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A BREGMAN HYBRID EXTRAGRADIENT METHOD FOR SOLVING PSEUDOMONOTONE EQUILIBRIUM AND FIXED POINT PROBLEMS

KITTISAK JANTAKARN, ANCHALEE KAEWCHAROEN*

Department of Mathematics, Faculty of Science, Naresuan University, Phitsanulok 65000, Thailand

Abstract. In this paper, using the concept of the Bregman distance, we propose a Bregman extragradient method for finding a common solution of a finite family of pseudomonotone equilibrium problems and the common fixed point problem of a finite family of Bregman relatively nonexpansive mappings. We introduce a generalized step size such that the algorithm does not require a prior knowledge of the operator norm. A strong convergence theorem was proved in the setting of reflexive Banach spaces and applied to variational inequality problems. Furthermore, we present some numerical examples to illustrate the consistency and accuracy of our algorithm and also to compare with the results in the literature.

Keywords. Bregman relatively nonexpansive mappings; Bregman hybrid extragradient methods; Fixed point; Pseudomonotone equilibrium problems.

1. Introduction

Let E be a real reflexive Banach space, and let $C \subset E$ be a nonempty, closed, and convex subset. Recall the equilibrium problem (EP) for a bifunction $G : C \times C \to \mathbb{R}$ as follows:

$$\text{Find } x^* \in C \text{ such that } G(x^*, y) \geq 0,\ \forall y \in C. \tag{1.1}$$

We denote the set of solutions of problem (1.1) by $EP(G)$.

In 1994, Blum and Oettli [4] revisited the EP, which is a fundamental concept and an important mathematical tool for solving many concrete problems. The EP can be considered as a general model and covers numerous fascinating and complicated problems in nonlinear analysis, such as fixed point problems, variational inequalities, and Nash equilibrium problems; see, e.g., [4, 16]. It is known that several problems arising in mathematics, economics, physics, computer science, and management science, can be modeled as an EP, and many fixed point methods have been investigated for the EP; see, e.g., [5, 9, 12, 15, 25] for details.

Let $T : C \to C$ be an operator. A point $x \in C$ is called a fixed point of T if $Tx = x$. We denote the set of fixed points of T by $F(T)$ from now on. The fixed point problem and its applications

*Corresponding author.

E-mail addresse: anchaleeka@nu.ac.th (A. Kaewcharoen)

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are great importance. Indeed, fixed point methods have been employed in many fields, such as, signal processing, image recovery, intelligent transportation systems, machine learning and so on.

In 2006, Tada and Takahashi [22] proposed the following algorithm for finding a common solution of the monotone equilibrium problem and the fixed point problem of a nonexpansive mapping in Hilbert spaces:

$$\begin{cases} x_0 \in C_0 = Q_0 = C, \\ z_n \in C \text{ such that } G(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C, \\ w_n = \alpha_n x_n + (1-\alpha_n) T z_n, \\ C_n = \{v \in C : \|w_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0. \end{cases} \tag{1.2}$$

At each step for determining the intermediate approximation z_n, one needs to solve a strongly monotone regularized equilibrium problem:

$$\text{Find } z_n \in C \text{ such that } G(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n \rangle \geq 0, \ \forall y \in C. \tag{1.3}$$

If G is only pseudomonotone, then subproblem (1.3) is not necessarily strongly monotone, even not pseudomonotone. In this case, algorithm (1.2) cannot be applied using the monotonicity of the subproblem. To overcome this difficulty, Ahn [2] proposed the following hybrid extragradient method for finding the common solutions of the pseudomonotone equilibrium problem and the fixed point problem of nonexpansive mappings. Indeed, he considered the following algorithm for finding a solution in $EP(G) \cap F(T)$ and established a strong convergence theorem of common solutions:

$$\begin{cases} x_0 \in C, \\ y_n = \arg\min\{\lambda_n G(x_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \\ t_n = \arg\min\{\lambda_n G(y_n, y) + \frac{1}{2}\|x_n - y\|^2 : y \in C\}, \\ z_n = \alpha_n x_n + (1-\alpha_n) T t_n, \\ C_n = \{v \in C : \|z_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_0 - x_n, v - x_n \rangle \leq 0\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0. \end{cases}$$

The resolvent of the bifunction G with respect to the Legendre function f is the operator $Res_G^f : E \to 2^C$ defined by $Res_G^f = \{z \in C : G(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \ \forall y \in C\}$.

In 1967, Bregman [6] introduced an elegant and effective technique based on a new distance, which is known as the Bregman distance function (defined in Definition 2.2). Recently, many new methods were introduced in the sense of the Bregman distance. In 2013, Agarwal et al. [1] proposed the following algorithm for finding the common solution of the monotone equilibrium problem and the fixed point problem of a weak Bregman relatively nonexpansive mapping in reflexive Banach spaces:

$$\begin{cases} x_0 \in C, \\ z_n = \nabla f^*(\beta_n \nabla f(Tx_n) + (1-\beta_n)\nabla f(x_n)), \\ y_n = \nabla f^*(\alpha_n \nabla f(x_0) + (1-\alpha_n)\nabla f(z_n)), \\ u_n = Res^f_G(y_n), \\ C_n = \left\{ v \in C_{n-1} \cap Q_{n-1} : D_f(z,u_n) \leq \alpha_n D_f(z,x_0) + (1-\alpha_n) D_f(z,x_n) \right\}, \\ Q_n = \{ v \in C_{n-1} \cap Q_{n-1} : \langle \nabla f(x_0) - \nabla f(x_n), v - x_n \rangle \leq 0 \}, \\ x_{n+1} = proj^f_{C_n \cap Q_n} x_0. \end{cases}$$

Under suitable conditions, they proved that the sequence $\{x_n\}$ converges strongly to $x^* = proj^f_{EP(G)\cap F(T)} x_0$.

In 2018, Eskandani et al. [11] proposed a new iterative process for solving the common element of the set of solutions of pseudomonotone equilibrium problems and the set of common fixed points of finite family of multi-valued Bregman relatively nonexpansive mappings via a Bregman hybrid extragradient method in the setting of reflexive Banach spaces. They introduced the Bregman Lipschitz-type condition for a pseudomonotone bifunction and provided the stepsize condition: $\{\lambda_n\} \subset [a,b] \subset (0,p)$, where $p = \min\left\{\frac{1}{c_1}, \frac{1}{c_2}\right\}$ is satisfied, where $c_1 = \max_{1\leq i\leq N}\{c_{i,1}\}$, and $c_2 = \max_{1\leq i\leq N}\{c_{i,2}\}$, and $c_{i,1}$ and $c_{i,2}$ are the Bregman Lipschitz constants of G_i for $i = 1,2,...,N$. Moreover, they proved the strong convergence theorem for the sequence $\{x_n\}$ generated by the following algorithm:

$$\begin{cases} x_0 \in C, \\ w^i_n = \arg\min\left\{ \lambda_n G_i(x_n,y) + D_f(y,x_n), y \in C \right\}, i = 1,2,...,N, \\ z^i_n = \arg\min\left\{ \lambda_n G_i(w^i_n,y) + D_f(y,x_n), y \in C \right\}, i = 1,2,...,N, \\ i_n \in \arg\max\left\{ D_f(z^i_n,x_n) : i = 1,2,...,N \right\}, \text{ set } \bar{z}_n := z^{i_n}_n, \\ y_n = \nabla f^*(\beta_{n,0}\nabla f(\bar{z}_n) + \sum_{r=1}^M \beta_{n,r} \nabla f(z_{n,r})), \ z_{n,r} \in T_r(\bar{z}_n), \\ x_{n+1} = proj^f_C(\nabla f^*(\alpha_n \nabla f(u_n) + (1-\alpha_n)\nabla f(y_n))). \end{cases} \tag{1.4}$$

Recently, Taiwo et al. [23] proposed the following parallel hybrid extragradient algorithm for approximating the common solution of pseudomonotone equilibrium problems and the split common fixed point problems of Bregman weak relatively nonexpansive mappings by using the concept of the Bregman W-mapping. They proved the strong convergence of the sequence $\{x_n\}$ generated by the following algorithm in p-uniformly convex and uniformly smooth Banach spaces:

$$\begin{cases} x_0 \in C_0 = Q_0 = C, \\ y^j_n = \arg\min\left\{ \lambda_n G_j(x_n,y) + \Delta_p(y,x_n), y \in C \right\}, j = 1,2,...,M, \\ z^j_n = \arg\min\left\{ \lambda_n G_j(y^j_n,y) + \Delta_p(y,x_n), y \in C \right\}, j = 1,2,...,M, \\ j_n \in \arg\max\left\{ \Delta_p(z^j_n,x_n) : j = 1,2,...,M \right\}, \text{ set } \bar{z}_n := z^{j_n}_n, \\ t_n = J_q^{E_1^*}(\beta_n J_p^{E_1} x_n + (1-\beta_n) J_p^{E_1} W_n \bar{z}_n), \\ C_n = \left\{ v \in C : \Delta_p(t_n,v) \leq \Delta_p(x_n,v) \right\}, \\ Q_n = \left\{ v \in C : \langle J_p^{E_1} x_0 - J_p^{E_1} x_n, v - x_n \rangle \leq 0 \right\}, \\ x_{n+1} = \prod_{C_n \cap Q_n} x_0. \end{cases}$$

Motivated and inspired by above results, we propose a Bregman extragradient method for finding a common solution of a finite family of pseudomonotone equilibrium problems and the common fixed point problem of a finite family of Bregman relatively nonexpansive mappings. Further, the stepsize of our algorithm is determined by a self-adaptive technique, and we prove that the sequences generated by the proposed iterative method are strongly convergent without prior estimate of the Bregman-Lipschitz constants. Moreover, we provide an application of our result to variational inequality problems and give some numerical examples to illustrate the strong convergence theorem. This paper mainly improves the works presented in [10, 11, 13, 14, 23].

2. Preliminaries

In this section, we recall some definitions and basic facts that we will need in the sequel. Let E be a reflexive Banach space with norm $\|\cdot\|$. We denote the dual space of E by E^*. Throughout this paper, we shall assume that $f : E \to (-\infty,+\infty]$ is a proper, lower semi-continuous, and convex function. We denote by $\text{dom}f := \{x \in E : f(x) < +\infty\}$ the domain of f. Let $x \in \text{int}(\text{dom}f)$, where $\text{int}(\text{dom}f)$ stands for the interior of the domain of f, and the subdifferential of f at x is the convex set defined by $\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y-x\rangle \leq f(y),\ \forall y \in E\}$, where the Fenchel conjugate of f is the function $f^* : E^* \to (-\infty,+\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}$. A function f on E is said to be *strong coercive* if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For $x \in \text{int}(\text{dom}f)$ and $y \in E$, define the directional derivative of f at x by

$$f^0(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}. \tag{2.1}$$

If the limit as $t \to 0^+$ in (2.1) exists for each y, then the function f is said to be *Gâteaux differentiable at* x. In this case, the gradient of f at x is the linear function $\nabla f(x)$, which is defined by $\langle \nabla f(x), y\rangle := f^0(x,y)$ for all $y \in E$. The function f is said to be *Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in \text{int}(\text{dom}f)$. When the limit as $t \to 0$ in (2.1) is attained uniformly for any $y \in E$ with $\|y\| = 1$, we say that f is *Fréchet differentiable* at x. Finally, f is said to be *uniformly Fréchet differentiable* on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. In this paper, we will take $f : E \to (-\infty,+\infty]$ to be an admissible function, that is, a proper, lower semi-continuous convex, and Gâteaux differentiable function. Under these conditions, we know that f is continuous in $\text{int}(\text{dom}f)$; see, e.g., [3] and the references therein. The Legendre function f is defined from a general Banach space E into $(-\infty,+\infty]$; see, e.g., [3]. It is known that, in reflexive spaces, f is the Legendre function if and only if it satisfies the following conditions:

(L_1) $\text{int}(\text{dom}f) \neq \emptyset$, f is Gâteaux differentiable on $\text{int}(\text{dom}f)$ and $\text{dom}\nabla f = \text{int}(\text{dom}f)$;
(L_2) $\text{int}(\text{dom}f^*) \neq \emptyset$, f^* is Gâteaux differentiable on $\text{int}(\text{dom}f^*)$ and $\text{dom}\nabla f^* = \text{int}(\text{dom}f^*)$.

Remark 2.1. [3] If E is a reflexive Banach space, and $f : E \to (-\infty,+\infty]$ is the Legendre function, then all of the following conditions are true:

(a) f is the Legendre function if and only if f^* is the Legendre function;
(b) $(\partial f)^{-1} = \partial f^*$;

(c) $\nabla f = (\nabla f^*)^{-1}$, $\text{ran}\nabla f = \text{dom}\nabla f^* = \text{int}(\text{dom} f^*)$ and $\text{ran}\nabla f^* = \text{dom}\nabla f = \text{int}(\text{dom} f)$;
(d) the functions f and f^* are strictly convex on the interior of respective domains.

The examples of Legendre functions were given in [3]. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is the Legendre function.

Definition 2.2. [6] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman distance with respect to f is the bifunction $D_f : \text{dom} f \times \text{int}(\text{dom} f) \longrightarrow [0, +\infty)$ defined by $D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle$.

It is obvious from the definition of D_f that

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle,$$

the three point identity

Definition 2.3. [6] Let C be a nonempty, closed, and convex subset of $\text{int}(\text{dom} f)$, and let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function. The Bregman projection with respect to f of $x \in \text{int}(\text{dom} f)$ onto C is defined as the necessarily unique vector $proj_C^f(x) \in C$, which satisfies $D_f(proj_C^f(x), x) = \inf\{D_f(y,x) : y \in C\}$.

Definition 2.4. [7] Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and convex function, and let $v_f : \text{int}(\text{dom} f) \times [0, +\infty) \longrightarrow [0, +\infty)$, be the modulus of total convexity of the function f at x defined by $v_f(x,t) := \inf\{D_f(y,x) : y \in \text{dom} f, \|y - x\| = t\}$. Then function f is said to be

(a) totally convex at a point $x \in \text{int}(\text{dom} f)$ if the modulus of the total convexity of function f at x is positive, that is, $v_f(x,t) > 0$ whenever $t > 0$;
(b) totally convex if it is totally convex at every point $x \in \text{int}(\text{dom} f)$. Let B be a nonempty bounded subset of E, and define the modulus of total convexity of the function f on the set B by $v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap \text{dom} f\}$;
(c) totally convex on bounded sets if the modulus of the total convexity of function f on set B is positive, $v_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$.

Let C be a nonempty, closed, and convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable and totally convex function on $\text{int}(\text{dom} f)$ and $x \in E$. It is known from [8] that $z = proj_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

We also have

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \quad \forall x \in E, y \in C.$$

Lemma 2.5. [17] *Let E be a Banach space, and $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function which is uniformly convex on bounded subsets of E. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in E. Then $\lim_{n\to\infty} D_f(x_n, y_n) = 0$ if and only if $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.6. *Let E be a reflexive Banach space. Let $f : E \to \mathbb{R}$ be a strong coercive function, and let $V_f : E \times E^* \longrightarrow [0, +\infty)$ be defined by $V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*)$, $\forall x \in E, x^* \in E^*$. Then the following assertions hold:*

(1) $V_f(x,x^*) = D_f(x, \nabla f^*(x^*)), \quad \forall x \in E, x^* \in E^*;$
(2) $V_f(x,x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*.$

Lemma 2.7. [17] *Let E be a Banach space. Let $t>0$ be a constant and $f:E\to\mathbb{R}$ be a uniformly convex function on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{n}\alpha_k x_k\right)\leq\sum_{k=0}^{n}\alpha_k f(x_k)-\alpha_i\alpha_j\rho_r\left(\|x_i-x_j\|\right),$$

for all $i,j\in(0,1,2,...,n)$, $x_k\in rB,\alpha_k\in(0,1)$ and $k=0,1,2,...,n$ with $\sum_{k=0}^{n}\alpha_k=1$, where ρ_r is the gauge of uniformly convexity of f.

Lemma 2.8. [18] *Let $f:E\to(-\infty,+\infty]$ be a proper, lower semicontinuous, and convex function. Then $f^*:E^*\to(-\infty,+\infty]$ is proper weak* lower semicontinuous and convex. Hence, V_f is convex in the second variable. Thus, for all $z\in E$,*

$$D_f\left(z,\nabla f^*\left(\sum_{i=1}^{N}t_i\nabla f(x_i)\right)\right)\leq\sum_{i=1}^{N}t_iD_f(z,x_i),$$

where $\{x_i\}_{i=1}^{N}\subset E$ and $\{t_i\}_{i=1}^{N}\subset(0,1)$ with $\sum_{i=1}^{N}t_i=1$.

Lemma 2.9. [19] *Let $f:E\to\mathbb{R}$ be a totally convex function. If $x_0\in E$ and the sequence $\{D_f(x_0,x_n)\}$ is bounded, then $\{x_n\}$ is bounded.*

Let C be a convex subset of $\text{int}(\text{dom}f)$, and let T be a self-mapping of C. A point p in C is said to be an *asymptotic fixed point* of T if there exists a sequence $\{x_n\}$ in C that it converges weakly to p and $\lim_{n\to\infty}\|x_n-Tx_n\|=0$. The set of asymptotic fixed points of T is denoted by $\hat{F}(T)$.

Definition 2.10. Let C be a nonempty, closed, and convex subset of E. A mapping $T:C\to C$ is said to be

(i) Bregman firmly nonexpansive (BFNE) if

$$\langle\nabla f(Tx)-\nabla f(Ty),Tx-Ty\rangle\leq\langle\nabla f(x)-\nabla f(y),Tx-Ty\rangle,\ \forall x,y\in C;$$

(ii) Bregman relatively nonexpansive if $\hat{F}(T)=F(T)\neq\emptyset$ and

$$D_f(p,Tx)\leq D_f(p,x),\ \forall x\in C,\ p\in F(T).$$

Next, $CB(C)$ is used to denote the family of nonempty closed bounded subsets of C.

Lemma 2.11. *([20]) Let E be a reflexive Banach space, and let $f:E\to\mathbb{R}$ be uniformly Fréchet differentiable and totally convex on bounded subsets of E. Let C be a nonempty, closed, and convex subset of $int(domf)$, and let $T:C\to CB(C)$ be a Bregman relatively nonexpansive mapping. Then $F(T)$ is closed and convex.*

Lemma 2.12. [19] *Let $f:E\to(-\infty,+\infty]$ be a Gâteaux differentiable and totally convex function. Let x_0 be an element in E, and let C be a nonempty, closed, and convex subset of E. Let $\{x_n\}$ be a bounded sequence and the weak limits of any subsequence of a sequence $\{x_n\}$ belong to $C\subset E$. If $D_f(x_n,x_0)\leq D_f(proj_C^f(x_0),x_0)$ for any $n\in\mathbb{N}$, then $\{x_n\}$ converges strongly to $proj_C^f(x_0)$.*

Lemma 2.13. [24] *Let C be a nonempty convex subset of E, and let $f:C\to\mathbb{R}$ be a convex and subdifferentiable function on C. Then, f attains its minimum at $x\in C$ if and only if $0\in\partial f(x)+N_C(x)$, where $N_C(x)$ is the normal cone of C at x, that is,*

$$N_C(x):=\{x^*\in E^*:\langle x-z,x^*\rangle\geq0,\ \forall z\in C\}.$$

Lemma 2.14. [11] *Let $f : E \to \mathbb{R}$ be a convex strong coercive lower semicontinuous Gâteaux differentiable and cofinite function. Let $G : C \times C \to (-\infty, +\infty]$ be a function such that $G(x,\cdot)$ is proper convex and lower semicontinuous on C for every fixed $x \in C$. Then, for every $\lambda \in (0,\infty)$ and $x \in C$, there exists $z \in C$ such that $z \in \arg\min\left\{\lambda G(x,y) + D_f(y,x) : y \in C\right\}$. Furthermore, if f is strictly convex, then this point is unique.*

Assumption 2.15. The bifunction $G : C \times C \to \mathbb{R}$ satisfies the following assumptions:

(A1) $G(x,x) = 0$, $\forall x \in C$;
(A2) G is pseudomonotone, i.e., $G(x,y) \geq 0$ and $G(y,x) \leq 0$ for all $x, y \in C$;
(A3) G is a Bregman Lipschitz condition, i.e., there exist two positive constants c_1, c_2 such that $G(x,y) + G(y,z) \geq G(x,z) - c_1 G(y,x) - c_2 G(z,y)$, $\forall x,y,z \in C$;
(A4) $G(\cdot,y)$ is continuous on C for all $y \in C$;
(A5) $G(x,\cdot)$ is convex lower semicontinuous and subdifferentiable on C for every fixed $x \in C$.

3. Main Results

In this section, we present our algorithm and prove a strong convergence theorem of common solutions. Let E be a real reflexive Banach space. Let C be a nonempty, closed, and convex subset of E. Let $G_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying (A1)-(A5) for $i = 1, 2, ..., N$. For $j = 1, 2, ..., M$, let $T_j : C \to C$ be a finite family of Bregman relatively nonexpansive mappings. Let $f : C \to \mathbb{R}$ be uniformly Fréchet differentiable, strong coercive, Legendre, totally convex, and bounded on bounded subsets of E. Suppose that the solution set $\Omega := \cap_{i=1}^{N} EP(G_i) \cap \cap_{j=1}^{M} F(T_j) \neq \emptyset$.

Algorithm 3.1. Choose $\lambda_0 > 0$ and $x_0 \in C$:

$$\left\{\begin{array}{l} y_n^i = \arg\min\{\lambda_n G_i(x_n, y) + D_f(y, x_n) : y \in C\} \quad i = 1, 2, ..., N, \\ z_n^i = \arg\min\{\lambda_n G_i(y_n^i, y) + D_f(y, x_n) : y \in C\} \quad i = 1, 2, ..., N, \\ i_n \in \arg\max\{D_f(z_n^i, x_n), i = 1, 2, ..., N\}, \;\; \bar{z}_n := z_n^{i_n}, \\ t_n = \nabla f^*(\alpha_n^0 \nabla f(x_n) + \sum_{j=1}^{M} \alpha_n^j \nabla f(T_j \bar{z}_n)), \\ C_n = \{v \in C : D_f(v, t_n) \leq D_f(v, x_n)\}, \\ Q_n = \{v \in C : \langle \nabla f(x_0) - \nabla f(x_n), v - x_n\rangle \leq 0\}, \\ x_{n+1} = proj_{C_n \cap Q_n}^{f} x_0, \end{array}\right. \tag{3.1}$$

and

$$\lambda_{n+1} = \left\{\begin{array}{l} \min\left\{\lambda_n, \min\limits_{1 \leq i \leq N}\left\{\dfrac{\mu\left(D_f(x_n, y_n^i) + D_f(y_n^i, z_n^i)\right)}{G(x_n, z_n^i) - G(x_n, y_n^i) - G(y_n^i, z_n^i)}\right\}\right\}, \\ \qquad\qquad \text{if } G(x_n, z_n^i) - G(x_n, y_n^i) - G(y_n^i, z_n^i) \neq 0 \\ \lambda_n, \quad \text{otherwise} \end{array}\right. \tag{3.2}$$

where $\{\alpha_n^j\} \subset (0,1), \sum_{j=0}^{M} \alpha_n^j = 1$, and $\liminf_{n\to\infty} \alpha_n^0 \alpha_n^j > 0$ for all $1 \leq j \leq M$ and $n \in \mathbb{N}$.

Lemma 3.2. *Let $\{x_n\}, \{y_n^i\}$, and $\{z_n^i\}$ be the sequences generated by Algorithm 3.1. Then, for $i = 1, 2, ..., N$, the following inequality holds for each $x^* \in EP(G_i)$*

$$D_f(x^*, z_n^i) \leq D_f(x^*, x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) - D_f(y_n^i, x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(z_n^i, y_n^i).$$

Proof. Using Lemma 2.14, we conclude that $\{y_n^i\}$ and $\{z_n^i\}$ are well-defined. From the definition of z_n^i and Lemma 2.13, we have $0 \in \lambda_n \partial_2 G_i(y_n^i, z_n^i) + \nabla_1 D_f(z_n^i, x_n) + N_C(z_n^i)$. Hence, there exist $w_n^i \in \partial_2 G_i(y_n^i, z_n^i)$ and $\bar{w}_n^i \in N_C(z_n^i)$ such that

$$0 = \lambda_n w_n^i + \nabla f(z_n^i) - \nabla f(x_n) + \bar{w}_n^i. \tag{3.3}$$

Using the definition of $\partial_2 G_i(y_n^i, z_n^i)$, we obtain $G_i(y_n^i, y) - G_i(y_n^i, z_n^i) \geq \langle w_n^i, y - z_n^i \rangle$, $i = 1, 2, \ldots, N$ for all $y \in C$. Replacing y with x^* in the above inequality, we arrive at

$$G_i(y_n^i, x^*) - G_i(y_n^i, z_n^i) \geq \langle w_n^i, x^* - z_n^i \rangle, \quad i = 1, 2, \ldots, N. \tag{3.4}$$

From the difinition of $N_C(\cdot)$ and (3.3), we obtain $\langle \nabla f(z_n^i) - \nabla f(x_n), y - z_n^i \rangle \geq \lambda_n \langle w_n^i, z_n^i - y \rangle$, for all $y \in C$. Substituting $y = x^*$ into (3.5), we obtain

$$\langle \nabla f(z_n^i) - \nabla f(x_n), x^* - z_n^i \rangle \geq \lambda_n \langle w_n^i, z_n^i - x^* \rangle. \tag{3.5}$$

Using (3.4) and (3.5), we have

$$\langle \nabla f(z_n^i) - \nabla f(x_n), x^* - z_n^i \rangle \geq \lambda_n \left(G_i(y_n^i, z_n^i) - G_i(y_n^i, x^*) \right) \geq \lambda_n G_i(y_n^i, z_n^i). \tag{3.6}$$

Similarly, since $y_n^i = \arg\min \left\{ \lambda_n G_i(x_n, y) + D_f(y, x_n) : y \in C \right\}$, we have $\langle \nabla f(y_n^i) - \nabla f(x_n), y_n^i - z_n^i \rangle \leq \lambda_n \left(G_i(x_n, z_n^i) - G_i(x_n, y_n^i) \right)$. Therefore, combining the last two inequalities with (3.2), we arrive at

$$\begin{aligned}
&\langle \nabla f(x_n) - \nabla f(z_n^i), x^* - z_n^i \rangle + \langle \nabla f(x_n) - \nabla f(y_n^i), z_n^i - y_n^i \rangle \\
&\geq \lambda_n \left(G(x_n, z_n^i) - G(x_n, y_n^i) - c_1 D_f(y_n^i, x_n)) \right) \\
&\geq \lambda_{n+1} (\langle \nabla f(y_n^i) - \nabla f(x_n), y_n^i - z_n^i \rangle - \lambda_n c_1 D_f(y_n^i, x_n) - \lambda_n c_2 D_f(z_n^i, y_n^i)).
\end{aligned}$$

Combining the last inequality with the three point identity, we conclude taht

$$D_f(x^*, x_n) \leq D_f(x^*, x_n) - D_f(y_n^i, x_n) - D_f(z_n^i, y_n^i) + \lambda_n \left(G_i(x_n, z_n^i) - G_i(x_n, y_n^i) - G_i(y_n^i, z_n^i) \right).$$

Additional, from the definition of λ_n, we have

$$\begin{aligned}
D_f(x^*, z_n^i) &\leq D_f(x^*, x_n) - D_f(y_n^i, x_n) - D_f(z_n^i, y_n^i) \\
&\qquad + \frac{\lambda_n}{\lambda_{n+1}} \lambda_{n+1} \left(G_i(x_n, z_n^i) - G_i(x_n, y_n^i) - G_i(y_n^i, z_n^i) \right) \\
&\leq D_f(x^*, x_n) - D_f(y_n^i, x_n) - D_f(z_n^i, y_n^i) + \frac{\lambda_n}{\lambda_{n+1}} \mu \left(D_f(y_n^i, x_n) + D_f(z_n^j, y_n^i) \right) \\
&= D_f(x^*, x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \mu \right) D_f(y_n^i, x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}} \mu \right) D_f(z_n^i, y_n^i).
\end{aligned}$$

□

Lemma 3.3. *For each $n \geq 0$, $\Omega \subset C_n \cap Q_n$, and $\{x_n\}$ is well-defined.*

Proof. In view of the definitions of Q_n and C_n, we easily conclude that both are closed and convex for all $n \geq 0$. Therefore, $C_n \cap Q_n$ is also closed and convex for all $n \geq 0$. Fix $p \in \Omega$.

Then

$$\begin{aligned} D_f(p,t_n) &\leq \alpha_n^0 D_f(p,x_n) + \sum_{j=1}^{M} \alpha_n^j D_f(p,T_j\bar{z}_n) \\ &\leq \alpha_n^0 D_f(p,x_n) + \sum_{j=1}^{M} \alpha_n^j D_f(p,\bar{z}_n) \\ &\leq \alpha_n^0 D_f(p,x_n) + \sum_{j=1}^{M} \alpha_n^j D_f(p,x_n) \\ &= D_f(p,x_n). \end{aligned} \tag{3.7}$$

This implies that $p \in C_n$ and $\Omega \subset C_n$.

Next, we show by induction that $\Omega \subset C_n \cap Q_n$ for all $n \geq 0$. The definition of Q_n yields that $\Omega \subset C_0 \cap Q_0$. Let $\Omega \subset C_k \cap Q_k$ for some $k > 0$. Since $C_k \cap Q_k$ is closed and convex, one concludes from the definition of the Bregman projection that there exists $x_{k+1} \in C_k \cap Q_k$ such that $x_{k+1} = proj^f_{C_k \cap Q_k} x_0$. From inequality (2.2), we have $\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - p \rangle \geq 0$, $\forall z \in C_k \cap Q_k$. Since $\Omega \subset C_k \cap Q_k$, one has $\langle \nabla f(x_0) - \nabla f(x_{k+1}), x_{k+1} - p \rangle \geq 0$, $\forall p \in \Omega$, and hence $p \in Q_{k+1}$. Since $\Omega \subset C_n$ for all $n \geq 0$, $\Omega \subset C_{k+1} \cap Q_{k+1}$. Thus, $\Omega \subset C_n \cap Q_n$. Furthermore $x_{n+1} = proj^f_{C_n \cap Q_n} x_0$ is well-defined for all $n \geq 0$. This means that $\{x_n\}$ is well-defined. □

Lemma 3.4. *If $x_{n+1} = x_n$ for all $n \in \mathbb{N}$, then $x_n \in \Omega$.*

Proof. If $x_{n+1} = proj^f_{C_n \cap Q_n} x_0 = x_n$, then the definition of C_n yields that $D_f(x_n,t_n) \leq D_f(x_n,x_n)$. Since f is the Legendre function and $D_f(x_n,t_n) = 0$, we have $x_n = t_n$. Using the relation $t_n = \nabla f^* \left(\alpha_n^0 \nabla f^*(x_n) + \sum_{j=1}^M \alpha_n^j \nabla f(T_j\bar{z}_n) \right)$, we obtain that $\nabla f(t_n) = \alpha_n^0 \nabla f(x_n) + \sum_{j=1}^M \alpha_n^j \nabla f(T_j\bar{z}_n)$, which implies that $\alpha_n^0 (\nabla f(t_n) - \nabla f(x_n)) = \sum_{j=1}^M \alpha_n^j \left(\nabla f(T_j\bar{z}_n) - \nabla f(t_n) \right)$, and $T_j\bar{z}_n = t_n$. On the other hand, let $x^* \in \Omega$. Then

$$\begin{aligned} D_f(x^*,t_n) &= D_f(x^*,T_j\bar{z}_n), \quad \forall j = 1,2,\ldots,M \\ &\leq D_f(x^*,\bar{z}_n) \\ &\leq D_f(x^*,x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(y_n^{i_n},x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(\bar{z}_n,y_n^{i_n}). \end{aligned} \tag{3.8}$$

This implies that

$$\left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(y_n^{i_n},x_n) - \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\mu\right) D_f(\bar{z}_n,y_n^{i_n}) \leq D_f(x^*,x_n) - D_f(x^*,t_n).$$

So, $D_f(y_n^{i_n},x_n) = 0$ and $D_f(\bar{z}_n,y_n^{i_n}) = 0$. Since f is the Legendre function, we conclude that $x_n = t_n = T_j\bar{z}_n = T_jx_n$ and hence $x_n \in F(T_j)$ for all $j = 1,2,\ldots,M$. Therefore, $x_n \in \bigcap_{j=1}^M F(T_j)$. Furthermore, from $x_n = y_n^i$, we have that $x_n = \arg\min \left\{ \lambda_n G_i(x_n,y) + D_f(y,x_n) : y \in C \right\}$, and also $x_n = \arg\min \left\{ \lambda_n G_i(y_n^i,y) + D_f(y,x_n) : y \in C \right\}$. Thus, we have shown that $x_n \in \bigcap_{i=1}^N EP(G_i) \cap \bigcap_{j=1}^M F(T_j)$. □

Lemma 3.5. *Let $\{x_n\}, \{y_n^i\}, \{z_n^i\}$, and $\{t_n\}$ be the sequences generated by Algorithm 3.1. Then, the following relations hold:* $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\lim_{n\to\infty} \|x_n - t_n\| = 0$, $\lim_{n\to\infty} \|x_n - y_n^i\| = 0$, $\lim_{n\to\infty} \|x_n - z_n^i\| = 0$, *and* $\lim_{n\to\infty} \|x_n - T_jx_n\| = 0$ *for all $i = 1,2,\ldots,N$ and $j = 1,2,\ldots,M$.*

Proof. Observe that $\Omega \subset C_n \cap Q_n$ for every $n \geq 0$ and $x_{n+1} = proj^f_{C_n \cap Q_n} x_0$. Let $w = proj^f_{\Omega} x_0$. It then follows that $D_f(x_0, x_{n+1}) \leq D_f(x_0, w)$, $\forall n \geq 0$. From the fact that $\{D_f(x_0, x_n)\}$ is bounded and Lemma 2.9, we have that $\{x_n\}$ is bounded. Similarly, from (3.7) and Lemma 2.9, we obtain that $\{t_n\}$ is also bounded. Since $x_{n+1} \in C_n \cap Q_n$ and $x_n = proj^f_{Q_n} x_0$, we have $D_f(x_n, x_0) \leq D_f(x_{n+1}, x_0)$, $\forall n \geq 0$. This shows that $\{D_f(x_n, x_0)\}$ is a nondecreasing bounded sequence of $\mathbb{R}$. It follows that $\lim_{n\to\infty} D_f(x_n, x_0)$ exists. Furthermore,

$$\begin{aligned} D_f(x_{n+1}, x_n) &= D_f(x_{n+1}, proj^f_{Q_n} x_0) \\ &\leq D_f(x_{n+1}, x_0) - D_f(proj^f_{Q_n} x_0, x_0) \\ &= D_f(x_{n+1}, x_0) - D_f(x_n, x_0) \longrightarrow 0 \text{ as } n \to \infty. \end{aligned} \tag{3.9}$$

Therefore, Lemma 2.5 sends us to $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$. From (3.9) and the fact that $x_{n+1} \in C_n \cap Q_n$, we obtain that $D_f(x_{n+1}, t_n) \leq D_f(x_{n+1}, x_n) \to 0$ as $n \to \infty$. Using Lemma 2.5 and above inequalities, we obtain that $\|x_{n+1} - t_n\| \to 0$ as $n \to \infty$. Using the fact above, we derive $\|x_n - t_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - t_n\| \to 0$ as $n \to \infty$. So, $\lim_{n\to\infty} |f(x_n) - f(t_n)| = \lim_{n\to\infty} \|\nabla f(x_n) - \nabla f(t_n)\| = 0$. Moreover,

$$\begin{aligned} &D_f(x^*, x_n) - D_f(x^*, t_n) \\ &= f(t_n) - f(x_n) - \langle \nabla f(x_n), x^* - x_n \rangle + \langle \nabla f(t_n), x^* - t_n \rangle \\ &= f(t_n) - f(x_n) - \langle \nabla f(x_n), x^* - t_n \rangle - \langle \nabla f(x_n), t_n - x_n \rangle + \langle \nabla f(t_n), x^* - t_n \rangle \\ &= f(t_n) - f(x_n) - \langle \nabla f(x_n) - \nabla f(t_n), x^* - t_n \rangle - \langle \nabla f(x_n), t_n - x_n \rangle. \end{aligned}$$

It follows that $\lim_{n\to\infty} \big(D_f(x^*, x_n) - D_f(x^*, t_n)\big) = 0$. Let $x^* \in \Omega$. From the Algorithm 3.1 and Lemma 2.7, we obtain

$$\begin{aligned} D_f(x^*, t_n) &= f(x^*) - \langle x^*, \alpha_n^0 \nabla f(x_n) + \sum_{j=1}^M \alpha_n^j \nabla f(T_j \bar{z}_n) \rangle + f^*(\alpha_n^0 \nabla f(x_n) + \sum_{j=1}^M \alpha_n^j \nabla f(T_j \bar{z}_n)) \\ &\leq f(x^*) - \alpha_n^0 \langle x^*, \nabla f(x_n) \rangle - \sum_{j=1}^M \alpha_n^j \langle x^*, \nabla f(T_j \bar{z}_n) \rangle \\ &\quad + \alpha_n^0 f^*(\nabla f(x_n)) + \sum_{j=1}^M \alpha_n^j f^*(\nabla f(T_j \bar{z}_n)) - \alpha_n^0 \alpha_n^j \rho_j^*(\|\nabla f(x_n) - \nabla f(T_j \bar{z}_n)\|) \\ &= \alpha_n^0 f(x^*) + \sum_{j=1}^M \alpha_n^j f(x^*) - \alpha_n^0 \langle x^*, \nabla f(x_n) \rangle - \sum_{j=1}^M \alpha_n^j \langle x^*, \nabla f(T_j \bar{z}_n) \rangle \\ &\quad + \alpha_n^0 f^*(\nabla f(x_n)) + \sum_{j=1}^M \alpha_n^j f^*(\nabla f(T_j \bar{z}_n)) - \alpha_n^0 \alpha_n^j \rho_j^*(\|\nabla f(x_n) - \nabla f(T_j \bar{z}_n)\|) \\ &\leq \alpha_n^0 D_f(x^*, x_n) + \sum_{j=1}^M \alpha_n^j D_f(x^*, \bar{z}_n) - \alpha_n^0 \alpha_n^j \rho_j^*(\|\nabla f(x_n) - \nabla f(T_j \bar{z}_n)\|) \\ &\leq \alpha_n^0 D_f(x^*, x_n) + \sum_{j=1}^M \alpha_n^j \left(D_f(x^*, x_n)\right) - \alpha_n^0 \alpha_n^j \rho_j^*(\|\nabla f(x_n) - \nabla f(T_j \bar{z}_n)\|) \\ &\leq D_f(x^*, x_n) - \alpha_n^0 \alpha_n^j \rho_j^*(\|\nabla f(x_n) - \nabla f(T_j \bar{z}_n)\|), \end{aligned}$$

which implies that $\alpha_n^0\alpha_n^j\rho_j^*(\|\nabla f(x_n)-\nabla f(T_j\bar{z}_n)\|)\leq \big(D_f(x^*,x_n)-D_f(x^*,t_n)\big)\to 0$ as $n\to\infty$. So, we have $\|\nabla f(x_n)-\nabla f(T_j\bar{z}_n)\|\to 0$ as $n\to\infty$, $\forall j=1,2,\dots,M$, which implies that $\|x_n-T_j\bar{z}_n\|\to 0$ as $n\to\infty$, $\forall j=1,2,\dots,M$. Hence, $D_f(x_n,T_j\bar{z}_n)\to 0$ as $n\to\infty$. From Lemma 3.4, we have $D_f(T_j\bar{z}_n,T_jx_n)\leq D_f(T_j\bar{z}_n,x_n)\to 0$ as $n\to\infty$, which implies that $\|T_j\bar{z}_n-T_jx_n\|\to 0$. Therefore,

$$\|x_n-T_jx_n\|\leq \|x_n-T_j\bar{z}_n\|+\|T_j\bar{z}_n-T_jx_n\|\to 0 \text{ as } n\to\infty. \tag{3.10}$$

From (3.8), we have

$$\left(1-\frac{\lambda_n}{\lambda_{n+1}}\mu\right)D_f(y_n^{i_n},x_n)+\left(1-\frac{\lambda_n}{\lambda_{n+1}}\mu\right)D_f(\bar{z}_n,y_n^{i_n})\leq D_f(x^*,x_n)-D_f(x^*,t_n).$$

Taking $n\to\infty$ in the inequality above, we have $\lim_{n\to\infty}D_f(y_n^{i_n},x_n)=\lim_{n\to\infty}D_f(\bar{z}_n,y_n^{i_n})=0$. This implies that $\lim_{n\to\infty}D_f(y_n^i,x_n)=\lim_{n\to\infty}D_f(z_n^i,y_n^i)=0$, $\forall i=1,2,\dots,N$. So, we have $D_f(y_n^i,x_n)\to 0$ as $n\to\infty$. Similarly, $D_f(z_n^i,y_n^i)\to 0$ as $n\to\infty$ for all $i=1,2,\dots,N$. By Lemma 2.5, we have that

$$\|y_n^i-x_n\|\to 0 \text{ and } \|z_n^i-y_n^i\|\to 0,\ \forall i=1,2,\dots,N. \tag{3.11}$$

Since f is uniformly continuous on bounded subsets of E, we have $\lim_{n\to\infty}|f(x_n)-f(y_n^i)|=0$ and $\lim_{n\to\infty}\|\nabla f(x_n)-\nabla f(y_n^i)\|=0$. Since $\|x_n-z_n^i\|\leq\|x_n-y_n^i\|+\|y_n^i-z_n^i\|$, we conclude from (3.11) that $\|x_n-z_n^i\|\to 0$ as $n\to\infty$, $\forall i=1,2,\dots,N$. □

Theorem 3.6. *Let $\{x_n\},\{y_n^i\}$, and $\{z_n^i\}$ be the sequences generated by Algorithm 3.1. Then there exists $x^*\in C$ such that $x^*\in\Omega:=\bigcap_{j=1}^M F(T_j)\cap\bigcap_{i=1}^N EP(G_i)$ and $\{x_n\}$ converges strongly to $x^*=proj_\Omega^f(x_0)$.*

Proof. We first show that there exists $x^*\in\bigcap_{j=1}^M F(T_j)$. Since $\{x_n\}$ is bounded and E is a reflexive Banach space, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$. It follows from (3.11), there exist subsequences $\{y_{n_k}^i\}$ of $\{y_n^i\}$ and $\{z_{n_k}^i\}$ of $\{z_n^i\}$ such that $y_{n_k}^i\rightharpoonup x^*$ and $z_{n_k}^i\rightharpoonup x^*$ for all $i=1,2,\dots,N$, respectively. In view of the asymptotic fixed point with (3.10), we conclude that x^* is an asymptotic fixed point of T_j. Since T_j is Bregman relatively nonexpansive, x^* is a fixed point of T_j for all $j=1,2,\dots,M$, then $x^*\in\bigcap_{j=1}^M F(T_j)$. Now, we show that $x^*\in\bigcap_{i=1}^N EP(G_i)$. Observe that $y_n^i=\arg\min\{\lambda_nG_i(x_n,y)+D_f(y,x_n):y\in C\}$. As (3.6), we obtain that $\lambda_n(G_i(x_n,y)-G_i(x_n,y_n^i))\geq\langle\nabla f(y_n^i)-\nabla f(x_n),y_n^i-y\rangle$, for all $y\in C$. This together with (3.11) and Assumption 2.15 concludes that $\lambda_n(G_i(x^*,y)-G_i(x^*,x^*))\geq 0$. The last inequality gives that $G_i(x^*,y)\geq 0$ for all $i=1,2,\dots,N$. Hence $x^*\in\bigcap_{i=1}^N EP(G_i)$. Thus, we have shown that $x^*\in\Omega$. Finally, we prove that $\{x_n\}$ converges strongly to $x^*=proj_\Omega^f(x_0)$. It follows from the definition of the Bregman projection together with the fact that Ω is a nonempty closed convex subset of E, we obtain that $proj_\Omega^f(x_0)$ is well-defined. Let $x^*=proj_\Omega^f(x_0)$. In view of $x_{n+1}=proj_{C_n\cap Q_n}^f(x_0)$ and $proj_\Omega^f(x_0)\in\Omega\subseteq C_n\cap Q_n$, we obtain that $D_f(x_{n+1},x_0)\leq D_f(proj_\Omega^f(x_0),x_0)$. It follows from Lemma 2.12 that $\{x_n\}$ converges strongly to $x^*=proj_\Omega^f(x_0)$. This completes the proof. □

4. Application to Variational Inequality Problems

In this section, we discuss an application of Theorem 3.6 to finding a solution of a variational inequality problem with a pseudomonotone mapping.

Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let E^* be dual of E, and let $A: C \to E^*$ be a nonlinear mapping. A is said to be pseudomonotone on C if

$$\langle Ax, y-x\rangle \geq 0 \Longrightarrow \langle Ax, x-y\rangle \leq 0,\ \forall x,y \in C.$$

We state the variational inequality problem as follows:

$$\text{Find } x^* \in C \text{ such that } \langle Ax^*, x-x^*\rangle \geq 0,\ \forall x \in C. \tag{4.1}$$

Note that if we take $G(x,y) := \langle Ax, y-x\rangle$ for all $x,y \in C$, then the equilibrium problem converts into the above variational inequality problem. If E is a real Hilbert space, it is well-known that x^* is a solution of (4.1) if and only if x^* solves the fixed point equation

$$x^* = P_C(x^* - \lambda Ax^*),\ \forall \lambda > 0.$$

It was shown in [11] that if $\{A_i\}_{i=1}^N$ is a finite family of pseudomonotone and L_i-Lipschitz continuous mappings from C to E^*, then $G_i(x,y) = \langle A_i x, y-x\rangle$ is pseudomonotone and Bregman Lipschitz continuous. The following lemma follows from [11, Lemma 4.1].

Lemma 4.1. *Let C be a nonempty, closed, and convex subset of a reflexive Banach space E. Let $A: C \to E^*$ be a mapping, and $f: E \to \mathbb{R}$ be a Legendre function. Then $proj_C^f(\nabla f^*(\nabla f(x) - \lambda A(y))) = \arg\min_{w\in C}\{\lambda\langle Ay, w-y\rangle + D_f(w,x)\}$, for all $x \in E, y \in C$ and $\lambda \in (0,+\infty)$.*

Let E be a real Banach space and $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$. For the p-uniformly convex space, the metric and Bregman distance have the following relation [21]:

$$\tau\|x-y\|^p \leq D_{\frac{1}{p}\|\cdot\|^p}(x,y) \leq \langle J_E^p(x) - J_E^p(y), x-y\rangle,$$

where $\tau > 0$ is a fixed number, and duality mapping $J_E^p : E \to 2^{E^*}$ is defined by

$$J_E^p(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^p, \|f\| = \|x\|^{p-1}\},$$

for all $x \in E$. We know that if E is a smooth strictly convex and reflexive Banach space, then J_E^p is a single-valued bijection. In this case, $J_E^p = (J_{E^*}^q)^{-1}$, where $J_{E^*}^q$ is the duality mapping of E^*. For $p = 2$, the duality mapping J_E^p is called the normalized duality and is denoted by J. The function $\phi : E^2 \to \mathbb{R}$ is defined by

$$\phi(y,x) = \|y\|^2 - 2\langle y, Jx\rangle + \|x\|^2, \tag{4.2}$$

for all $x,y \in E$. The generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x),\ \forall x \in E,$$

where C is a nonempty, closed, and convex subset of E.

We consider the case that $p = 2$ and the Lyapunov function (4.2). Furthermore, if A_i is L_i-Lipschitz continuous, then G_i is Bregman-Lipschitz-type continuous with $c_{i,1} = c_{i,2} = \frac{L_i}{2\tau}$. Let E be a uniformly smooth and uniformly convex Banach space, and $f(\cdot) = \frac{1}{2}\|\cdot\|^2$. Then, $\nabla f = J, D_{\frac{1}{2}\|\cdot\|^2}(x,y) = \frac{1}{2}\phi(x,y)$ and $proj_C^{\frac{1}{2}\|\cdot\|^2} = \Pi_C$. In particular, if E is a Hilbert space, then $\nabla f = I, D_{\frac{1}{2}\|\cdot\|^2}(x,y) = \frac{1}{2}\|x-y\|^2$ and $proj_C^{\frac{1}{2}\|\cdot\|^2} = P_C$, where P_C is the metric projection. Therefore, the following corollary follows from our main result.

Corollary 4.2. *Let C be a nonempty, closed, and convex subset of a uniformly smooth and 2-uniformly convex Banach space E. Suppose that $\{A_i\}_{i=1}^N$ is a finite family of pseudomonotone and L_i-Lipschitz continuous mappings from C to E^*. Assume that, for each $1 \le j \le M$, $T_j : C \to C$ is a relatively nonexpansive mapping, that is, $F(T_j) = \hat{F}(T_j) \neq \emptyset$ and $\phi(z, T_j x) \le \phi(z,x)$, for all $x \in C$ and $z \in F(T_j)$. Let $\Omega = \bigcap_{j=1}^M F(T_j) \cap \bigcap_{i=1}^N VIP(A_i, C) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated by the following algorithm: For arbitrary $x_0 \in C_0 = C$:*

$$\begin{cases} y_n^i = \Pi_C(J^{-1}(J(x_n) - \lambda_n A_i(x_n))) \ \ i = 1,2,\ldots,N, \\ z_n^i = \Pi_C(J^{-1}(J(x_n) - \lambda_n A_i(x_n))) \ \ i = 1,2,\ldots,N, \\ i_n \in \arg\max\{\phi(z_n^i, x_n), i = 1,2,\ldots,N\}, \ \ \bar{z}_n := z_n^{i_n}, \\ t_n = J^{-1}(\alpha_n^0 J(x_n) + \sum_{j=1}^M \alpha_n^j J(T_j \bar{z}_n)), \\ C_n = \{v \in C : \phi(v, t_n) \le \phi(v, x_n)\}, \\ Q_n = \{v \in C : \langle J(x_0) - J(x_n), v - x_n \rangle \le 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}^f x_0, \end{cases}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\lambda_n, \min\limits_{1 \le i \le N}\left\{\dfrac{\mu\left(D_f(x_n, y_n^i) + D_f(y_n^i, z_n^i)\right)}{\langle A_i(x_n) - A_i(y_n^i), z_n^i - y_n^i\rangle}\right\}\right\}, \\ \qquad\qquad \text{if } \langle A_i(x_n) - A_i y_n^i, z_n^i - y_n^i \rangle \neq 0 \\ \lambda_n, \quad \text{otherwise} \end{cases}$$

where $\{\alpha_n^j\} \subset (0,1), \sum_{j=0}^M \alpha_n^j = 1$, and $\liminf_{n\to\infty} \alpha_n^0 \alpha_n^j > 0$ for all $i = 1,2,\ldots,N$. Then, $\{x_n\}$ converges strongly to $\Pi_\Omega x_0$.

5. Numerical Example

In this section, we present some numerical examples for Algorithm 3.1. For all the test, we use the stopping criterion $\|x_n - x^*\| \le 10^{-4}$ to study the convergence result of all the algorithms, where x^* is the solution of the considered problem, and $\{x_n\}$ is the sequence generated by each algorithm.

In the first example, we consider the convex function $f : \mathbb{R}_+^m \to \mathbb{R}$, which is defined by $f(x) = \sum_{k=1}^m x_k \ln(x_k)$, given over the set $\mathbb{R}_+^m := \{x \in \mathbb{R}^m : x_k > 0\}$. In this case, we have $\nabla f(x) = (1 + \ln(x_1), 1 + \ln(x_2), \ldots, 1 + \ln(x_m))^T$ and $\nabla f^*(x) = (\exp(x_1 - 1), \exp(x_2 - 1), \ldots, \exp(x_m - 1))^T$ such that $D_f(x,y) = \sum_{k=1}^m (y_k - x_k + x_k(\ln(x_k) - \ln(y_k)))$ for all $x, y \in \mathbb{R}_+^m$.

Example 5.1. Let $E = \mathbb{R}^m$ and C be defined by $C = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}_+^m : |x_k| \le 5, \ k = 1,2,\ldots,m\}$. Consider the problem:

$$\text{Find } x \in \Omega := \left(\bigcap_{j=1}^M F(T_j)\right) \cap \left(\bigcap_{i=1}^N EP(G_i)\right),$$

where $T_j : C \to C$ is defined by $T_j(x) = \frac{2x}{1+2j}$, for all $j = 1,2,\ldots,M$ and $x \in \mathbb{R}_+^m$. It can be easily shown that T_j is a Bregman relatively nonexpansive mapping. Also, $G_i : C \times C \to \mathbb{R}$ is defined as $G_i(x,y) = \sum_{k=1}^m (p_{ik} y_k^2 - p_{ik} x_k^2)$, $k = 1,2,\ldots,m$, where $p_{ik} \in (0,1)$ is randomly generated for all $i = 1,2,\ldots,N$, $k = 1,2,\ldots,m$. It is easy to see that the Conditions $(A1) - (A5)$ are satisfied. Moreover, $\Omega := \{x^*\}$, where $x^* = (0,0,\ldots,0)^T$. For each $n \in \mathbb{N}$, we choose $\alpha_n^0 = \alpha_n^j = \frac{1}{M+1}$ and μ, λ_0 generated by randomly in $(0,1)$. We present the performance of Algorithm 3.1 by using the convex function defined above for the following values of M and N:

Case I: $M=5$ and $N=5$;
Case II: $M=7$ and $N=10$;
Case III: $M=10$ and $N=5$.

The initial point x_0 is generated by 10 random starting points, and the present results are on average the numerical results shown in the Table 1. The number of iterations of $\{x_n\}$ is illustrated in Figure 1, Figure 2, and Figure 3.

TABLE 1. Computational result for Example 5.1

$\mathbb{R}^m$		Algorithm 3.1			Algorithm 1.4		
(m)		Case I	Case II	Case III	Case I	Case II	Case III
5	Iter.	9	9	8	46	45	45
	Time (s)	0.622	1.469	1.521	2.125	4.482	5.128
7	Iter.	10	9	10	60	60	61
	Time (s)	0.654	1.641	2.170	5.253	6.949	6.788
10	Iter.	11	10	10	79	80	78
	Time (s)	0.803	1.073	1.228	6.822	7.623	8.215

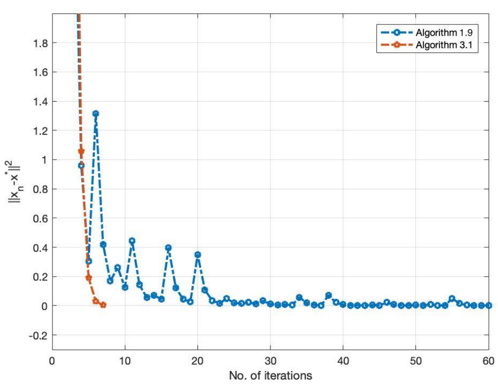


Figure 1: Numbers of iterations in $\mathbb{R}^5$

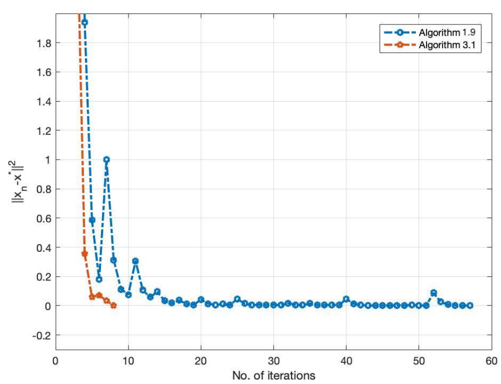


Figure 2: Numbers of iterations in $\mathbb{R}^7$

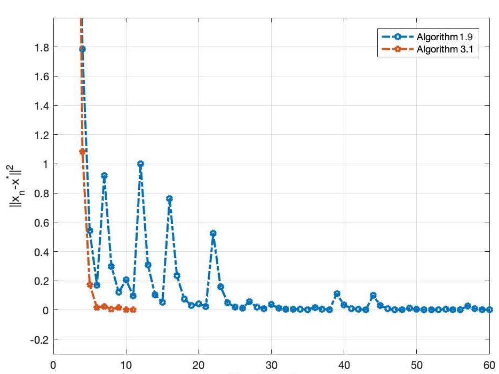


Figure 3: Numbers of iterations in $\mathbb{R}^{10}$

The following example, we present in the case of the convex function $f:\mathbb{R}^m\to\mathbb{R}$ with $f(x)=\frac{1}{2}\|x\|^2$, $\forall x\in\mathbb{R}^m$ such that $\nabla f=\nabla f^*=I$ and $D_f(x,y)=\frac{1}{2}\|x-y\|^2$.

Example 5.2. Let $E=\mathbb{R}^m$ and $C=\{x\in\mathbb{R}^m : -2\le x_k\le 5,\ k=1,2,\dots,m\}$. In this situation, the bifunction $G_i : C\times C\to\mathbb{R}$ can be formulated in the form: $G_i(x,y)=\langle P_ix+Q_iy+q_i, y-x\rangle$, where $q_i\in\mathbb{R}^m$ for $i=1,2,\dots,N$, P_i and Q_i are symmetric positive semidefinite matrices such that Q_i is symmetric positive semidefinite and Q_i-P_i is symmetric negative semidefinite. In this numerical experiment, the matrices P_i and Q_i are generated by randomly in $[-2,5]$. Indeed, the bifunction G_i is pseudomonotone and satisfies a Lipschitz-type condition with $c_{1,i}=c_{2,i}=\frac{1}{2}\|Q_i-P_i\|$. For $j=1,2,\dots,M$, let $T_j : C\to C$ be defined by $T_j(x)=proj^f_{D_j}(x)$, where D_j is randomly generated in the interval $[-1,3]$. So, we obtain that T_j is nonexpansive and so T_j is a Bregman relatively nonexpansive mapping. Moreover, $\Omega:=\{0\}$, for $n\in\mathbb{N}$, we choose the parameter $\alpha_n^0=\alpha_n^j=\frac{1}{M+1}$ and generate the parameters μ,λ_0 randomly in $(0,1)$. The following three cases of the values M and N are considered:

Case I: $M=5$ and $N=5$;

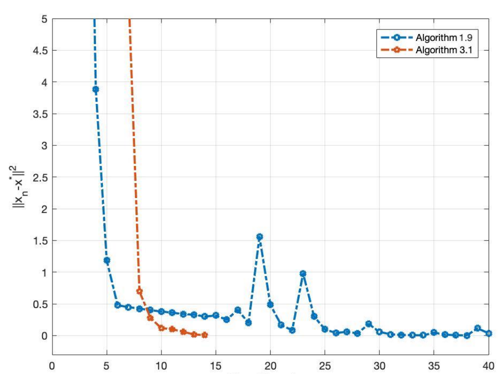
Case II: $M=10$ and $N=15$;

Case III: $M=15$ and $N=10$.

We present the performance of Algorithm 3.1 by randomly generating 10 starting points and shown the results are on average in Table 2. The number of iterations of $\{x_n\}$ is illustrated in Figure 4, Figure 5, and Figure 6.

TABLE 2. Computational result of Example 5.2

$\mathbb{R}^m$		Algorithm 3.1			Algorithm 1.4		
(m)		Case I	Case II	Case III	Case I	Case II	Case III
10	Iter.	15	15	14	87	87	84
	Time (s)	1.23	1.29	1.49	4.34	4.59	4.65
15	Iter.	18	18	16	96	95	96
	Time (s)	1.25	2.01	3.67	4.86	5.51	5.71
20	Iter.	26	26	26	99	97	97
	Time (s)	1.86	2.39	2.55	5.27	8.91	8.85



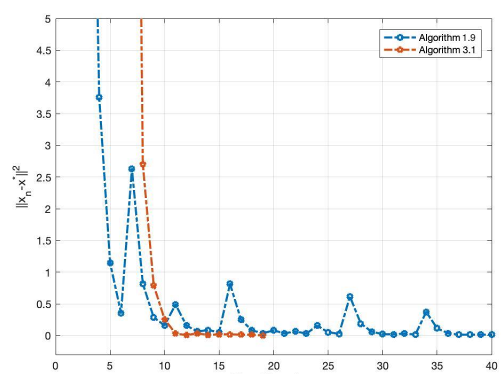


Figure 4: Numbers of iterations in $\mathbb{R}^{10}$

Figure 5: Numbers of iterations in $\mathbb{R}^{15}$

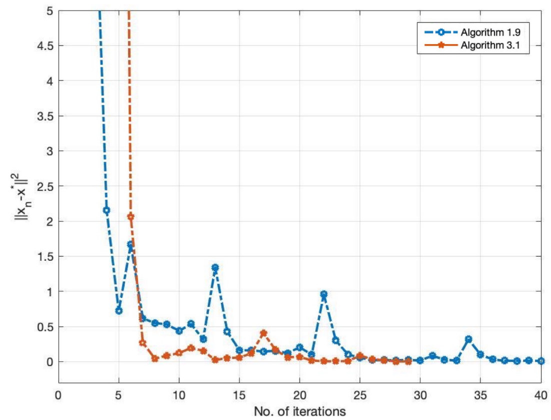


Figure 6: Numbers of iterations in $\mathbb{R}^{20}$

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REFERENCES

[1] R.P. Agarwal, J.W. Chen, Y.J. Cho, Strong convergence theorems for equilibrium problems and weak Bregman relatively nonexpansive mappings in Banach spaces, J. Inequal. Appl. 2013 (2013) 118.

[2] P.N. Anh, A hybrid extragradient method for pseudomonotone equilibrium problems and fixed point problem, Bull. Malays. Math. Sci. Soc. 36 (2013) 107-116.

[3] H.H. Bauschke, J.M. Borwein, P.L. Combettes, Essential smoothness, essential strict convexity, and Legendre function in Banach spaces, Commun. Contemp. Math. 3 (2001) 615–647.

[4] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, Math. Stud. 63 (1994) 123-145.

[5] G. Bigi, M. Castellani, M. Papssacantando, Nonlinear Programming Technique for Equilibria, Springer, Cham, 2019.

[6] L.M. Bregman, The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming, USSR. Comput. Math. Math. Phys. 7 (1967) 200-217.

[7] D. Butnariu, A.N. Iusem, Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, Applied Optimization, Kluwer Academic, Dordrecht, 2000. doi: 10.1007/978-94-011-4066-9.

[8] D. Butnariu, E. Resmerita, Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, Abstr. Appl. Anal. 2006 (2006) 084919.

[9] L.C. Ceng, A subgradient-extragradient method for bilevel equilibrium problems with the constraints of variational inclusion systems and fixed point problems, Commun. Optim. Theory, 2021 (2021) 4.

[10] V. Dadashi, O.S. Iyiola, Y. Shehu, The subgradient extragradient method for pseudomonotone equilibrium problems, Optimization 69 (2020) 901-923.

[11] G.Z. Eskandani, M. Raeisi, T.M. Rassias, A hybrid extragradient method for solving pseudomonotone equilibrium problems using Bregman distance, J. Fixed Point Theory Appl. 20 (2018) 132.

[12] D.V. Hieu, H.N. Duong, B.H. Thai, Convergence of relaxed inertial methods for equilibrium problems, J. Appl. Numer. Optim. 3 (2021) 215-229.

[13] D.V. Hieu, P.K. Anh, Parallel hybrid extragradient methods for pseudomonotone equilibrium problems and nonexpansive mappings, Numer. Algo. 73 (2016) 197-217.

[14] D.V. Hieu, P.K. Quy, Explicit iterative algorithms for solving equilibrium problems, Calcolo, 56 (2019) 11.

[15] P.T. Hoai, N.T. Thuong, N.T. Vinh, Golden ratio algorithms for solving equilibrium problems in Hilbert spaces, J. Nonlinear Var. Anal. 5 (2021) 493-518.

[16] L.D. Muu, W. Oettli, Convergence of an adaptive penalty scheme for finding constrained equilibria, Nonlinear Anal. 18 (1992) 1159-1166.

[17] E. Naraghirad, J.C. Yao, Bregman weak relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2013 (2013) 141.

[18] R.P. Phelps, Convex Functions, Monotone Operators, and Differentiability, Springer, Berlin, 1993.

[19] S. Reich, S. Sabach, Two strong convergence theorems for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim. 31 (2010) 22-44.

[20] N. Shahzad, H. Zegeye, Convergence theorem for common fixed points of a finite family of multi-valued Bregman relatively nonexpansive mappings, Fixed Point Theory Appl. 2014 (2014) 152.

[21] F. Schöpfer, T. Schuster, A.K. Louis, An iterative regularization method for the solution of the split feasibility problem in Banach spaces, Inverse Probl. 24 (2008) 055008.

[22] A. Tada, W. Takahashi, Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem, J. Optim. Theory Appl. 133 (2007) 359–370.

[23] A. Taiwo, L.O. Jolaoso, O.T. Mewomo, Parallel hybrid algorithm for solving pseudomonotone equilibrium and split common fixed point problems, Bull. Malays. Math. Sci. Soc. 43 (2020) 1893-1918.

[24] J.V. Tiel, Convex Analysis: An Introductory Text, Wiley, New York, 1984.

[25] L.H. Yen, L.D. Muu, A normal-subgradient algorithm for fixed point problems and quasiconvex equilibrium problems, Appl. Set-Valued Anal. Optim. 2 (2020) 329-337.