max\{\|x_0 - p\|, \frac{\delta_0 N_1 + \|f(p)-p\|}{(1-k)}\}$. Thus, we have that $\{x_n\}$ is bounded. □

Theorem 4.3. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.2. Then, under the Assumption 3.1, $\{x_n\}$ converges strongly to $p^* \in \Gamma$, where $p^* = P_\Gamma \circ f(p^*)$.*

Proof. Let $p \in \Omega$. Observe that

$$\begin{aligned}
\|w_n - p\|^2 &= \|x_n - p\|^2 + 2\theta_n\langle x_n - p, S_n x_n - S_n x_{n-1}\rangle + \theta_n^2\|S_n x_n - S_n x_{n-1}\|^2 \\
&\le \|x_n - p\|^2 + 2\theta_n\|x_n - p\|\|x_n - x_{n-1}\| + \theta_n^2\|x_n - x_{n-1}\|^2 \\
&\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|[2\|x_n - p\| + \beta_n N_1] \\
&\le \|x_n - p\|^2 + \theta_n\|x_n - x_{n-1}\|N_2, \qquad (4.4)
\end{aligned}$$

for some $N_2 > 0$. Using (4.3), we have

$$\begin{aligned}
\|u_n - p\|^2 &= \gamma_n\|w_n - p\|^2 + (1-\gamma_n)\|q_n - p\|^2 - \gamma_n(1-\gamma_n)\|w_n - q_n\|^2 \\
&\le \gamma_n\|w_n - p\|^2 + (1-\gamma_n)\|w_n - p\|^2 - \gamma_n(1-\gamma_n)\|w_n - q_n\|^2 \\
&\le \|w_n - p\|^2. \qquad (4.5)
\end{aligned}$$

Furthermore, using (4.4) and (4.5), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \|\eta_n(x_n - p) + \delta_n(Su_n - p)\|^2 + 2\beta_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \eta_n^2\|x_n - p\|^2 + \delta_n^2\|Su_n - p\|^2 + 2\delta_n\eta_n\|x_n - p\|\|Su_n - p\| + 2\beta_n\langle f(x_n) - p, x_{n+1} - p\rangle \\
&\le \eta_n(\delta_n + \eta_n)\|x_n - p\|^2 + \delta_n(\eta_n + \delta_n)\|u_n - p\|^2 + 2\beta_n\langle f(x_n) - f(p), x_{n+1} - p\rangle \\
&\quad + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le \eta_n(\delta_n + \eta_n)\|x_n - p\|^2 + \delta_n(\eta_n + \delta_n)\|w_n - p\|^2 - \gamma_n(1-\gamma_n)\delta_n(\eta_n + \delta_n)\|w_n - q_n\|^2 \\
&+ 2\beta_n\langle f(x_n) - f(p), x_{n+1} - p\rangle + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (\delta_n + \eta_n)^2\|x_n - p\|^2 + \delta_n(\eta_n + \delta_n)\theta_n\|x_n - x_{n-1}\|N_2 - \gamma_n(1-\gamma_n)\delta_n(\eta_n + \delta_n)\|w_n - q_n\|^2 \\
&+ \beta_n k\|x_n - p\|^2 + \beta_n k\|x_{n+1} - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle \\
&\le (1 - 2\beta_n + \beta_n k)\|x_n - p\|^2 + \beta_n^2\|x_n - p\|^2 + \delta_n(1-\beta_n)\theta_n\|x_n - x_{n-1}\|N_2 \\
&\quad - \gamma_n(1-\gamma_n)\delta_n(1-\beta_n)\|w_n - q_n\|^2 + \beta_n k\|x_{n+1} - p\|^2 + 2\beta_n\langle f(p) - p, x_{n+1} - p\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\leq \left(1-\frac{2\beta_n(1-k)}{1-\beta_n k}\right)\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_n k}\left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_n N_3}{2(1-k)}\right. \\
&\left.-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\beta_n)}{2\beta_n(1-k)}\|w_n-q_n\|^2+\frac{1}{1-k}\langle f(p)-p,x_{n+1}-p\rangle\right] \\
&=\left(1-\frac{2\beta_n(1-k)}{1-\beta_n k}\right)\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_n k}\Psi_n, \qquad (4.6)
\end{aligned}$$

where $N_3=\sup_{n\in\mathbb{N}}\{\|x_n-p\|^2 : n\geq\mathbb{N}\}$ and

$$\begin{aligned}
\Psi_n=&\frac{\delta_n(1-\alpha_n)\theta_n}{2\alpha_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\alpha_n N_3}{2(1-k)}-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\alpha_n)}{2\alpha_n(1-\alpha_n k)(1-k)}\|w_n-q_n\|^2 \\
&+\frac{1}{1-k}\langle f(p)-p,x_{n+1}-p\rangle.
\end{aligned}$$

According to Lemma 2.1, it is sufficient to show that $\limsup_{k\to\infty}\Psi_n\leq 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfies the condition

$$\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\}\geq 0. \qquad (4.7)$$

To show $\limsup_{k\to\infty}\Psi_n\leq 0$, we suppose that for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ such that (4.7) holds. Then, $\liminf_{k\to\infty}\{\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\}\geq 0$. From (4.6), we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\leq \left(1-\frac{2\beta_n(1-k)}{1-\beta_n k}\right)\|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_n k}\left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_n N_3}{2(1-k)}\right. \\
&\left.\quad-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\beta_n)}{2\alpha_n(1-\alpha_n k)(1-k)}\|w_n-q_n\|^2+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\right] \\
&\leq \|x_n-p\|^2+\frac{2\beta_n(1-k)}{1-\beta_n k}\left[\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_n N_3}{2(1-k)}\right. \\
&\left.\quad-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\beta_n)}{2\beta_n(1-k)}\|w_n-q_n\|^2+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\gamma_{n_k}(1-\gamma_{n_k})\delta_{n_k}(1-\beta_{n_k})\|w_{n_k}-q_{n_k}\|^2\right) \\
&\leq \limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2+\frac{1}{1-\beta_{n_k}k}\left[\beta_{n_k}\delta_{n_k}(1-\beta_{n_k})\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2\right.\right. \\
&\quad\left.\left.+\beta_{n_k}^2N_3+2\beta_n\langle f(p)-p,x_{n_k+1}-p\rangle\right]-\|x_{n_k+1}-p\|^2\right] \\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0,
\end{aligned}$$

and hence $\lim_{k\to\infty}\|w_{n_k}-q_{n_k}\|=0$. From (4.4) and (4.5), we obtain

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq \|\eta_n(x_n-p)+\delta_n(Su_n-p)\|^2+2\beta_n\langle f(x_n)-p,x_{n+1}-p\rangle\\
&\leq \eta_n^2\|x_n-p\|^2+\delta_n^2\|Su_n-p\|^2+2\delta_n\eta_n\|x_n-p\|\|Su_n-p\|+2\beta_n\langle f(x_n)-p,x_{n+1}-p\rangle\\
&\leq \eta_n^2\|x_n-p\|^2+\delta_n^2\|u_n-p\|^2+\delta_n\eta_n(\|x_n-p\|^2+\|u_n-p\|^2)+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&\quad +2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq \eta_n\|x_n-p\|^2+\delta_n\gamma_n\|w_n-p\|^2+(1-\gamma_n)\|q_n-p\|^2-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2\\
&\quad +2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq \eta_n\|x_n-p\|^2+\delta_n\|w_n-p\|^2+\delta_n(1-\gamma_n)[(1-\lambda_n\alpha_n(2-L))\|w_n-p\|^2\\
&\quad -(1-\frac{\mu\lambda_n}{\lambda_{n+1}}-\lambda_n\alpha_nL)\|z_n-w_n\|]-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle\\
&\quad +2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq \eta_n\|x_n-p\|^2+\delta_n[\|x_n-p\|^2+\theta_n\|x_n-x_{n-1}\|N_2]-\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}-\lambda_n\alpha_nL\right)(1-\gamma_n)\|z_n-w_n\|^2]\\
&\quad -\gamma_n(1-\gamma_n)\|w_n-q_n\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle\\
&\leq \|x_n-p\|^2+\delta_n\theta_n\|x_n-x_{n-1}\|N_2-\left(1-\frac{\mu\lambda_n}{\lambda_{n+1}}-\delta_n\lambda_n\alpha_nL\right)(1-\gamma_n)\|z_n-w_n\|^2\\
&-\gamma_n(1-\gamma_n)\|w_n-q_n\|^2+2\beta_n\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\beta_n\langle f(p)-p,x_{n+1}-p\rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\limsup_{k\to\infty}\left(\left(1-\frac{\mu\lambda_{n_k}}{\lambda_{n_k+1}}-\delta_{n_k}\lambda_{n_k}\alpha_{n_k}L\right)(1-\gamma_{n_k})\|z_{n_k}-w_{n_k}\|^2\right)\\
&\leq \limsup_{k\to\infty}\Bigg[\|x_{n_k}-p\|^2+\delta_{n_k}\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2-\gamma_{n_k}(1-\gamma_{n_k})\|w_{n_k}-q_{n_k}\|^2\\
&+2\beta_{n_k}\langle f(x_{n_k})-f(p),x_{n_k+1}-p\rangle+2\beta_{n_k}\langle f(p)-p,x_{n_k+1}-p\rangle-\|x_{n_k+1}-p\|^2\Bigg]\\
&\leq -\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0,
\end{aligned}
$$

and hence $\lim_{k\to\infty}\|z_{n_k}-w_{n_k}\|=0$. Similarly, we can obtain $\lim_{k\to\infty}\|z_{n_k}-q_{n_k}\|=0$. Observe that

$$
\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\leq \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|u_n-p\|^2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|w_n-p\|^2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \beta_n\|f(x_n)-p\|^2+\eta_n\|x_n-p\|^2+\delta_n\|x_n-p\|^2+\delta\theta_n\|x_n-x_{n-1}\|N_2-\eta_n\delta_n\|x_n-Su_n\|^2\\
&\leq \|x_n-p\|^2+\beta_n\|f(x_n)-p\|^2+\delta\theta_n\|x_n-x_{n-1}\|N_2-\eta_n\delta_n\|x_n-Su_n\|^2,
\end{aligned}
$$

which implies that

$$\limsup_{k\to\infty}\left(\eta_n\delta_n\|x_{n_k}-Su_{n_k}\|^2\right)\leq\limsup_{k\to\infty}\Big[\|x_{n_k}-p\|^2+\delta\beta_{n_k}\frac{\theta_{n_k}}{\beta_{n_k}}\|x_{n_k}-x_{n_k-1}\|N_2$$
$$+\beta_n\|f(x_n)-p\|^2-\|x_{n_k+1}-p\|^2\Big]$$
$$\leq-\liminf_{k\to\infty}[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2]\leq 0.$$

Using (4.6), we have that $\lim_{k\to\infty}\|x_{n_k}-Su_{n_k}\|=0$. It is easy to see that, as $k\to\infty$,

$$\|w_{n_k}-x_{n_k}\|=\theta_{n_k}||x_{n_k}-x_{n_k-1}||=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}||x_{n_k}-x_{n_k-1}||\to 0.$$

In addition, we have that

$$\|u_{n_k}-w_{n_k}\|\leq\gamma_{n_k}\|w_{n_k}-w_{n_k}\|+(1-\gamma_{n_k})\|q_{n_k}-w_{n_k}\|\to 0\text{ as }k\to\infty,$$

$$\|u_{n_k}-x_{n_k}\|\leq\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0\text{ as }k\to\infty,\tag{4.8}$$

and

$$\|u_{n_k}-Su_{n_k}\|\leq\|u_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|+\|x_{n_k}-Su_{n_k}\|\to 0\text{ as }k\to\infty.$$

Thus,

$$\|x_{n_k+1}-x_{n_k}\|\leq\beta_n\|f(x_{n_k})-x_{n_k})\|+\eta_n\|x_n-x_{n_k}\|+\delta_{n_k}\|Su_{n_k}-x_{n_k}\|\to 0\text{ as }k\to\infty.$$

Since $\{x_{n_k}\}$ is bounded, then there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $\{x_{n_{k_j}}\}$ converges weakly to $x^*\in H$. In addition, using (4.8) and the boundedness of $\{u_{n_k}\}$, there exists a subsequence $\{u_{n_{k_j}}\}$ of $\{u_{n_k}\}$ such that $\{u_{n_{k_j}}\}$ converges weakly to $x^*\in H$. Since S is demiclosed, we have $x^*\in F(S)$. Furthermore,

$$\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\lim_{j\to\infty}\langle f(p)-p,x_{n_{k_j}}-p\rangle=\langle f(p)-p,x^*-p\rangle.$$

Since p is a unique solution of $RVIP$, we obtain that $\limsup_{k\to\infty}\langle f(p)-p,x_{n_k}-p\rangle=\langle f(p)-p,x^*-p\rangle\leq 0$, which implies that $\limsup_{k\to\infty}\langle f(p)-p,x_{n_k+1}-p\rangle\leq 0$. Using our assumption, we have that $\Psi_n=\frac{\delta_n(1-\beta_n)\theta_n}{2\beta_n(1-k)}\|x_n-x_{n-1}\|N_2+\frac{\beta_nN_3}{2(1-k)}-\frac{\gamma_n(1-\gamma_n)\delta_n(1-\beta_n)}{2\beta_n(1-\beta_nk)(1-k)}\|w_n-q_n\|^2+\frac{1}{((1-k)}\langle f(p)-p,x_{n+1}-p\rangle\leq 0$. Thus, it follows from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-p\|=0$. From Lemma 2.2 (iii), we obtain that $\|p-p^*\|\to 0$ as $n\to\infty$. Thus $\|x_n-p^*\|\leq\|x_n-p\|+\|p-p^*\|\to 0$ as $n\to\infty$. Hence, $\{x_n\}$ converges strongly to $p^*\in\Gamma$. □

5. Numerical Experiment

In this section, we present some numerical experiments to show the efficiency and applicability of our method in comparison with our Algorithm without $\{S_n\}$ in the inertial term in the framework of infinite and finite dimensional Hilbert spaces.

Example 5.1. Let $H=L_2([0,1])$ be equipped with inner product $\langle x,y\rangle=\int_0^1 x(t)y(t)dt$, $\forall x,y\in L_2([0,1])$ and $\|x\|^2:=\int_0^1|x(t)|^2dt\ \forall x,y,\in L_2([0,1])$. Let $F;A;f:L_2([0,1])\to L_2([0,1])$ be defined by $Ax(t)=\max\{0,x(t)\}$, $t\in[0,1]$, $Fx(t)=fx(t)=\frac{x(t)}{2}$. It is easy to see that A is 1-Lipschitz continuous and monotone, F γ-strongly monotone, and f is a contraction on $L_2([0,1])$.

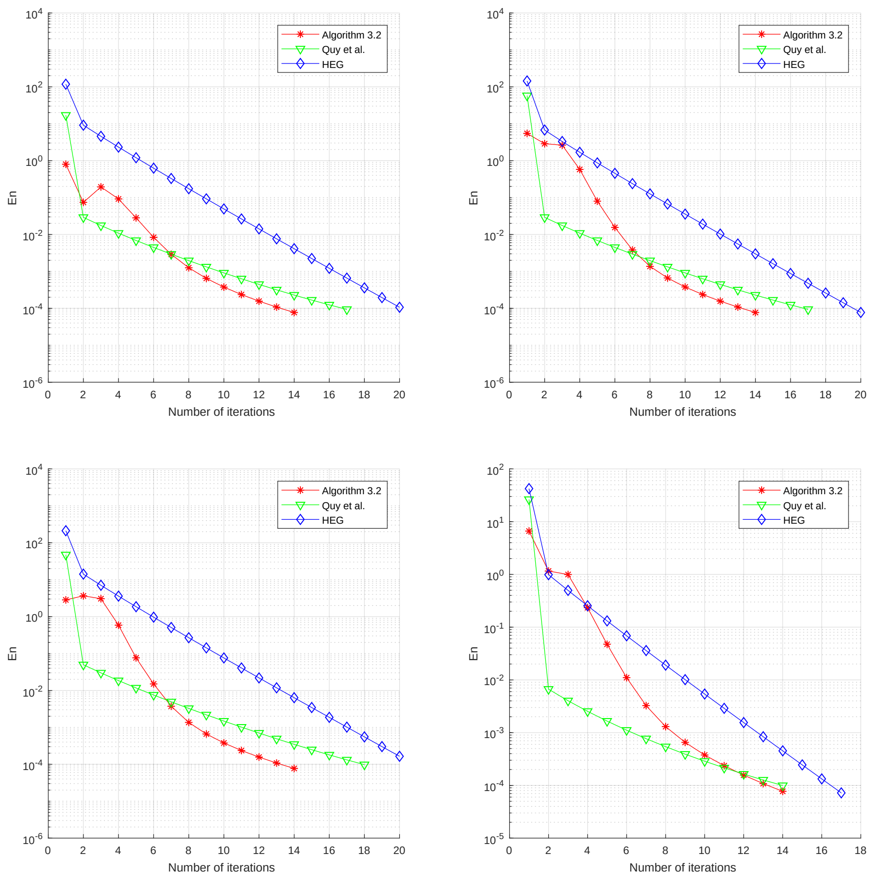

FIGURE 1. Example 5.1, Top Left: Case I; Top Right: Case II; Bottom left: case III; Bottom right: Case IV.

Let $S_n; S : L_2([0,1]) \to L_2([0,1])$ be defined by $Sx(s) = \int_0^1 t^n x(s)ds \; \forall t \in L_2([0,1])$ and $S_n x(t) = \int_0^1 \sin x(t)$. Let C be defined by $C = \{x \in L_2 : \langle a, x\rangle = b\}$ where $a \neq 0$ and $b = 2$. Thus, we have

$$P_C(\bar{x}) = \max\left\{0, \frac{b - \langle a, \bar{x}\rangle}{\|a\|^2}\right\} a + \bar{x}.$$

We choose $\zeta_n = 0.25, \mu = 0.5, \theta_n = \overline{\theta}, \alpha_n = \frac{1}{n+6}, \beta_n = \frac{1}{5n+6}, \eta_n = \frac{2}{3n+2}, \delta_n = 1 - \eta_n - \beta_n$, and $\varepsilon_n = \frac{10^{20}}{n^2}$ for all $n \in \mathbb{N}$. It is easy to verify that all the hypothesis of Theorem 4.3 are satisfied. We implement our algorithm for different values of x_0, x_1 as follows

Case I: $x_0(t) = 2t^2 + t + 2,\ x_1(t) = t + 2$;
Case II: $x_0(t) = 2t^2,\ x_1(t) = -5t + 2$;
Case III: $x_0(t) = t,\ x_1(t) = \log(t)$;
Case IV: $x_0(t) = 5t + 1, x_1(t) = 3t^2$.

Example 5.2. Let $H = \mathbb{R}^2$, and consider a nonlinear operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $A(x_1, x_2) = (x_1 + x_2 + \cos(x_1), -x_1 + x_2 + \cos(x_2))$. Let $f(x) = \frac{x}{2}, F(x) = \sin x$,and C be defined as $C = [-1,1] \times [-1,1]$. It is easy to see that A is 3-Lipschitz continuous and monotone, F γ-strongly monotone, and f is a contraction on $\mathbb{R}^2$. Let Y be a 2×2 matrix defined by $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. We

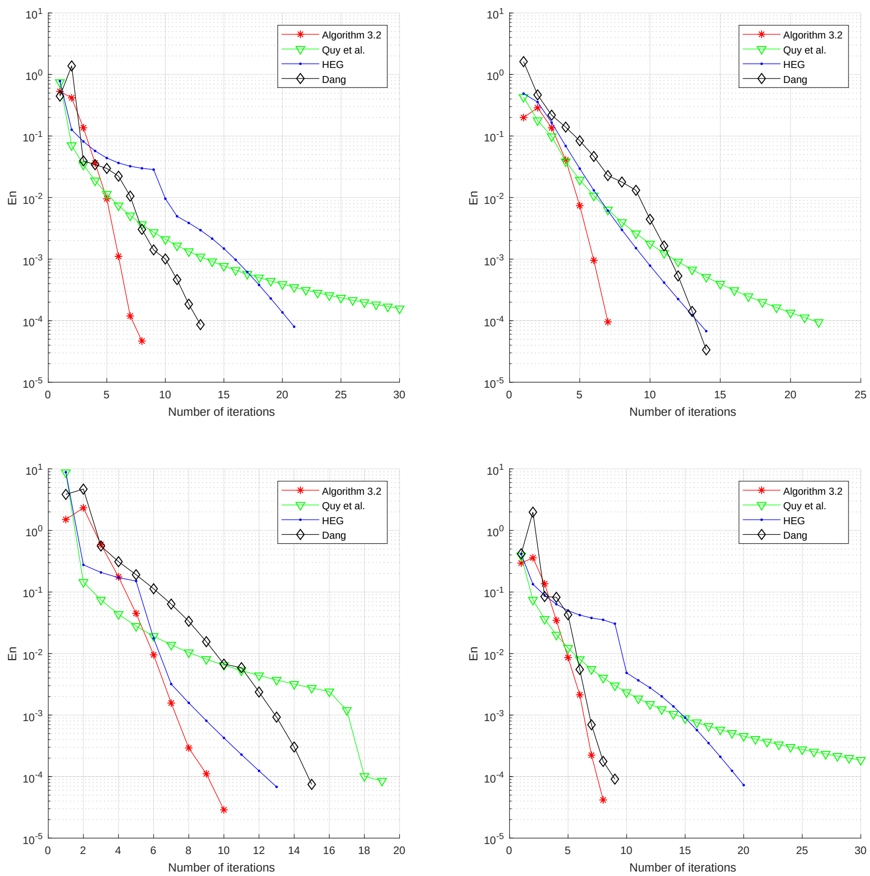

FIGURE 2. Example 5.2, Top Left: Case I; Top Right: Case II; Bottom left: case III; Bottom right: Case IV

define the mapping $S:\mathbb{R}^2\to\mathbb{R}^2$ by $Sx=\|Y\|^{-1}Yx$, where $x=(x_1,x_2)^T$. It is easily see that S is a nonexpansive mapping. We choose $\zeta_n=\frac{1}{(n+1)^2}, \mu=0.5, \theta_n=\overline{\theta}=\frac{1}{3}, \alpha_n=\frac{1}{50n+13}$, $\beta_n=\frac{1}{5n+6}, \eta_n=\frac{2n}{3n+2}, \delta_n=1-\eta_n-\beta_n, \varepsilon_n=\frac{1}{n^2}$, and $\gamma_n=\frac{1}{2n+1}$. For, Algorithm 3.2 and Dang *et al.* [17], we choose $\lambda_1=0.75$, the HEG of [23], and we choose $\lambda_n=\frac{1}{4.5}$. It is easy to verify that all the hypothesis of Theorem 4.3 are satisfied. We implement our algorithm for different values of x_0,x_1 as follows

Case I: $x_0=(1,2)'$, $x_1=(1.2,0.5)$;
Case II: $x_0=(1,0)'$, $x_1=(0,1)'$;
Case III: $x_0=(0.98,1.02)'$, $x_1=(1.50,2.36)'$;
Case IV: $x_0=(-2,-4)'$, $x_1=(1,0.5)'$;

6. CONCLUSION

In this paper, we introduced a generalized inertial extrapolation iterative method with regularization for solving a fixed point problem and a variational inequality problem involving a monotone and Lipschitz continuous operator in frame work of real Hilbert spaces. Our method uses the stepsizes that are generated at each iteration by some simple computations. This make our method efficient without the prior knowledge of the operator norm or the coefficient of an underlying operator. Furthermore, we proved that the proposed method converges strongly to a solution of problem (1.2) in real Hilbert spaces. In addition, we present some numerical experiments to show the efficiency and implementation of our method in the framework of infinite and finite dimensional Hilbert spaces. Our comparison shows that our method speeds up the convergence for the case without $\{S_n\}$. Our highlights are the regularization approach, the generalized inertial introduced, and the new proof for the strong convergence.

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