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A FULL-NEWTON STEP FEASIBLE INTERIOR-POINT ALGORITHM BASED ON A SIMPLE KERNEL FUNCTION FOR $P_*(\kappa)$-HORIZONTAL LINEAR COMPLEMENTARITY PROBLEM

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Abstract. In this paper, we propose and analyze an interior-point algorithm with full-Newton steps for $P_*(\kappa)$-horizontal linear complementarity problem based on a kernel function which was first used to design the interior-point algorithm for linear optimization by Zhang et al. (J Nonlinear Funct. Anal. Article ID 31, 2021). The method mainly based on exploiting the search direction by the kernel function. By using the properties of the kernel function, we prove that the complexity of the algorithm coincides with the currently best known iteration complexity of feasible interior-point methods for $P_*(\kappa)$-horizontal linear complementarity problem. Finally, some computational results are demonstrated, which show that our algorithm is efficient and promising.

Keywords. Horizontal linear complementarity problem; Interior-point method; Kernel function; Polynomial iteration complexity.

1. INTRODUCTION

Given $Q,R \in R^{n\times n}$, $q \in R^n$, the horizontal linear complementarity problem (HLCP) is to seek a vector pair $(x,s) \in R^n \times R^n$ such that

$$Qx + Rs = q, \quad xs = 0, \quad (x,s) \geq 0, \tag{P}$$

where xs denotes the coordinatewise product of the vectors x and s. It worth noting that the HLCP becomes the standard linear complementarity problem (LCP) if $R = -I$, which I is unit matrix. HLCPs cover fairly general classes of mathematical programming and equilibrium problems. For instance, by exploiting the first-order optimality conditions of the optimization problem, we can formulate any linear optimization (LO) and convex quadratic optimization (CQO) into a HLCP, and HLCPs are also closed related to variational inequalities. The variable inequality

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problem was widely used in the study of equilibrium problems in a number of disciplines, such as economics, mathematical physics, geometry, an so on. For a comprehensive learning about the basic theory, algorithms, and applications, we refer to [3].

In this paper, we consider the HLCP with $\{Q,R\}$ being a $P_*(\kappa)$-pair, denoted by $P_*(\kappa)$-HLCP. Note that $\{Q,R\}$ is called a $P_*(\kappa)$-pair if $Qx+Rs=0$ implies that

$$(1+4\kappa)\sum_{i\in I_+(x)} x_is_i + \sum_{i\in I_-(x)} x_is_i \geq 0,$$

where $\kappa>0$, $I_+(x) := \{i\in I : x_is_i \geq 0\}$, $I_-(x) := \{i \in I : x_is_i \leq 0\}$, and $I := \{1,2,\dots,n\}$. There are numerous methods for solving $P_*(\kappa)$-HLCP. Among them, interior-point methods (IPMs) received more attention since they not only have polynomial complexity but also highly efficient in practice. As the HLCP is closely related to the LO, various interior-point methods (IPMs) designed for the LO have been extended to $P_*(\kappa)$-HLCP. Some relevant references can be founded in [1, 4, 13, 15].

It has clear that a suitable kernel function play a powerful role for solving optimization problems with primal-dual IPMs based on kernel functions. Roos et al. [12] introduced primal-dual feasible IPM based on the classical logarithmic kernel function, and obtained the currently best known iteration bound for LO, namely $O(\sqrt{n}log\frac{n}{\varepsilon})$. The advantage of their method is that it uses only full-Newton steps (instead of damped steps) at each iteration. Thus the calculation of step size is avoided and no line searches are needed. In addition, the iterates of their algorithm always lie in the quadratic convergence neighborhood under some mild assumptions. Moveover, Roos [14] proposed a full-Newton step infeasible IPM(IIPM) for LO. The algorithm still has best known iteration complexity for LO, which is $O(nlog\frac{n}{\varepsilon})$. Subsequently, Wang et al. [16] extended Roos et al.'s full-Newton step primal-dual IPM for LO, presented in [12], to $P_*(\kappa)$-LCP and derived the currently best known iteration bound for $P_*(\kappa)$-LCP, namely $O((1+4\kappa)\sqrt{n}log\frac{n}{\varepsilon})$. The majority of primal-dual IPMs, say the above mentioned ones, are based on the classical logarithmic kernel function, which leads to the Newton search directions. However, some authors, such as Peng et al. [11] and Bai et al. [2], presented some different kernel function based IPMs for LO, which reduce the gap between the practical behavior of the algorithms and the theoretical performance results. Recently, Zhang and Xu [17] presented a full-Newton step interior-point algorithm based on modified Newton direction for LO. Their algorithm has polynomial iteration complexity $O(\sqrt{n}log\frac{n}{\varepsilon})$, which matches with the currently best known iteration bound for LO. Liu and Sun [10] and Kheirfam [5, 6] respectively, extended some variants of the infeasible IPM to LO, LCP, and symmetric optimization (SCO), whose feasibility step is induced by a specific kernel function. Lesaja et al. [8] presented IPM for cartesian $P_*(\kappa)$-LCP over symmetric cones based on the eligible kernel functions. Recently, Zhang et al. [18] first proposed and analyzed a full-Newton IIPM based on a kernel function for LO. This kernel function has a linear growth term, while the usual kernel function in that its growth term is a quadratic (or higher degree). Therefore, the kernel function has the simplest possible form compared with existing kernel functions. They also obtained the best complexity result. For an overview of these related results, we refer to [1, 5, 6, 8, 10, 16, 18, 19] and the references therein.

Motivated by the above mentioned works, we propose a new full-Newton feasible IPM for $P_*(\kappa)$-HLCP based on the search directions, which are defined by the kernel function with linear growth term [18]. We take full steps along these search directions to avoid the calculation of the step size. We demonstrate the iteration bound coincides with the currently best known complexity bound for $P_*(\kappa)$-HLCP. To the best of our knowledge, we are the first to solve $P_*(\kappa)$-HLCP based on the kernel function. The analysis of the algorithm is more complicated than in the LO case mainly due to the fact that we loose the orthogonality of the search directions in the $P_*(\kappa)$-HLCP case. We also give few computational results to verify the efficiency of our kernel function.

The remainder paper is organized as follows. In Section 2, we present a feasible IPM with full-Newton steps for $P_*(\kappa)$-HLCP based on the search directions. In Section 3, we analyze the algorithm and derive the complexity bound. Moveover, some computational results are presented in Section 4. Finally, concluding remarks are given in Section 5.

Some of the notations used in this paper are as follows. For $x=(x_1,x_2,\cdots,x_n)^T\in R^n$, x_{min} denotes the minimum value of the components of x, e denotes a vector of length n that is all-one, i.e., $e=(1,1,...,1)^T$. X is the diagonal matrix whose diagonal elements are the coordinates of vector x, i.e., X=diag(x). The index set I is $I=\{1,2,\cdots,n\}$. Moreover, $\parallel x\parallel=\sqrt{x^Tx}$ denotes the 2-norm of the vector x. $x^Ty=\sum_{i=1}^n x_iy_i$ for $x,y\in R^n$.

2. The full-Newton step feasible IPM for $P_*(\kappa)$-HLCP

In this section, we introduce a kernel function with linear growth term and the central path for $P_*(\kappa)$-HLCP. The full-Newton step feasible IPM for $P_*(\kappa)$-HLCP based on the kernel function is also proposed.

2.1. The central path and the new search directions.

The solution for $P_*(\kappa)$-HLCP is equivalent to finding the solution of the following system

$$\left\{\begin{array}{lr} Qx+Rs=q, & x\geq 0,\\ xs=0, & s\geq 0.\end{array}\right.$$

The basic idea of prime-dual IPMs is to replace the complementarity condition $xs=0$ by the parameterized equation $xs=\mu e$ with $\mu>0$. Therefore, we may consider the following system

$$\left\{\begin{array}{lr} Qx+Rs=q, & x\geq 0,\\ xs=\mu e, & s\geq 0.\end{array}\right. \tag{2.1}$$

Without loss of generality, we assume that (P) satisfies the IPC, i.e., there exists $(x_0,s_0)>0$ such that $Qx_0+Rs_0=q$. Then the above system has a unique solution for each $\mu>0$ ([7, Lemma 4.3]), that is denoted by $(x(\mu),s(\mu))$. We call $\{(x(\mu),s(\mu))|\mu>0\}$ the μ-central of $P_*(\kappa)$-HLCP. The set of such centers (with μ running through all positive real numbers) is called the central path of $P_*(\kappa)$-HLCP. As μ goes to zero, the limit of the central path exists and since the limit point satisfies $xs=0$, the limit yields the optimal solution for $P_*(\kappa)$-HLCP.

A natural way to define search directions is to use Newton's method on system (2.1). This leads to the following system

$$\left\{\begin{array}{l} Q\Delta x+R\Delta s=0,\\ x\Delta s+s\Delta x=\mu e-xs.\end{array}\right. \tag{2.2}$$

To simplify the analysis, we introduce the following notations

$$\upsilon:=\sqrt{\frac{xs}{\mu}},\quad dx:=\frac{\upsilon\Delta x}{x},\quad ds:=\frac{\upsilon\Delta s}{s}. \tag{2.3}$$

It follows from (2.3) that system (2.2) transforms to the following system

$$\begin{cases} \overline{Q}d_x + \overline{R}d_s = 0, \\ d_x + d_s = \upsilon^{-1} - \upsilon, \end{cases} \tag{2.4}$$

where $\overline{Q} := QXV^{-1}$, $\overline{R} := RSV^{-1}$. We derive the search directions $(d_x,\ d_s)$ by solving (2.4) and then we compute $(\Delta x,\ \Delta s)$ via (2.3). The new iterates with a full-Newton step are given by

$$x^+ := x + \Delta x, \quad s^+ := s + \Delta s.$$

Definition 2.1 ([2])**.** A twice differentiable function $\psi : (0, +\infty) \to [0, +\infty)$ is called a kernel function if it is satisfied with the following conditions:

$$\psi(1) = \psi'(1) = 0, \qquad \psi''(t) > 0,\ \ \forall t > 0, \qquad \lim_{t \downarrow 0} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty.$$

The right of second equation in (2.4) is the negative gradient of the classical logarithmic barrier function

$$\Psi_c(\upsilon) := \sum_{i=1}^{n} \psi_c(\upsilon_i), \quad \upsilon_i = \sqrt{\frac{x_i s_i}{\mu}},$$

where $\psi_c(t) := \frac{1}{2}(t^2 - 1) - \log t$ with $t > 0$, i.e., $d_x + d_s = -\nabla\Psi_c(v)$.

In this paper, we consider the kernel function $\psi(t)$ and its barrier function $\Psi(\upsilon)$ as follows

$$\psi(t) := 2(t-1) - 2\log t, \qquad \Psi(\upsilon) := \sum_{i=1}^{n} \psi(\upsilon_i). \tag{2.5}$$

The kernel function differs from the usual kernel function in that its growth term is linear in t. It was first introduced [18] and was used to determine the iteration direction of the algorithm for LO.

Thus we modify the search directions by new system

$$\begin{cases} \overline{Q}d_x + \overline{R}d_s = 0, \\ d_x + d_s = -\nabla\Psi(\upsilon), \end{cases} \tag{2.6}$$

where $\Psi(\upsilon)$ and $\psi(t)$ are defined by (2.5).

By using $\psi'(t) = 2 - \frac{2}{t}$, we can rewrite system (2.6) as the following system

$$\begin{cases} \overline{Q}d_x + \overline{R}d_s = 0, \\ \upsilon(d_x + d_s) = 2e - 2\upsilon. \end{cases}$$

2.2. **Generic full-Newton step feasible IPM for $P_*(\kappa)$-HLCP.** For the analysis of the proposed algorithm and based on [20], we use a norm-based proximity measure $\delta(x, s; \mu)$ defined by

$$\delta(\upsilon) := \delta(x, s; \mu) = \frac{1}{2} \parallel \upsilon\nabla\Psi(\upsilon) \parallel = \parallel v - e \parallel .$$

One can easily verify that $\delta(v) = 0$ iff $\upsilon = e$, i.e., $xs = \mu e$. Thus $\delta(v)$ is a distance which measure the closeness of a point (x, s) to the central path.

The following lemma gives a lower and upper bound for υ in terms of $\delta(\upsilon)$.

Lemma 2.2 ([9] Lemma 4.2)**.** *Let* $\delta := \delta(x, s; \mu)$. *Then*

$$1 - \delta \le \upsilon_i \le 1 + \delta, \qquad i = 1, 2, \cdots, n.$$

Next, we estimate the lower and the upper bounds concerning the scaled directions $d_x^T d_s$. We consider the following system

$$\begin{cases} \overline{Q}d_x + \overline{R}d_s = 0, \\ d_x + d_s = 2\upsilon^{-1} - 2e. \end{cases} \tag{2.7}$$

Since $(\overline{Q}, \overline{R})$ is a $P_*(\kappa)$-pair, then we have

$$\begin{aligned} d_x^T d_s &\geq -4\kappa \sum_{i\in I^+} [d_x]_i[d_s]_i \geq -\kappa \sum_{i\in I^+} ([d_x]_i + [d_s]_i)^2 \\ &\geq -\kappa \sum_{i=1}^{n} ([d_x]_i + [d_s]_i)^2 = -\kappa ||d_x + d_s||^2 \\ &= -4\kappa||\upsilon^{-1}(e-\upsilon)||^2 \geq -4\kappa \frac{\delta^2}{\upsilon_{min}^2} \geq -\frac{4\kappa\delta^2}{(1-\delta)^2}, \end{aligned}$$

where $I^+ := \{i : [d_x]_i[d_s]_i \geq 0\}$. On the other hand, we have

$$||d_x + d_s||^2 = ||d_x||^2 + 2d_x^T d_s + ||d_s||^2 = ||d_x - d_s||^2 + 4d_x^T d_s,$$

which implies that

$$d_x^T d_s \leq \frac{1}{4}||d_x + d_s||^2 = ||\upsilon^{-1}(e-\upsilon)||^2 \leq \frac{1}{\upsilon_{min}^2}||e-\upsilon||^2 \leq \frac{\delta^2}{(1-\delta)^2}. \tag{2.8}$$

The framework of the algorithm for $P_*(\kappa)$-HLCP is shown as follows.

Algorithm 1: A full-Newton step IPM

Input:
- Accuracy parameter $\varepsilon > 0$;
- threshold parameter $0 < \tau < 1$;
- barrier update parameter θ, $0 < \theta < 1$;
- strictly feasible pair (x^0, s^0) with $\mu^0 > 0$ such that $\delta(x^0, s^0; \mu^0) \leq \tau$.

begin
- $x := x^0$; $s := s^0$; $\mu := \mu^0$;
- **while** $x^T s > \varepsilon$
  - solve (2.7) and use (2.3) to obtain $(\Delta x, \Delta s)$, and let
  - $x := x + \Delta x$;
  - $s := s + \Delta s$;
  - μ-update: $\mu := (1-\theta)\mu$;
- **end**

end

3. Analysis of the Algorithm

In this section, we first find a condition for feasibility of the full-Newton step. Subsequently, we study the local quadratic convergence of the full-Newton step to the central path. Finally, we derive the iteration bound of the proposed algorithm.

3.1. **Feasibility of the full-Newton step.** By using (2.3) and the second equation of system (2.7), we obtain

$$\begin{aligned} x^+s^+ &= (x+\Delta x)(s+\Delta s) \\ &= xs+(s\Delta x+x\Delta s)+\Delta x\Delta s \\ &= \mu\upsilon^2+\mu\upsilon(d_x+d_s)+\mu d_xd_s \\ &= \mu\upsilon^2+2\mu(e-\upsilon)+\mu d_xd_s \\ &= \mu[(\upsilon-e)^2+e+d_xd_s]. \end{aligned} \tag{3.1}$$

Lemma 3.1. *A new iterates (x^+,s^+) are strictly feasible iff $(\upsilon-e)^2+e+d_xd_s>0$.*

The proof is similar to the Lemma 4.1 in [18]. Thus we omit the proof.

3.2. **Local quadratic convergence of the full-Newton step.** The following lemma proceeds to prove the local quadratic convergence of the full-Newton step to the target point $(x(\mu),s(\mu))$.

Lemma 3.2. *If $\delta:=\delta(x,s;\mu)<\frac{2^{\frac14}}{2^{\frac14}+\sqrt{3+4\kappa}}$ and $n\geq 2$, then the full-Newton step is strictly feasible. Moreover,*

$$\delta(x^+,s^+;\mu)\leq\frac{(3+4\kappa)\delta^2}{\sqrt2(1-\delta)^2}.$$

Proof.

$$\begin{aligned} \delta(x^+,s^+;\mu) &= ||\upsilon^+-e||=\frac{\|(\upsilon^+)^2-e\|}{\|\upsilon^++e\|}\leq\frac{\|(\upsilon^+)^2-e\|}{\|e\|} \\ &= \frac{\|(\upsilon^+)^2-e\|}{\sqrt n}\leq\frac{\|(\upsilon^+)^2-e\|}{\sqrt2}=\frac{1}{\sqrt2}||(\upsilon-e)^2+e+d_xd_s-e|| \\ &= \frac{1}{\sqrt2}||(\upsilon-e)^2+d_xd_s||\leq\frac{1}{\sqrt2}(||\upsilon-e||^2+||d_xd_s||) \\ &\leq \frac{1}{\sqrt2}||\upsilon-e||^2+\frac{1}{2\sqrt2}(||d_x||^2+||d_s||^2) \\ &= \frac{1}{\sqrt2}||\upsilon-e||^2+\frac{1}{2\sqrt2}(||d_x+d_s||^2-2d_x^Td_s) \\ &\leq \frac{1}{\sqrt2}||\upsilon-e||^2+\frac{1}{2\sqrt2}[\frac{1}{\upsilon_{min}^2}||\upsilon(d_x+d_s)||^2-2d_x^Td_s] \\ &\leq \frac{1}{\sqrt2}||\upsilon-e||^2+\frac{2}{\sqrt2(1-\delta)^2}||\upsilon-e||^2+\frac{2\kappa\delta^2}{(1-\delta)^2} \\ &= \frac{1}{\sqrt2}(\delta^2+\frac{2\delta^2}{(1-\delta)^2}+\frac{4\kappa\delta^2}{(1-\delta)^2}) \\ &\leq \frac{(3+4\kappa)\delta^2}{\sqrt2(1-\delta)^2}. \end{aligned}$$

This completes the second part of the proof. As for the first part, we know that if

$$\delta<\frac{2^{\frac14}}{2^{\frac14}+\sqrt{3+4\kappa}},$$

then

$$||d_x d_s|| \leq \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2} < 1.$$

Thus $(\upsilon - e)^2 + e + d_x d_s > 0$. This completes the proof. □

Corollary 3.3. *If $\delta := \delta(x,s;\mu) < \frac{1}{4\kappa+4}$, then*

$$\delta(x^+, s^+; \mu) \leq \frac{8\sqrt{2}}{3}(1+\kappa)\delta^2,$$

which shows the local quadratical convergence of the full-Newton step to the central path.

Proof. Observe that

$$\begin{aligned}
\delta(x^+, s^+; \mu) &\leq \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2} \\
&\leq \frac{1}{\sqrt{2}(1-\frac{1}{4\kappa+4})^2}(3+4\kappa)\delta^2 \\
&= \frac{(4+4\kappa)^2}{\sqrt{2}(3+4\kappa)}\delta^2 \\
&\leq \frac{8\sqrt{2}}{3}(1+\kappa)\delta^2.
\end{aligned}$$

The proof is complete. □

The following lemma describes the effect of a μ-update and a full-Newton step on the proximity measure.

Lemma 3.4. *Let x, s be strictly feasible and $\mu > 0$. If*

$$\delta := \delta(x,s;\mu) < \frac{2^{\frac{1}{4}}}{2^{\frac{1}{4}} + \sqrt{3+4\kappa}},$$

$\mu^+ := (1-\theta)\mu$ *for* $0 < \theta < 1$, *then*

$$\delta(x^+, s^+; \mu^+) \leq \frac{\delta^2 + \theta\sqrt{n} + \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}{\sqrt{1-\theta}\left(\sqrt{1-\theta} + \sqrt{1 - 4\delta + \delta^2 - \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}\right)}.$$

Proof. Since $\upsilon^+ = \sqrt{\frac{x^+s^+}{\mu}}$, one has

$$\begin{aligned}
\delta(x^+,s^+;\mu^+) &= ||\sqrt{\frac{x^+s^+}{\mu^+}} - e|| \\
&= \frac{1}{\sqrt{1-\theta}}||\sqrt{1-\theta}e - \upsilon^+|| \\
&= \frac{1}{\sqrt{1-\theta}}||\frac{(1-\theta)e - (\upsilon^+)^2}{\sqrt{1-\theta}e + \upsilon^+}|| \\
&\leq \frac{1}{\sqrt{1-\theta}}\frac{||e-\theta e - (\upsilon^+)^2||}{\sqrt{1-\theta} + min(\upsilon^+)} \\
&= \frac{1}{\sqrt{1-\theta}}\frac{||e-\theta e - (\upsilon - e)^2 - e - d_xd_s||}{\sqrt{1-\theta} + min(\upsilon^+)} \\
&\leq \frac{1}{\sqrt{1-\theta}}\frac{||\theta e|| + ||\upsilon - e||^2 + ||d_xd_s||}{\sqrt{1-\theta} + min(\upsilon^+)} \\
&\leq \frac{1}{\sqrt{1-\theta}}\frac{\theta\sqrt{n} + \delta^2 + \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}{\sqrt{1-\theta} + min(\upsilon^+)},
\end{aligned}$$

where the last inequation is due to the relation $\|d_xd_s\| \leq \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}$, which can be obtained from the proof of the lemma above. According to

$$\begin{aligned}
min(\upsilon^+)^2 &= min((\upsilon - e)^2 + e + d_xd_s) \\
&= min(\upsilon^2 - 2\upsilon + 2e + d_xd_s) \\
&\geq (1-\delta)^2 - 2(1+\delta) + 2 - ||d_xd_s||_\infty \\
&\geq 1 - 4\delta + \delta^2 - \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2},
\end{aligned}$$

can obtains

$$min(\upsilon^+) \geq \sqrt{1 - 4\delta + \delta^2 - \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}.$$

Hence

$$\delta(x^+,s^+;\mu^+) \leq \frac{\delta^2 + \theta\sqrt{n} + \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}{\sqrt{1-\theta}(\sqrt{1-\theta} + \sqrt{1 - 4\delta + \delta^2 - \frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}})}.$$

This completes the proof. □

3.3. **Iteration bound.** To give the complexity of the algorithm, we need to find a threshold parameter τ and an update parameter θ, such that at the start of an iteration the iterate satisfies $\delta(x,s;\mu) < \tau$. After a full-Newton step and μ-update, we still have $\delta(x^+,s^+;\mu^+) < \tau$. It follows from Lemma 3.4 that its sufficient condition

$$\frac{\delta^2+\theta\sqrt{n}+\frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}{\sqrt{1-\theta}(\sqrt{1-\theta}+\sqrt{1-4\delta+\delta^2-\frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}})}<\tau. \tag{3.2}$$

Notes that the left-hand side of the above formula is monotonically increasing with respect to δ, this means that

$$\begin{aligned}&\frac{\delta^2+\theta\sqrt{n}+\frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}}{\sqrt{1-\theta}(\sqrt{1-\theta}+\sqrt{1-4\delta+\delta^2-\frac{(3+4\kappa)\delta^2}{\sqrt{2}(1-\delta)^2}})}\\ \leq\ &\frac{\tau^2+\theta\sqrt{n}+\frac{(3+4\kappa)\tau^2}{\sqrt{2}(1-\tau)^2}}{\sqrt{1-\theta}(\sqrt{1-\theta}+\sqrt{1-4\tau+\tau^2-\frac{(3+4\kappa)\tau^2}{\sqrt{2}(1-\tau)^2}})}.\end{aligned}$$

To keep $\delta(x^+,s^+;\mu^+)<\tau$, it suffices that

$$\frac{\tau^2+\theta\sqrt{n}+\frac{(3+4\kappa)\tau^2}{\sqrt{2}(1-\tau)^2}}{\sqrt{1-\theta}(\sqrt{1-\theta}+\sqrt{1-4\tau+\tau^2-\frac{(3+4\kappa)\tau^2}{\sqrt{2}(1-\tau)^2}})}<\tau.$$

By some elementary calculation, if we set $\tau=\frac{2^{\frac{1}{4}}}{3(3+4\kappa)}$, $\theta=\frac{2}{5(3+4\kappa)\sqrt{n}}$, then inequality (3.2) is satisfied, which means that $(x,s)>0$ and $\delta(x,s;\mu)\leq\frac{2^{\frac{1}{4}}}{2^{\frac{1}{4}}+\sqrt{3+4\kappa}}$ are maintained during the algorithm. Thus the algorithm is well-defined.

Lemma 3.5. *Suppose that (x^0,s^0) are strictly feasible, and $\delta(x^0,s^0;\mu^0)\leq\frac{2^{\frac{1}{4}}}{3(3+4\kappa)}$ with $\mu^0=\frac{(x^0)^Ts^0}{n}$. Let (x^k,s^k) be the iterates obtained after k iterations of Algorithm 1. Then inequality $(x^k)^Ts^k\leq\varepsilon$ is satisfied after at most*

$$k\geq\frac{1}{\theta}\log\frac{21n\mu^0}{20\varepsilon}$$

iterations.

Proof. By using (2.8) and (3.1), it follows that

$$\begin{aligned}(x^+)^Ts^+ &= e^T(x^+s^+)=\mu e^T[(\upsilon-e)^2+e+d_xd_s]\\ &= \mu[e^T(\upsilon-e)^2+e^Te+d_x^Td_s]\\ &\leq \mu(||e||||\upsilon-e||^2+n+d_x^Td_s)\\ &\leq \mu(\sqrt{n}\delta^2+n+\frac{\delta^2}{(1-\delta)^2})\\ &\leq \frac{21}{20}n\mu,\end{aligned}$$

where the last inequalities is established because $\delta \leq \frac{2^{\frac{1}{4}}}{3(3+4\kappa)}$. Thus, we have

$$(x^k)^T s^k \leq \frac{21}{20} n\mu^k = \frac{21}{20} n(1-\theta)^k \mu^0.$$

Then, the inequality $(x^k)^T s^k \leq \varepsilon$ is satisfied if

$$\frac{21}{20} n(1-\theta)^k \mu^0 \leq \varepsilon.$$

Taking logarithms at both sides, we obtain

$$k\log(1-\theta) + \log(\frac{21}{20} n\mu^0) \leq \log\varepsilon. \tag{3.3}$$

By using (3.3) and $-\log(1-\theta) \geq \theta$, we observe that the above inequality holds if

$$k\theta \geq \log(\frac{21}{20} n\mu^0) - \log\varepsilon = \log\frac{21n\mu^0}{20\varepsilon}.$$

Hence, the result of this lemma holds. □

The above discussion leads to the following main result of the paper, which gives an upper bound on the number of iterations for Algorithm 1.

Theorem 3.6. *Let* $\tau = \frac{2^{\frac{1}{4}}}{3(3+4\kappa)}$ *and* $\theta = \frac{2}{5(3+4\kappa)\sqrt{n}}$. *Then Algorithm 1 requires at most*

$$O\left((3+4\kappa)\sqrt{n}\log\frac{n\mu^0}{\varepsilon}\right),$$

iterations to find an ε*-solution of* $P_*(\kappa)$*-HLCP.*

4. Computational Examples

In this section, we present preliminary computational results to compare the proposed algorithm with two full-Newton step algorithms. The first is based on the classical logarithmic kernel function [12], and the second is based on the positive asymptotic kernel function [19]. The algorithms are tested by using MATLAB 2018b on an Intel Core i5(2.40Ghz) with 8GB RAM. We consider the following three test examples. The first two are standard LCP, and the last one is HLCP.

Example 4.1 [9]

$$Q = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, \quad q = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad x^0 = \begin{bmatrix} 0.4 \\ 0.45 \end{bmatrix}.$$

Example 4.2 [21]

$$Q = \begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 2 & 5 & 6 & \cdots & 6 \\ 2 & 6 & 9 & \cdots & 10 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 6 & 10 & \cdots & 4n-3 \end{bmatrix}, \quad q = \begin{bmatrix} -1 \\ -1 \\ \vdots \\ -1 \\ -1 \end{bmatrix}, \quad x^0 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}.$$

Example 4.3 By using MATLAB 2018b, we randomly generate matrices $Q, R \in R^{n\times n}$ with row full rank, $x^0 = [1,1,\cdots,1]^T$ and $s^0 = [1,1,\cdots,1]^T$. Thus we obtain a corresponding test problem.

Table 1 demonstrates the computational results of our algorithm for Examples 4.1, 4.2, and 4.3. For Example 4.1, we take $\varepsilon = 10^{-4}, \tau = 0.8, \theta = 0.1, 0.2, 0.3, 0.4$, and 0.5. For Example 4.2 and 4.3, we take $\varepsilon = 10^{-4}, \theta = 0.2, \tau = 0.8, n = 10, 20, 50, 100, 200$, and 300. Table 2, 3, and 4 demonstrate the computational results of other algorithms for these three examples.

From the tables, we conclude that:

- In most cases, our algorithm use fewer iterations and much less time to solve these examples, compared with the other algorithms.

- Tables 1 and 2 demonstrate that the number of iterations of the proposed algorithms is related to the parameter θ. Generally speaking, the number of iterations decreases with the increase of θ.

- From Tables 1, 3, and 4, we see that as n increases, the number of iterations of our method increases slowly, which means that our algorithm is suitable for solving some large-scale problems.

TABLE 1. Computational results for proposed algorithm

θ	Example 4.1		n	Example 4.2		Example 4.3	
	Iter	CPU time(sec)		Iter	CPU time(sec)	Iter	CPU time(sec)
0.1	107	0.018250	10	19	0.021073	77	0.187183
0.2	59	0.017685	20	22	0.024335	82	0.290253
0.3	42	0.017073	50	27	0.047902	103	1.247159
0.4	35	0.015862	100	33	0.126952	124	4.818016
0.5	31	0.015507	200	36	0.299768	132	14.77525
			300	42	0.863025	146	40.903032

TABLE 2. Computational results for Example 4.1 using algorithms [12] and [19]

θ	Algorithm [12]		Algorithm [19]	
	Iter	CPU time(sec)	Iter	CPU time(sec)
0.1	125	0.019158	110	0.021093
0.2	63	0.018166	61	0.019135
0.3	43	0.017920	44	0.018130
0.4	32	0.017650	36	0.017912
0.5	25	0.017112	31	0.017377

TABLE 3. Computational results for Example 4.2 using algorithms [12] and [19]

n	Algorithm [12]		Algorithm [19]	
	Iter	CPU time(sec)	Iter	CPU time(sec)
10	23	0.024928	71	0.252338
20	26	0.025757	73	0.256345
50	40	0.075107	77	0.980675
100	44	0.204511	78	1.251524
200	48	0.561675	80	3.214563
300	51	1.246898	83	7.365054

TABLE 4. Computational results for Example 4.3 using algorithms [12] and [19]

n	Algorithm [12]		Algorithm [19]	
	Iter	CPU time(sec)	Iter	CPU time(sec)
10	81	0.235081	79	0.208289
20	84	0.235990	91	0.325721
50	109	2.164347	117	1.478392
100	126	5.001609	124	4.912536
200	134	15.35888	133	15.49288
300	152	41.82299	149	40.98413

5. CONCLUDING REMARKS

We presented and analyzed a new full-Newton step feasible IPM for solving $P_*(\kappa)$-HLCP based on a simple kernel function. We proved that complexity bound coincides with the currently best known iteration bound of feasible IPMs for $P_*(\kappa)$-HLCP. Moreover, some preliminary computational results suggested the efficiency and reliability of the algorithm. Our further work may includes two folds. One is to design the kernel function based IIPM, and the other is to make computational experiments for more test examples.

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