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PMICA - PARALLEL MULTI-STEP INERTIAL CONTRACTING ALGORITHM FOR SOLVING COMMON VARIATIONAL INCLUSIONS

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Abstract. Our study is focused on a common variational inclusion problem in real Hilbert spaces. A parallel inertial algorithm for solving the inclusion problem is proposed and analysed. Our new convergence theorem has several theoretical advantages over some related works in the literature, and primary numerical experiments illustrate the practical potential of our iterative algorithm.

Keywords. Common variational inclusion; Fixed point; Inertial contracting algorithm; Resolvent; Subgradient extragradient method.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. We consider the following common variational inclusion (CVI) problem:

$$\text{find } u^* \in H \quad \text{such that} \quad 0 \in F_i(u^*) + A_i(u^*), \tag{1.1}$$

where $A_i : H \to 2^H$ is a multi-valued operator, and $F_i : H \to H$ is a monotone operator for each $i = 1,2,\ldots,N$. Common variational inclusions are quite general since many nonlinear problems, such as fixed point problems, zero point problems and so on, can be modeled in such a way. If $N = 1$, then the above CVI problem (1.1) is reduced to the following variational inclusions:

$$\text{find } u^* \in H \quad \text{such that} \quad 0 \in F(u^*) + A(u^*), \tag{1.2}$$

where $A : H \to 2^H$ is a multi-valued operator, and $F : H \to H$ is a monotone operator.

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Recently, Yambangwai *et al.* [14] successfully used (1.1) for image restoration problems. As an example for the generality of the CVI (1.1), consider the problem:

$$\min_{x\in\mathbb{R}^n} f_i(x)+g_i(x), \tag{1.3}$$

where, for all $i=1,2,\ldots,N$, $f_i:\mathbb{R}^n\to\mathbb{R}$ is convex and differentiable function, and $g_i:\mathbb{R}^n\to\mathbb{R}$ is proper, convex, and lower semi-continuous function. Using the first order optimality condition, problem (1.3) is equivalent to:

$$\text{find } u^*\in H \quad \text{such that} \quad 0\in\nabla f_i(u^*)+\partial g_i(u^*).$$

Thus, this translates to the CVI (1.1) with $F_i=\nabla f_i$ and $A_i=\partial g_i$, for all $i=1,2,\ldots,N$.

Another interesting example is the common solutions to variational inequality (CSVI) problems. Let $F_i:H\to H$ be a given mapping for each $i=1,2,\ldots,N$. Let K_i be a nonempty, closed, and convex subset of H, and let $A_i=N_{K_i}$, $i=1,2,\ldots,N$ where N_{K_i} is the normal cone of K_i. The CVI problem (1.1) becomes

$$\text{find } u^*\in H \quad \text{such that} \quad 0\in F_i(u^*)+N_{K_i}(u^*), \quad =1,2,\ldots,N. \tag{1.4}$$

Recall that the normal cone of K_i is defined at x as:

$$N_{K_i}(x):=\begin{cases}\{z\in H\mid\langle z,y-x\rangle\le 0 \text{ for all } y\in K_i\}, & \text{if } x\in K_i,\\ \emptyset, & \text{otherwise.}\end{cases}$$

Then the CSVI problem (1.4) can be remodeled as finding $u^*\in K=\cap_{i=1}^N K_i$ such that

$$\langle F_i(u^*),u-u^*\rangle\ge 0,\ \ \forall u\in K_i, i=1,2,\ldots,N, \tag{1.5}$$

where K_i is a nonempty, closed, and convex subset of H, and $F_i:H\to H$ is a monotone operator for each $i=1,2,\ldots,N$.

One of the common practical algorithms for solving variational inclusion (1.2) is the projection and contraction method, which was originally introduced in Euclidean spaces, and was studied and extended to infinite dimension Hilbert spaces in many different ways recently; see, e.g., [3, 9, 10]. In this regards, Zhang *et al.* [16] introduced the following proximal algorithm for solving monotone variational inclusions.

$$\begin{cases} y_k=J_{\lambda_k}^A\left(u_k-\lambda_k F\left(u_k\right)\right),\\ d\left(u_k,y_k\right)=\left(u_k-y_k\right)-\lambda_k\left(F\left(u_k\right)-F\left(y_k\right)\right),\\ u_{k+1}=u_k-\gamma\beta_k d\left(u_k,y_k\right),\end{cases}$$

where $\gamma\in(0,2)$, $\beta_k=\frac{\phi(u_k,y_k)}{\|d(u_k,y_k)\|^2}$, and $\phi\left(u_k,y_k\right)=\langle u_k-y_k,d\left(u_k,y_k\right)\rangle$. They established the weak convergence of the iterative algorithm.

Recently, Censor *et al.* [4] proposed a hybrid method for solving the common variational inequalities (1.5), but the method needs to compute projections at each iteration. In [8], the following parallel hybrid subgradient extragradient method was presented

$$\begin{cases} u_0 \in H, \\ v_n^i = P_{K_i}\left(u_n - \lambda F_i\left(u_n\right)\right), \quad i = 1, \ldots, N, \\ z_n^i = P_{T_i^n}\left(u_n - \lambda F_i\left(v_n\right)\right), \quad i = 1, \ldots, N, \\ where \quad T_i^n = \left\{v \in H : \left\langle u_n - \lambda F_i\left(u_n\right) - v_n^i, v - v_n^i \right\rangle \leq 0\right\}, \\ i_n = \arg\max\left\{\left\|z_n^i - u_n\right\| : i = 1, \ldots, N\right\}, \bar{z}_n = z_n^{i_n}, \\ C_n = \left\{t \in H : \|t - \bar{z}_n\| \leq \|t - u_n\|\right\}, \\ Q_n = \left\{t \in H : \langle t - u_n, u_n - u_0 \rangle \geq 0\right\}, \\ u_{n+1} = P_{C_n \cap Q_n}\left(u_0\right), n \geq 1, \end{cases}$$

where $K_i, i = 1, \ldots, N$ is a finite family of nonempty, closed, and convex subset of H.

In 2018, Dong *et al.* [5] introduced two hybrid projection and contraction algorithms for finding common solutions of variational inequality problems (1.5) and established their strong convergence. The two algorithms read as follows:

$$\begin{cases} y_k^i = P_{K_i}(x_k - \lambda F_i(x_k)), \quad i = 1, \ldots, N, \\ d_k^i = (x_k - y_k^i) - \lambda(F_i(x_k) - F_i(y_k^i)), \quad i = 1, \ldots, N, \\ z_k^i = x_k - \gamma\rho_k^i d_k^i, \quad i = 1, \ldots, N, \\ C_k^i = \{v \in H : \|z_k^i - v\|^2 \leq \|x_k - v\|^2 - \gamma(2-\gamma)\rho_k^i\varphi_k^i\}, \quad i = 1, \ldots, N, \\ Q_k = \{v \in H : \langle x_k - v, x_k - x_0 \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \bigcap Q_k} x_0, \end{cases}$$

and

$$\begin{cases} y_k^i = P_{K_i}(x_k - \lambda F_i(x_k)), \quad i = 1, \ldots, N, \\ w_k^i = P_{K_i}(x_k - \lambda\gamma\rho_k^i F_i(y_k^i)), \quad i = 1, \ldots, N, \\ C_k^i = \{v \in H : \|w_k^i - v\|^2 \leq \|x_k - v\|^2 - \gamma(2-\gamma)\rho_k^i\varphi_k^i\}, \quad i = 1, \ldots, N, \\ Q_k = \{v \in H : \langle x_0 - x_k, v - x_k \rangle \leq 0\}, \\ x_{k+1} = P_{C_k \bigcap Q_k} x_0, \end{cases}$$

where $C_k = \bigcap_{i=1}^N C_k^i, \gamma \in (0,2)$, $\rho_k^i = \frac{\varphi_k^i}{\|d_k^i\|^2}$, and $\varphi_k^i = \langle x_k - y_k^i, d_k^i \rangle$.

Motivated by the above and [5, 8, 14, 15, 16], we adopt the advantages of known techniques, such as inertia, contraction and hybrid methods, and propose a simple strong convergent algorithm for solving CVIs. Our method converges under weaker assumptions than related results; see, e.g., [14], and the numerical experiments emphasize the practical potential of the scheme.

The outline of the paper is as follows. Basic definitions and results are presented in Section 2. Our proposed method is presented and analyzed in Section 3. In Section 4, two primary numerical experiments in finite and infinite dimensional spaces demonstrate and compare the performance of the algorithm. The last section, Section 5 ends this paper.

2. Preliminaries

Let H be a real Hilbert space. Denote by $\rightharpoonup$ and $\to$ the weak and strong convergence, respectively. The weak $\omega-$limit set of a sequence $\{u_n\}_{n=0}^{\infty}$ is denoted by $\omega_w(u_n) = \{u : \exists u_{n_j} \rightharpoonup u\}$.

Definition 2.1. Let $T : H \to H$ be a mapping on H.

(1) The fixed point set of T is denoted by $\text{Fix}(T) := \{x \in H \mid T(x) = x\}$.

(2) T is said to be L-Lipschitz continuous if and only if, for all $x,y \in H$,

$$\|Tx - Ty\| \leq L\|x - y\|,$$

where $L > 0$ is the Lipschitz constant. If $L = 1$, then T is said to be nonexpansive and contractive if $L < 1$.

(3) T is said to be firmly nonexpansive if and only if $2T - I$ is nonexpansive, or equivalently, for all $x,y \in H$,

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2.$$

(4) T is said to be monotone if and only if, for all $x,y \in H$,

$$\langle Tx - Ty, x - y\rangle \geq 0.$$

Definition 2.2. Let $B : H \to 2^H$ be a point-to-set operator. B is called a maximal monotone operator if B is monotone, i.e., $\langle u - v, x - y\rangle \geq 0$, $\forall u \in B(x)$, $v \in B(y)$, and the graph $\text{gra}(B)$ of B, $\text{gra}(B) = \{(x,u) \in H \times H \,|\, u \in B(x)\}$, is not properly contained in the graph of any other monotone operator.

It is clear that a monotone mapping B is maximal if and only if, for any $(x,u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for all $(y,v) \in \text{gra}(B)$ implies $u \in B(x)$. Let the set-valued mapping $A : H \to 2^H$ be maximal monotone. Define the resolvent operator J_r^A by $J_r^A = (I + rA)^{-1}, \quad r > 0$. It is worth mentioning that the resolvent operator J_r^A is single-valued and firmly nonexpansive.

Definition 2.3. Let C be a nonempty, closed, and convex subset of H. P_C is called the metric projection of H onto C if, for any point $u \in H$, there exists a unique point $P_C u \in C$ such that

$$\|u - P_C u\| \leq \|u - y\|, \; \forall y \in C.$$

It is well known that P_C is a firmly nonexpansive mapping of H onto C (see, [1, Proposition 4.16]). Furthermore, $P_C x \in C$, and $\langle x - P_C x, P_C x - y\rangle \geq 0$, $\forall y \in C$.

Lemma 2.4. [2] *Let $A : H \to 2^H$ be a maximal monotone mapping, and let $F : H \to H$ be a Lipschitz continuous mapping. Then the mapping $B = A + F$ is maximal monotone.*

Lemma 2.5. [11] *Let C be a closed and convex subset of H. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{u_n\}_{n=0}^{\infty}$ satisfies $\omega_w(u_n) \subset C$ and $\|u_n - u\| \leq \|u - q\|$, $\forall n \geq 1$, then $\{u_n\}_{n=0}^{\infty}$ converges strongly to q.*

Lemma 2.6. [1, Proposition 23.39] *Let $A : H \to 2^H$ be maximally monotone and $\text{zer}(A) = A^{-1}0 = \{x \in H \,|\, 0 \in Ax\}$. Then $\text{zer}(A)$ is closed and convex.*

3. Main Results

For common variational inclusion problem (1.1), assume the following:

(C1) The solution set of the CVI (1.1), denoted by $\Omega = \bigcap_{i=1}^{N} (F_i + A_i)^{-1}(0)$, is nonempty.

(C2) The mapping F_i is monotone and Lipschitz continuous with constant $L_i > 0$, for each $i = 1,2,\ldots,N$ on H.

(C3) The mapping $A_i : H \to 2^H$ is maximal monotone, for each $i = 1,2,\ldots,N$.

We are now ready to present our new method.

Algorithm 1

Step 0. Fix $s, M \in \mathbb{N}$ and denote $S := \{0, \ldots, s\}$. Let $\left|\theta_{k,n}\right| < M$ and $\delta \in (0,2)$ for $k \in S$ and $n \in \mathbb{N}$. Choose $u_0 \in H$ and let $u_{-k} = u_0$, for all $k \in S$. Set $n := 0$.
Step 1. Compute

$$v_n = u_n + \sum_{k\in S} \theta_{k,n}(u_{n-k} - u_{n-k-1}).$$

Step 2. Generate i_n and $\bar{y}_n$ by

$$\begin{cases} z_n^i = J_{r_n^i}^{A_i}(v_n - r_n^i F_i(v_n)), \\ d(v_n, z_n^i) = (v_n - z_n^i) - r_n^i(F_i(v_n) - F_i(z_n^i)), \\ y_n^i = v_n - \delta\beta_n^i d\left(v_n, z_n^i\right), \\ i_n = \arg\max\{\|y_n^i - u_n\| \mid i = 1, 2, \ldots, N\},\ \bar{y}_n = y_n^{i_n}, \end{cases}$$

where

$$\beta_n^i = \begin{cases} \dfrac{\phi(v_n, z_n^i)}{\|d(v_n, z_n^i)\|^2} & d(v_n, z_n^i) \neq 0; \\ c & d(v_n, z_n^i) = 0, \end{cases}$$

and $c > 1$ is an arbitrary constant and $\phi(v_n, z_n^i) = \langle v_n - z_n^i, d(v_n, z_n^i)\rangle$.
Step 3. Set

$$\begin{cases} C_n = \{t \in H \mid \|\bar{y}_n - t\|^2 \leq \|v_n - t\|^2 - \delta(2-\delta)\beta_n^{i_n}\phi(v_n, z_n^{i_n})\}, \\ Q_n = \{t \in H \mid \langle u_0 - u_n, u_n - t\rangle \geq 0\}, \end{cases}$$

and compute

$$u_{n+1} = P_{C_n \cap Q_n} u_0.$$

Step 4. Set $n \leftarrow n+1$, and go to Step 1.

We start our analysis with the following definition.

Definition 3.1. Let $c_1, c_2 > 0$ be given constants in $(0,1)$. r_n^i is said to satisfy the stepsize conditions, for each $i = 1, 2, \ldots, N$, if r_n^i satisfies

$$c_1\|v_n - z_n^i\|^2 \leq \phi(v_n, z_n^i),$$

$$\beta_n^i \geq c_2,$$

and

$$\inf_{n\geq 0}\{r_n^i\} \geq \underline{r}^i > 0.$$

From [12, Lemma 5.2] and [17, Lemma 3.4], we have the following result.

Lemma 3.2. *Consider the CVI (1.1) and assume that Conditions* (C1)–(C3) *hold. Let* $\{u_n\}_{n=0}^{\infty}$ *be the sequence generated by Algorithm* 1. *Then* r_n^i *satisfies the stepsize conditions if* $r_n^i \in [a,b] \subset (0, 1/L_i)$ *or* r_n^i *updated adaptively via* $r_n^i = \sigma\eta^{m_n^i}$, $\sigma > 0$, $\eta \in (0,1)$, *where* m_n^i *is the smallest nonnegative integer such that* $r_n^i\|F_i(u_n) - F_i(z_n^i)\| \leq \nu_i\|u_n - z_n^i\|$, $i = 1, \ldots, N$, *where* $\nu_i \in (0,1)$ *is given.*

Similarly to [15, Lemma 3.7], we have the following result.

Lemma 3.3. *If r_n^i satisfy the stepsize conditions, then*

$$\|v_n - z_n^i\|^2 \le \frac{1}{c_1 c_2 \delta}\|v_n - y_n^i\|^2. \tag{3.1}$$

Lemma 3.4. *Let Conditions* (C1)–(C3) *hold. Let* $\lim_{n\to+\infty}\|v_n - u_n\| = 0$ *and* $\lim_{n\to+\infty}\left\|v_n - z_n^i\right\| = 0$ *for each* $i = 1,2,\ldots,N$. *If* $\{u_n\}_{n=0}^{\infty}$ *is bounded, then* $\omega_w(u_n) \subset \Omega$.

Proof. Since $\{u_n\}_{n=0}^{\infty}$ is bounded, one has $\omega_w(u_n) \ne \emptyset$. Taking $\hat{\omega} \in \omega_w(u_n)$ arbitrarily, one has that there exists a subsequence $\{u_{n_j}\}_{j=0}^{\infty}$ of $\{u_n\}_{n=0}^{\infty}$, which converges weakly to $\hat{\omega}$. From the assumption, it follows that $\{v_{n_j}\}_{j=0}^{\infty}$ and $\{z_{n_j}^i\}_{j=0}^{\infty}$, $i = 1,\ldots,N$, converge weakly to $\hat{\omega}$.

Next we show that $\hat{\omega}$ is a solution of (1.1), that is, $\hat{\omega} \in \Omega$. Let $(v,\tau) \in \mathrm{gra}(A_i + F_i)$, i.e., $\tau - F_i(v) \in A_i(v)$, $i = 1,\ldots,N$. From the definition of z_n^i, we have

$$v_{n_j} - r_{n_j}^i F_i(v_{n_j}) \in (I + r_{n_j}^i A_i)(z_{n_j}^i),$$

that is,

$$\frac{v_{n_j} - z_{n_j}^i}{r_{n_j}^i} - F_i(v_{n_j}) \in A_i(z_{n_j}^i).$$

By virtue of the maximal monotonicity of A_i, we obtain

$$\langle v - z_{n_j}^i, \tau - F_i(v) - \frac{v_{n_j} - z_{n_j}^i}{r_{n_j}^i} + F_i(v_{n_j})\rangle \ge 0.$$

Hence,

$$\begin{aligned}
\langle v - z_{n_j}^i, \tau\rangle &\ge \langle v - z_{n_j}^i, F_i(v) + \frac{v_{n_j} - z_{n_j}^i}{r_{n_j}^i} - F_i(v_{n_j})\rangle \\
&= \langle v - z_{n_j}^i, F_i(v) - F_i(z_{n_j}^i) + F_i(z_{n_j}^i) - F_i(v_{n_j}) + \frac{v_{n_j} - z_{n_j}^i}{r_{n_j}^i}\rangle \\
&\ge \langle v - z_{n_j}^i, F_i(z_{n_j}^i) - F_i(v_{n_j})\rangle + \langle v - z_{n_j}^i, \frac{v_{n_j} - z_{n_j}^i)}{r_{n_j}^i}\rangle,
\end{aligned}$$

where the last inequality comes from the monotonicity of F_i. Since $\lim_{n\to+\infty}\|v_n - z_n^i\| = 0$, F_i is Lipschitz continuous, and $\inf_{n\ge 0}\{r_n^i\} \ge \underline{r}^i > 0$, we have

$$\lim_{j\to+\infty}\langle v - z_{n_j}^i, \tau\rangle = \langle v - \hat{\omega}, \tau\rangle \ge 0.$$

From conditions (C2)-(C3) and Lemma 2.4, we have that $A_i + F_i$ is maximal monotone for each $i = 1,\ldots,N$. Then $0 \in (A_i + F_i)(\hat{\omega})$, $\forall i = 1,\ldots,N$, that is, $\hat{\omega} \in \Omega$. Therefore, we have $\omega_w(u_n) \subset \Omega$. This completes the proof. □

Theorem 3.5. *Assume that Conditions* (C1)–(C3) *hold and* r_n^i *satisfies the stepsize conditions,* $i = 1,\ldots,N$ *and* $n \in \mathbb{N}$. *Then the sequence* $\{u_n\}_{n=0}^{\infty}$ *generated by Algorithm* 1 *converges strongly to* $\bar{\omega} = P_\Omega u_0$.

Proof. For simplicity, we divide the proof into four steps.

Step 1. $P_{C_n\cap Q_n}$ is well defined.

From Conditions (C2)-(C3), Lemma 2.4, and Lemma 2.6, we obtain that Ω is closed and convex set. It is easy to know that C_n is convex and closed for $n\in\mathbb{N}$ (see [11, Lemma 1.3]). Since Q_n is either a half-space or the all space H, it is closed convex for $n\in\mathbb{N}$.

Now, we show that $\Omega\subseteq C_n$ for all $n\in\mathbb{N}$. Take $u\in\Omega$ arbitrarily. According to the definition of y_n^i and β_n^i, we have, for $i=1,\ldots,N$,

$$\begin{aligned}\left\|y_n^i-u\right\|^2&=\left\|v_n-\delta\beta_n^i d\left(v_n,z_n^i\right)-u\right\|^2\\&=\|v_n-u\|^2+\delta^2{\beta_n^i}^2\left\|d\left(v_n,z_n^i\right)\right\|^2-2\delta\beta_n^i\left\langle v_n-u,d\left(v_n,z_n^i\right)\right\rangle\\&=\|v_n-u\|^2+\delta^2\beta_n^i\phi\left(v_n,z_n^i\right)-2\delta\beta_n^i\left\langle v_n-u,d\left(v_n,z_n^i\right)\right\rangle,\end{aligned}\tag{3.2}$$

where the last equality comes from the definition of β_n^i. By the definition of ϕ, we obtain

$$\begin{aligned}\left\langle v_n-u,d(v_n,z_n^i)\right\rangle&=\left\langle v_n-z_n^i,d\left(v_n,z_n^i\right)\right\rangle+\left\langle z_n^i-u,d\left(v_n,z_n^i\right)\right\rangle\\&=\phi\left(v_n,z_n^i\right)+\left\langle z_n^i-u,d\left(v_n,z_n^i\right)\right\rangle.\end{aligned}\tag{3.3}$$

Since $J_{r_n^i}^{A_i}$ is firmly nonexpansive, it follows that

$$\begin{aligned}\langle z_n^i-u,(I-r_n^iF_i)v_n-(I-r_n^iF_i)u\rangle&=\langle J_{r_n^i}^{A_i}(I-r_n^iF_i)v_n-J_{r_n^i}^{A_i}(I-r_n^iF_i)u,(I-r_n^iF_i)v_n-(I-r_n^iF_i)u\rangle\\&\geq\|J_{r_n^i}^{A_i}(I-r_n^iF_i)v_n-J_{r_n^i}^{A_i}(I-r_n^iF_i)u\|^2\\&=\|z_n^i-u\|^2.\end{aligned}$$

Using the above inequality, we have

$$\begin{aligned}\langle z_n^i-u,v_n-z_n^i-r_n^iF_i(v_n)\rangle&=\langle z_n^i-u,(I-r_n^iF_i)v_n-(I-r_n^iF_i)u+(I-r_n^iF_i)u-z_n^i\rangle\\&\geq\|z_n^i-u\|^2+\langle z_n^i-u,u-z_n^i\rangle+\langle z_n^i-u,-r_n^iF_i(u)\rangle\\&=-\langle z_n^i-u,r_n^iF_i(u)\rangle,\end{aligned}$$

which implies

$$\langle z_n^i-u,v_n-z_n^i-r_n^i(F_i(v_n)-F_i(u))\rangle\geq 0.\tag{3.4}$$

From the monotonicity of F_i and $r_n^i>0$, we have

$$\langle z_n^i-u,r_n^i(F_i(z_n^i)-F_i(u))\rangle\geq 0.\tag{3.5}$$

Adding (3.4) and (3.5), we obtain

$$\left\langle z_n^i-u,d\left(v_n,z_n^i\right)\right\rangle=\left\langle z_n^i-u,\left(v_n-z_n^i\right)-r_n^i\left(F_i\left(v_n\right)-F_i\left(z_n^i\right)\right)\right\rangle\geq 0.\tag{3.6}$$

Using (3.2), (3.3) and (3.6), it follows that

$$\|y_n^i-u\|^2\leq\|v_n-u\|^2-\delta(2-\delta)\beta_n^i\phi(v_n,z_n^i)\quad i=1,\ldots,N.$$

Therefore,

$$\|\bar{y}_n-u\|^2\leq\|v_n-u\|^2-\delta(2-\delta)\beta_n^{i_n}\phi\left(v_n,z_n^{i_n}\right),$$

which implies $\Omega\subseteq C_n$ for all $n\in\mathbb{N}$.

Next we show that $\Omega\subseteq Q_n$ for all $n\in\mathbb{N}$ by induction. For $n=0$, we have $\Omega\subseteq H=Q_0$. Assume $\Omega\subseteq Q_n$. Since $u_{n+1}=P_{C_n\cap Q_n}u_0$, we have

$$\langle u_0-u_{n+1},u_{n+1}-u\rangle\geq 0\quad\text{for all } u\in C_n\cap Q_n.$$

Since $\Omega \subseteq C_n \cap Q_n$, we obtain

$$\langle u_0 - u_{n+1}, u_{n+1} - u \rangle \geq 0 \quad \text{for all } u \in \Omega.$$

which implies that $\Omega \subseteq Q_{n+1}$. Hence, $\Omega \subseteq Q_n$ for all $n \in \mathbb{N}$. Then, $\Omega \subseteq C_n \cap Q_n$ for all $n \in \mathbb{N}$. Furthermore, $P_{C_n \cap Q_n}$ is well defined.

Step 2. $\{u_n\}_{n=0}^{\infty}$ is bounded and $\lim\limits_{n \to \infty} \|u_{n+1} - u_n\| = 0$.

Since Ω is a nonempty, closed, and convex subset of H, there exists a unique element $\bar{\omega} \in \Omega$ such that $\bar{\omega} = P_\Omega u_0$. From $u_{n+1} = P_{C_n \cap Q_n} u_0$ and $\Omega \in Q_n$, we have

$$\|u_{n+1} - u_0\| \leq \|p - u_0\| \quad \text{for all } p \in \Omega.$$

Due to $\bar{\omega} \in \Omega \subset C_n \cap Q_n$, we have

$$\|u_{n+1} - u_0\| \leq \|\bar{\omega} - u_0\|, \tag{3.7}$$

which implies $\{u_n\}_{n=0}^{\infty}$ is bounded. The fact that $u_{n+1} \in Q_n$ implies that

$$\langle u_{n+1} - u_n, u_0 - u_n \rangle \leq 0.$$

So, we obtain

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &= \|u_{n+1} - u_0\|^2 - \|u_0 - u_n\|^2 + 2\langle u_{n+1} - u_n, u_0 - u_n \rangle \\ &\leq \|u_{n+1} - u_0\|^2 - \|u_n - u_0\|^2. \end{aligned} \tag{3.8}$$

From (3.7) and (3.8), we obtain

$$\begin{aligned} \sum_{n=0}^{K} \|u_{n+1} - u_n\|^2 &\leq \sum_{n=0}^{K} \left(\|u_{n+1} - u_0\|^2 - \|u_n - u_0\|^2 \right) \\ &= \|u_{K+1} - u_0\|^2 \leq \|\bar{\omega} - u_0\|^2, \end{aligned}$$

which yields

$$\sum_{n=0}^{\infty} \|u_{n+1} - u_n\|^2 < +\infty.$$

Therefore,

$$\lim_{n \to +\infty} \|u_{n+1} - u_n\| = 0. \tag{3.9}$$

Step 3. $\lim\limits_{n \to +\infty} \|v_n - z_n^i\| = 0$, $i = 1, \ldots, N$.

From the definition of v_n, $|\theta_{k,n}| < M$, and the trigonometric inequality of the norm, it follows that

$$\begin{aligned} \|v_n - u_n\| &= \|\sum_{k \in S} \theta_{k,n}(u_{n-k} - u_{n-k-1})\| \\ &\leq \sum_{k \in S} |\theta_{k,n}| \|u_{n-k} - u_{n-k-1}\| \\ &\leq M \sum_{k \in S} \|u_{n-k} - u_{n-k-1}\|. \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we obtain

$$\lim_{n \to +\infty} \|v_n - u_n\| = 0. \tag{3.11}$$

Therefore, it follows from (3.9) and (3.11) that

$$\lim_{n \to +\infty} \|v_n - u_{n+1}\| = 0. \tag{3.12}$$

Since $u_{n+1} \in C_n$, $\delta \in (0,2)$ and Definition 3.1, we have

$$\|\bar{y}_n - u_{n+1}\|^2 \leq \|v_n - u_{n+1}\|^2 - \delta(2-\delta)\beta_n^{i_n}\phi(v_n, z_n^{i_n}) \leq \|v_n - u_{n+1}\|^2. \tag{3.13}$$

From (3.12) and (3.13), we obtain

$$\lim_{n\to+\infty} \|\bar{y}_n - u_{n+1}\| = 0,$$

which together with (3.9) yields $\lim_{n\to+\infty} \|\bar{y}_n - u_n\| = 0$. From the definition of i_n and $\bar{y}_n$, we have

$$\lim_{n\to+\infty} \|y_n^i - u_n\| = 0 \quad i = 1,\ldots,N. \tag{3.14}$$

Observe that

$$\|y_n^i - v_n\| \leq \|y_n^i - u_n\| + \|u_n - v_n\|. \tag{3.15}$$

Using (3.11), (3.14), and (3.15), we obtain

$$\lim_{n\to+\infty} \|y_n^i - v_n\| = 0 \quad i = 1,\ldots,N. \tag{3.16}$$

According to (3.1) and (3.16), we conclude

$$\lim_{n\to+\infty} \|v_n - z_n^i\| = 0 \quad i = 1,\ldots,N. \tag{3.17}$$

Step 4. The sequence $\{u_n\}_{n=0}^{\infty}$ converges strongly to $\bar{\omega} = P_\Omega u_0$.

Notice that $\{u_n\}_{n=0}^{\infty}$ is bounded. Using (3.11), (3.17), and Lemma 3.4, it follows that $\bar{\omega}_w(u_n) \subset \Omega$, which together with (3.7) and Lemma 2.5 guarantees that $\{u_n\}_{n=0}^{\infty}$ converges strongly to $\bar{\omega} = P_\Omega u_0$. This completes the proof. □

4. Numerical Experiments

In this section, we consider two numerical examples in [7] on CSVI problem (1.4) for testing and comparing the performance of our scheme with [5, Algorithms 4.1 and 4.2]. Since $J_{r_n^i}^{N_{K_i}} = P_{K_i}$, when solving the CSVI problem (1.4), Algorithm 1 can be transformed into the following form in the following two examples:

$$\begin{cases} v_n = u_n + \sum\limits_{k\in S} \theta_{k,n}(u_{n-k} - u_{n-k-1}), \\ z_n^i = P_{K_i}(v_n - r_n^i F_i(v_n)), \\ d(v_n, z_n^i) = (v_n - z_n^i) - r_n^i(F_i(v_n) - F_i(z_n^i)), \\ y_n^i = v_n - \delta\beta_n^i d\left(v_n, z_n^i\right), \\ i_n = \arg\max\{\|y_n^i - u_n\| \mid i = 1,2,\ldots,N\},\ \bar{y}_n = y_n^{i_n}, \\ C_n = \{t \in H \mid \|\bar{y}_n - t\|^2 \leq \|v_n - t\|^2 - \delta(2-\delta)\beta_n^{i_n}\phi(v_n, z_n^{i_n})\}, \\ Q_n = \{t \in H \mid \langle u_0 - u_n, u_n - t\rangle \geq 0\}, \\ u_{n+1} = P_{C_n\cap Q_n} u_0, \end{cases}$$

where $S := \{0,\ldots,s\}$. Let $\left|\theta_{k,n}\right| < M$ for $k \in S$ and $n \in \mathbb{N}, \delta \in (0,2)$

$$\beta_n^i = \begin{cases} \dfrac{\phi(v_n, z_n^i)}{\|d(v_n, z_n^i)\|^2} & d(v_n, z_n^i) \neq 0; \\ c & d(v_n, z_n^i) = 0, \end{cases}$$

where $c > 1$ is an arbitrary constant and $\phi(v_n, z_n^i) = \langle v_n - z_n^i, d(v_n, z_n^i)\rangle$.

TABLE 1. Numerical results for Algorithms 1, 4.1 and 4.2 with $l = 20$.

	Iter.			CPU in s		
m	Alg. 1	Alg. 4.1 in [5]	Alg. 4.2 in [5]	Alg. 1	Alg. 4.1 in [5]	Alg. 4.2 in [5]
5	457	611	657	0.6141	0.8344	1.7109
10	850	997	1056	4.0922	5.4703	12.2578
15	1049	1280	1240	5.5266	6.1547	13.2297

In the numerical results listed in the following tables, 'Iter.' and 'CPU in s' denote the number of iterations and the execution time in seconds, respectively.

Example 4.1. [7] We consider a affine variational inequality in Euclidean space. Let the operators $F_i(x) = M_i x + q_i$ (see [6, 7]), where

$$M_i = B_i B_i^T + C_i + D_i, \quad \forall i = 1, \ldots, N,$$

and B_i is an $m \times m$ matrix, C_i is an $m \times m$ skew-symmetric matrix, D_i is an $m \times m$ diagonal matrix, whose diagonal entries are nonnegative (so M_i is positive semi-definite), and q_i is a vector in $\mathbb{R}^m$. The feasible set $K_i = K \subset \mathbb{R}^m$ is a closed convex subset defined by

$$K = \{x \in \mathbb{R}^m \mid Qx \leq b\},$$

where Q is an $l \times m$ matrix and b is a nonnegative vector. It is clear that F_i is monotone and L_i-Lipschitz continuous with $L_i = \|M_i\|, i = 1, \ldots, N$ and $L = \max\{\|M_i\|, i = 1, \ldots, N\}$. Let $q = 0$. Then, the solution set $\Omega = \{0\}$.

In this example, the starting points are $u_0 = u_{-1} = (1, \ldots, 1)^T \in \mathbb{R}^m$ and the number of subproblems N is 10. The matrices Q, B_i, C_i, D_i and the vector b are generated randomly. The stopping criteria is $\|x_n\| \leq 0.001$. The choice of $\theta_{k,n}$ in Algorithm 1 is $[-0.7500, -0.500, -0.2500, -0.1250, -0.0625, -0.0312]$, for $s = 5$. Other parameters are chosen as follows:

Algorithm 1: $\delta = 0.6$, $s = 5$, $r_n^i = \frac{0.68}{L_i}$;

Algorithms 4.1 and 4.2 in [5]: $\lambda = \frac{1}{4L}$, $\gamma = 1.5$.

The numerical results listed in Table 1 and Table 2 are 10 times average values of the required iteration steps and the elapsed CPU times with regard to running repeatedly the corresponding algorithm. The Table 1 illustrates that the execution time and the number of iterations of three algorithms all become bigger as m increases. Furthermore, combined with Table 2, it was observed that our Algorithm 1 performs better than Algorithms 4.1 and 4.2 in [5] in the number of iterations and the CPU time.

Example 4.2. [7] Let H be the function space $L^2[0,1]$, and let K_i be the unit ball $B[0,1] \subset H$. In this example, we consider the operators $F_i : K_i \to H$ defined by

$$F_i(x)(t) = \int_0^1 [x(t) - H_i(t,s)h_i(x(s))]ds + g_i(t)$$

for all $x \in K$, $t \in [0,1]$ and $i = 1, 2$, where

$$H_1(t,s) = \frac{2tse^{t+s}}{e\sqrt{e^2-1}}, \quad h_1(x) = \cos x, \quad g_1(t) = \frac{2te^t}{e\sqrt{e^2-1}},$$

TABLE 2. Numerical results for Algorithms 1, 4.1 and 4.2 with $m = 10$.

l	Iter.			CPU in s		
	Alg. 1	Alg. 4.1 in [5]	Alg. 4.2 in [5]	Alg. 1	Alg. 4.1 in [5]	Alg. 4.2 in [5]
30	768	954	1091	7.3266	7.8672	15.9219
40	861	940	975	9.1438	11.5172	25.3078
50	943	1096	1265	14.4141	16.2047	35.7188

$$H_2(t,s) = \frac{\sqrt{21}}{7}(t+s), \quad h_2(x) = \exp(-x^2), \quad g_2(t) = \frac{\sqrt{21}}{7}(t+\frac{1}{2}).$$

One can verify that F_i is monotone and 2-Lipschitz continuous (see [13, p.168] and [7]). Moreover, the solution set of the CVI for the operators F_i on $B[0,1]$ is $\Omega = \{0\}$.

We choose the starting point $u_0(t) = 1$ and the sets $K_i = B[0,1]$. The stopping criteria is $\|x_n\| \leq$ Error. In Algorithm 1, we take $\delta = 1.8$, $r_n^i = \frac{0.7}{L}$ and $\theta_{k,n}$ is $[-0.7500, -0.500, -0.2500, -0.1250, -0.0625, -0.0313]$, for $s = 5$. In Algorithms 4.1 and 4.2 in [5], we choose $\lambda = \frac{1}{3}$, $\gamma = 1.9$.

As shown in Table 3, we see that our Algorithm 1 outperforms Algorithms 4.1 and 4.2 in [5] from running time or the number of iterations.

TABLE 3. Numerical results for Algorithms 1, 4.1 and 4.2 in Example 4.2.

Error	Iter.			CPU in s		
	Alg. 1	Alg. 4.1 in [5]	Alg. 4.2 in [5]	Alg. 1	Alg. 4.1 in [5]	Alg. 4.2 in [5]
0.01	188	322	398	0.1094	0.7188	0.2031
0.005	222	456	606	0.2656	0.7813	0.2813
0.001	583	1826	1394	0.4844	1.4844	0.6406
0.0005	1013	2590	2664	0.8438	1.6875	1.2656

5. CONCLUSIONS

We introduced the so-called parallel multi-step inertial contracting algorithm (PMiCA) for solving common variational inclusions in real Hilbert spaces. The proposed algorithm combines several concepts and known techniques, such as the inertial technique and the contraction and hybrid methods. Under suitable conditions, a strong convergence theorem was established, and primary numerical experiments illustrated the performances and advantages of the new method by comparing with related results in the literature.

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