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THE SECOND-ORDER CAUCHY PROBLEM IN A SCALE OF BANACH SPACES WITH VECTOR-VALUED MEASURES OF NONCOMPACTNESS AND AN APPLICATION TO KIRCHHOFF EQUATIONS

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Abstract. In the paper, by using Darbo-Sadovskii fixed point theorem for condensing operators on Fréchet spaces with respect to vector-valued measure of noncompactness, we prove the existence results for the second-order Cauchy problem $u''(t) = f(t,u(t))$, $t \in (0,T)$, $u(0) = u_0$, $u'(0) = u_1$, in a scale of Banach spaces. The result is applied to the Kirchhoff equations in Gevrey class.

Keywords. Condensing operator; Kirchhoff equation; Measures of noncompactness; Scale of Banach spaces.

1. Introduction

The technique of measure of noncompactness (MNC, for short) has a very important role in functional analysis. Based on the MNC, Darbo's fixed point theorem demonstrates the existence of fixed points for a condensing map and become a significant extension of the Schauder fixed point theorem. Recently, some of generalized Darbo's fixed point theorems have been established based on the concept of generalized MNC (see [6]) and applied to the existence results for solving various types of nonlinear differential equations, nonlinear functional integral equations as well as their infinite systems (see, e.g., [2, 9, 12, 13]). This paper focuses on fixed point theorems for condensing operators on Fréchet spaces with respect to vector-valued MNC. Up to our knowledge, this concept was only introduced in [5], where the authors proved the existence results for the Cauchy problem with delay in a scale of Banach spaces $(X_s, |.|_s)$, $s \in [a,b]$

$$\frac{du}{dt} = f(t,u(t),u(h(t))),\ t \in (0,T),\ u(0) = u_0,$$

where $h(t) < t^{1/p}$ for some $p \in (0,1)$. The important point to note here is that the Cauchy problem with the right-hand side operator acting in a scale of Banach spaces is known as the abstract version of the Cauchy-Kovalevskaya theorem (see [14, 15]) and has a wide range of real applications in the study of PDE problems, including the Navier–Stokes equations for viscous

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incompressible flows, the integrable Camassa–Holm type equations, and the birth–and–death stochastic dynamics in the continuum; see, e.g., [3, 4, 7, 11]. The main difficulties in carrying out the differential equations in a scale of Banach spaces are that the operator f does not act from each space X_s into itself, but from X_s into larger spaces $X_r \supset X_s$, $r < s$. These difficulties demonstrate the advantages of using a vector–valued MNC instead of a real–valued MNC when we define the value of MNC as a function of parameter of the scale. One knows that the condensation of the operator F with respect to the measure of noncompactness Φ means that

$$\Phi(\Omega) \leq \Phi(F(\Omega)) \text{ implies compactness of } \overline{\Omega}. \tag{1.1}$$

So the relation between two functions $\Phi(\Omega), \Phi(F(\Omega))$ in (1.1) gives more information than in the case when Φ takes values in $[0,\infty)$.

In this paper, we investigate the following Cauchy problem in a scale of Banach spaces

$$\frac{d^2u}{dt^2} = f(t,u(t)),\ t \in (0,T),\ u(0) = u_0,\ u'(0) = u_1, \tag{1.2}$$

where f is a continuous operator from $[0,T) \times X_{s'}$ into X_s, $s < s'$and

$$|f(t,u)|_s \leq \frac{L|u|_{s'}}{(s'-s)^2}, \forall (t,u) \in [0,T) \times X_{s'};$$

$$\alpha_s\big(f(t,\Omega)\big) \leq \frac{L\alpha_{s'}(\Omega)}{(s'-s)^2}, \forall t \in [0,T) \text{ and bounded subsets } \Omega \subset X_{s'}.$$

The abstract results will be applied to generalized Kirchhoff equation

$$D_t^2 u(t,x) = f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\Delta_x u(t,x),\ (t,x) \in \Omega_T = [0,T] \times \Omega,$$
$$u(0,x) = u_0(x), u_t(0,x) = u_1(x), x \in \Omega, \tag{1.3}$$

where P, Ω are open subsets in $\mathbb{R}^n$ and $P \subset \Omega$ is bounded. (1.3) was considered in [8] and generalized in [10] where the approaches relied on the sequence of successive approximations and Schauder's fixed point theorem. However, our proof involves considering (1.3) so far for abstract Cauchy problem and the concept of Darbo–Sadovskii fixed point theorem for condensing operators on Fréchet spaces with respect to vector–valued MNC. In Section 2, we represent the existence of fixed points for a condensing map with respect to a vector-valued MNC in locally convex spaces which was established in [5]. The proof of the existence results of (1.2) is given in Section 3. Section 4 is devoted to applying the abstract results to the generalized Kirchhoff equation.

2. Preliminaries

2.1. The fixed point of a condensing map.

Definition 2.1. [1] Let E be a locally convex space, and let $\mathscr{M}$ be a family of subsets of E such that $\overline{conv}(\Omega) \in \mathscr{M}$ whenever $\Omega \in \mathscr{M}$. Let $(Q, \leq)$ be an ordered Banach space, and let K be a cone, contained in the positive cone. An operator $\Phi : \mathscr{M} \to K$ is called a MNC if

$$\Phi(\overline{conv}(\Omega)) = \Phi(\Omega), \text{ for all } \Omega \in \mathscr{M}.$$

The MNC Φ is said to be regular if it satisfies the following condition

$$\Phi(\Omega) = 0_Q \text{ if and only if } \Omega \text{ is relatively compact.}$$

Definition 2.2. [5] Let Φ be a MNC defined on a family $\mathscr{M}$ of subsets of the locally convex space E. An operator $F : D \subset E \to E$ is said to be condensing with respect to Φ (or $\Phi-$ *condensing*) if

(i) for every $\Omega \subset D$ such that $\Omega \in \mathscr{M}$, $F(\Omega) \in \mathscr{M}$;
(ii) moreover, for every $\Omega \subset D$ such that $\Omega \in \mathscr{M}$, if $\Phi(\Omega) \leq \Phi(F(\Omega))$, then $\overline{\Omega}$ is compact.

Theorem 2.3. [5] *Let E be a Fréchet space. Let $C \subset E$ be a nonempty, convex, and closed set. Let $F : C \to C$ be a continuous operator, and let Φ be a MNC defined on a family $\mathscr{M}$ of subsets of E. Moreover, assume that*

(1) If $\Omega \in \mathscr{M}, \{u\} \in \mathscr{M}$ and $\Omega_1 \subset \Omega$ then $\Omega_1 \in \mathscr{M}, \Omega \cup \{u\} \in \mathscr{M}$ and one has $\Phi(\Omega \cup \{u\}) = \Phi(\Omega)$.
(2) The operator F is $\Phi-$condensing and $F(C) \in \mathscr{M}$.

Then F has at least one fixed point in C.

Proof. Let us choose a point $u \in \overline{conv}(F(C))$ and denote by Σ the class of all closed and convex subsets Ω of C such that $\Omega \in \mathscr{M}, u \in \Omega$ and $F(\Omega) \subset \Omega$. Set

$$B = \bigcap_{\Omega \in \Sigma} \Omega, \;\; K = \overline{conv}\left(F(B) \cup \{u\}\right).$$

Obviously, $\overline{conv}(F(C)) \in \Sigma$ by condition 2 and Definition 2.2, and $B \in \mathscr{M}$ by condition 1. Furthermore, from $F(\Omega) \subset \Omega$, $\forall \Omega \in \Sigma$, it follows that $F(B) \subset B$. Hence $K \in \mathscr{M}$ by condition 1. We now claim $B = K$. Indeed, since $u \in B$ and $F(B) \subset B$, it follows that $K \subset B$, which implies $F(K) \subset F(B) \subset K$. Hence, $K \in \Sigma$, and then $B \subset K$. It follows from condition 1 that

$$\Phi(B) = \Phi(K) = \Phi\left(F(B) \cup \{u\}\right) = \Phi\left(F(B)\right).$$

Since F is $\Phi-$*condensing*, it follows that B is compact. Thus we conclude from the Schauder–Tychonoff theorem that there is a fixed point for the operator $F : C \to C$. ☐

2.2. **The MNC in a scale of Banach spaces.** We list below some of the properties of the Kuratowski measure of noncompactness, which can be found in [1, 1.1.4; 1.1.6; 4.1.6].

Proposition 2.4. *Let E be a Banach space, and let α be the Kuratowski measure of noncompactness in E, which is defined by, for each bounded subset $\Omega \subset E$,*

$$\alpha(\Omega) = \inf\{d > 0 : \Omega \textit{ is covered by a finite family of subsets with diameter less than } d\}.$$

Then

(1) α is regular;
(2) $\alpha(\Omega_1 \cup \Omega_2) = \max\{\alpha(\Omega_1), \alpha(\Omega_2)\}$; $\alpha(\lambda\Omega) = |\lambda|\alpha(\Omega)$;
(3) If $\dim E = \infty$ and B is a ball with radius R, then $\alpha(B) = 2R$;
(4) If $A \subset C([0,T], E)$ is a equicontinuous set such that the set $A(t) = \{u(t) : u \in A\}$ is bounded for all $t \in [0,T]$, then the function $t \mapsto \alpha(A(t))$ is continuous and one has

$$\alpha\left(\left\{\int_0^t u(\tau)d\tau : u \in A\right\}\right) \leq \int_0^t \alpha(A(\tau))d\tau.$$

Let $(X_s, |.|_s), s \in [a,b]$ be a scale of Banach spaces, that is,

$$X_{s'} \subset X_s, \ |u|_s \le |u|_{s'}, \ \text{ for } s < s'. \tag{2.1}$$

The requirement on (1.2) is that $u_0, u_1 \in X_b$. In what follows, $\overline{B}_s(u_0, r)$ stands for the closed ball centered at u_0 with radius r in X_s and α_s denotes the Kuratowski measure of noncompactness on X_s. We conclude from (2.1) that

$$\alpha_s(\Omega) \le \alpha_{s'}(\Omega) \text{ for a bounded subset } \Omega \subset X_{s'} \text{ and } s < s'. \tag{2.2}$$

We now set up the MNC in $\{X_s\}$ which is used to solve (1.2) for $t \in [0,T]$. Fix $\lambda > \max\{(b-a)/T, (b-a)\}$, which will be determined later. We define the set

$$\Delta = \left\{ (t,s) : s \in [a,b), t \in \left[0, \frac{b-s}{\lambda}\right) \right\},$$

and the Fréchet space

$$E = \left\{ u \in C([0,T_\lambda), X_a) : u|_{[0,t]} \in C([0,t], X_s), \ \forall (t,s) \in \Delta \right\}, \ T_\lambda = \frac{b-a}{\lambda},$$

whose topology is induced by the following countable separating family of seminorms

$$p_n(u) = \sup_{t \in [0,t_n]} |u(t)|_{s_n},$$

where the sequence $\{(t_n, s_n)\}_n$ is dense in Δ. Here and subsequently, for simplicity of notation, we use the same letter u for the restriction of $u \in E$ to $[0,t]$.

Let β be a positive constant and denote by Q the space of functions $g : \Delta \to \mathbb{R}$ such that

(Q1) The function $t \mapsto g(t,s)$ is continuous on $\left[0, \frac{b-s}{\lambda}\right)$ for every $s \in [a,b)$;
(Q2) $\|g\| = \sup_{(t,s) \in \Delta} (b-s-\lambda t)^\beta |g(t,s)| < \infty$.

It is clear that $(Q, \|.\|)$ is a Banach space. We will consider the partial order in Q, which is defined by the cone of nonnegative functions, that is, $g_1 \le g_2$ if $g_1(t,s) \le g_2(t,s), \ \forall (t,s) \in \Delta$. We also define in Q the cone K of nonnegative functions $g \in Q$ such that

(K) The function $s \mapsto g(t,s)$ is nondecreasing on $[a, b-\lambda t)$ for every $t \in [0, T_\lambda)$.

Further, let us denote by $\mathscr{M}$ the family of subsets $\Omega \subset E$ satisfying:

(M1) There exists $R > 0$ such that $\sup_{(t,s) \in \Delta} (b-s-\lambda t)^\beta |u(t)|_s \le R, \ \forall u \in \Omega$;
(M2) Ω is equicontinuous in $C([0,t], X_s)$ for all $(t,s) \in \Delta$.

Now, we introduce an operator $\Phi : \mathscr{M} \to K$ defined by

$$\Phi(\Omega)(t,s) = \alpha_s\big(\Omega(t)\big), \ (t,s) \in \Delta, \ \Omega \in \mathscr{M}.$$

Lemma 2.5. [5] *The operator Φ is a regular MNC and satisfies the condition 1 in Theorem* (2.3).

3. The Results

We consider the following assumption for the functions involved in (1.2):

(H) There exists a positive number L such that, for $a \le s < s' \le b$, function f is continuous from $[0,T) \times X_{s'}$ into X_s and

$$|f(t,u)|_s \le \frac{L|u|_{s'}}{(s'-s)^2}, \forall (t,u) \in [0,T) \times X_{s'};$$

$$\alpha_s\big(f(t,\Omega)\big) \le \frac{L\alpha_{s'}(\Omega)}{(s'-s)^2}, \forall t \in [0,T) \text{ and bounded subsets } \Omega \subset X_{s'}.$$

Lemma 3.1. *Assume that conditions (H) are satisfied. Consider the operator*

$$Fu(t) = \bar{u}(t) + \int_0^t d\tau \int_0^\tau f(r,u(r))dr,$$

where $\bar{u}(t) = u_0 + tu_1$ and the convex set C, defined as follows

$$C = \left\{ u \in E : \sup_{(t,s)\in\Delta} (b-s-\lambda t)^\beta |u(t)|_s \le R \right\}.$$

Then we can choose λ and R sufficiently large such that

(1) The operator F acts continuously from C to C.
(2) If $\Omega \subset C$, then $F(\Omega) \in \mathscr{M}$. In particular, $F(C) \in \mathscr{M}$.

Proof. First, we prove claim 1.

To verify that $Fu \in E$ if $u \in C$, we fix $(t,s) \in \Delta$ and choose $s' \in (s, b-\lambda t)$. Then $(t,s') \in \Delta$ and $u \in C([0,t], X_{s'})$. Therefore, function $r \mapsto f(r,u(r))$ belongs to $C([0,t],X_s)$, so does Fu. Let $u_n, u \in C$ and $u_n \to u$ as $n \to \infty$. To demonstrate $F(u_n) \to F(u)$ as $n \to \infty$, we need to prove $p_m(F(u_n) - F(u)) \to 0$ for each $m \in \mathbb{N}^+$. Choosing $k \in \mathbb{N}^+$ such that $t_m < t_k, s_m < s_k$, one has

$$0 \le \sup_{r\in[0,t_m]} |u_n(r) - u(r)|_{s_m} \le \sup_{r\in[0,t_k]} |u_n(r) - u(r)|_{s_k} \to 0 \text{ as } n \to \infty.$$

Therefore, the set $A = \{u_n(r), u(r) : \tau \in [0,t_m], n \in \mathbb{N}^+\}$ is compact in X_{s_k}. As a consequence, function f is uniformly continuous from $[0,t_m] \times A$ into X_{s_m}. Given $\varepsilon > 0$, let $\delta > 0$ be chosen such that $(t,u),(t,v) \in [0,t_m] \times A$, $|u-v|_{s_k} < \delta$, which implies $|f(t,u) - f(t,v)|_{s_m} < \varepsilon$. Then, for n so large, $\sup_{r\in[0,t_m]} |u_n(r) - u(r)|_{s_k} < \delta$, we obtain

$$\begin{aligned} p_m(F(u_n) - F(u)) &= \sup_{t\in[0,t_m]} |Fu_n(t) - Fu(t)|_{s_m} \\ &\le \int_0^{t_m} d\tau \int_0^\tau |f(r,u_n(r)) - f(r,u(r))|_{s_m} dr \\ &\le \varepsilon \frac{(t_m)^2}{2}. \end{aligned}$$

Thus $p_m(F(u_n) - F(u)) \to 0$ as $n \to \infty$ for each $m \in \mathbb{N}^+$, or equivalently, $F(u_n) \to F(u)$ as $n \to \infty$. We next claim that $F(C) \subset C$. Set $M = \sup_{t\in[0,T]} |u_0 + tu_1|_b$. If $u \in C$ and $(t,s) \in \Delta$, then we have by the assumption (H) that

$$|Fu(t)|_s \le |\overline{u}(t)|_b + \int_0^t d\tau \int_0^\tau \frac{L|u(r)|_{s(r)}}{(s(r)-s)^2} dr$$
$$M + L\int_0^t d\tau \int_0^\tau \frac{(b-s(r)-\lambda r)^{-\beta} R}{(s(r)-s)^2} dr,$$

when $s < s(r) < b - \lambda r$. By setting $s(r) = (b+s-\lambda r)/2$, one has $s(r) - s = b - s(r) - \lambda r = \frac{b-s-\lambda r}{2}$, which yields that

$$\begin{aligned} |Fu(t)|_s &\le M + L\int_0^t d\tau \int_0^\tau \frac{R2^{\beta+2}}{(b-s-\lambda r)^{\beta+2}} dr \\ &\le M + LR2^{\beta+2}\int_0^t \frac{(b-s-\lambda\tau)^{-1-\beta}}{\lambda(\beta+1)} d\tau \\ &\le M + \frac{LR2^{\beta+2}}{\lambda^2\beta(\beta+1)(b-s-\lambda t)^\beta}. \end{aligned}$$

We thus obtain

$$(b-s-\lambda t)^\beta |Fu(t)|_s \le Mb^\beta + \frac{LR2^{\beta+2}}{\lambda^2\beta(\beta+1)}. \tag{3.1}$$

We can choose λ sufficiently large such that $L2^{\beta+2} < \lambda^2\beta(\beta+1)$ and R sufficiently large such that the right–hand side of (3.1) is less than R.

We proceed to prove claim 2. The definition of the set C and $F(C) \subset C$ follows that if $\Omega \subset C$, then $F(\Omega)$ satisfies condition (M1). Let $u \in \Omega$ and $(t,s) \in \Delta$. Then, for $t_1, t_2 \in [0,t]$, $t_1 < t_2$, the assumption (H) demonstrates that

$$|Fu(t_1) - Fu(t_2)|_s \le |\overline{u}(t_1) - \overline{u}(t_2)|_b + \int_{t_1}^{t_2} d\tau \int_0^\tau \frac{L|u(r)|_{s(r)}}{(s(r)-s)^2} dr.$$

Letting $s(\tau) = (b+s-\lambda\tau)/2$, we deduce that

$$\begin{aligned} |Fu(t_1) - Fu(t_2)|_s &\le |u_1|_b|t_1 - t_2| + \int_{t_1}^{t_2} d\tau \int_0^\tau \frac{LR2^{\beta+2}}{(b-s-\lambda r)^{\beta+2}} dr \\ &\le |u_1|_b|t_1 - t_2| + \frac{LR2^{\beta+2}}{\lambda(\beta+1)} \int_{t_1}^{t_2} (b-s-\lambda\tau)^{-1-\beta} d\tau \\ &\le |u_1|_b|t_1 - t_2| + LR2^{\beta+2}(b-s-\lambda t)^{-1-\beta}|t_1 - t_2|. \end{aligned}$$

We conclude that set $F(\Omega)$ is equicontinuous in $C([0,t], X_s)$. The condition (M2) is satisfied and we have $F(\Omega) \subset C$. This completes the proof. □

Lemma 3.2. *Let B be an operator which is defined on K by*

$$B(g)(t,s) = \int_0^t d\tau \int_0^\tau \frac{Lg(r, S(r))dr}{(S(r)-s)^2},\ (t,s) \in \Delta,\ g \in K,$$

where $S(r) = (b+s-\lambda r)/2$. Then B is an increasing operator from K into K and satisfies

$$\|B^n(g)\| \le \|g\| \left(\frac{L2^{\beta+2}}{\lambda^2\beta(\beta+1)}\right)^n,\ n \in \mathbb{N}^+. \tag{3.2}$$

Proof. For each $g \in K$ and a fixed $s \in [a,b)$, we prove that function $B(g)$ is well defined and continuous with respect to the variable t on $\left[0, \frac{b-s}{\lambda}\right)$ by showing that the function $P(r) := g(r,S(r))(S(r)-s)^{-2}$ is Lebesgue measurable and bounded in each interval $[0,t']$ with $t' < (b-s)/\lambda$. Fix $t'' \in [0,(b-s)/\lambda)$. Setting $\phi(\rho) = g(t'',\rho)$, one has $g(t'',S(r)) = (\phi \circ S)(r)$. Since ϕ is nondecreasing, $\phi^{-1}(-\infty,\alpha)$ is an interval. From this and the continuity of S, the set

$$\{r \in [0,t'] : \phi \circ S(r) < \alpha\} = S^{-1}\big(\phi^{-1}(-\infty,\alpha)\big) \tag{3.3}$$

is Lebesgue measurable, so is the function $r \mapsto g(t'',S(r))$. We conclude that $(t'',r) \mapsto g(t'',S(r))$ is a Carathéodory function. Therefore, the function $r \mapsto g(r,S(r))$ is measurable, so is the function $P(r)$. The boundedness of $P(r)$ on $[0,t']$ with $t' < (b-s)/\lambda$ follows from $S(r)-s = b-S(r)-\lambda r = (b-s-\lambda r)/2$ and

$$\begin{aligned} P(r) &\le \frac{\|g\|}{[b-S(r)-\lambda r]^{\beta}}(S(r)-s)^{-2} \\ &\le \frac{\|g\|2^{\beta+2}}{(b-s-\lambda r)^{\beta+2}} \le \frac{\|g\|2^{\beta+2}}{(b-s-\lambda t')^{\beta+2}}, \ \forall r \in [0,t']. \end{aligned}$$

Thus, $B(g)(t,s)$ is well defined and continuous in $t \in [0,(b-s)/\lambda)$. Moreover, $B(g)(t,s)$ is nondecreasing in the variable s, which is due to the fact that the functions $s \mapsto (b+s-\lambda r)/2, s \mapsto (b-s-\lambda r)^{-2}$ and $\rho \mapsto g(t'',\rho)$ are nondecreasing.

To prove (3.2), we will prove by induction that

$$B^n(g)(t,s) \le \|g\| \left(\frac{L2^{\beta+2}}{\lambda^2\beta(\beta+1)}\right)^n \frac{1}{(b-s-\lambda t)^{\beta}}, \ (t,s) \in \Delta. \tag{3.4}$$

Indeed, from the definition of B and space Q, we have

$$\begin{aligned} B(g)(t,s) &\le \int_0^t d\tau \int_0^{\tau} \frac{L\|g\|(b-S(r)-\lambda r)^{-\beta}}{(S(r)-s)^2} dr \\ &\le L\|g\|2^{\beta+2} \int_0^t d\tau \int_0^{\tau} \frac{dr}{(b-s-\lambda r)^{\beta+2}} \\ &\le \|g\| \frac{L2^{\beta+2}}{\lambda^2\beta(\beta+1)} \frac{1}{(b-s-\lambda t)^{\beta}}. \end{aligned}$$

That is (3.4) for $n=1$. Assume that (3.4) holds for degree n. Then,

$$\begin{aligned} B^{n+1}(g)(t,s) &= \int_0^t d\tau \int_0^{\tau} \frac{LB^n g(r,S(r))dr}{(S(r)-s)^2} \\ &\le L\|g\| \left(\frac{L2^{\beta+2}}{\lambda^2\beta(\beta+1)}\right)^n \int_0^t d\tau \int_0^{\tau} \frac{(b-S(r)-\lambda r)^{-\beta}}{(S(r)-s)^2} dr \\ &\le L\|g\|2^{\beta+2} \left(\frac{L2^{\beta+2}}{\lambda^2\beta(\beta+1)}\right)^n \int_0^t d\tau \int_0^{\tau} \frac{dr}{(b-s-\lambda r)^{\beta+2}} \\ &\le \|g\| \left(\frac{L2^{\beta+2}}{\lambda^2\beta(\beta+1)}\right)^{n+1} \frac{1}{(b-s-\lambda t)^{\beta}}, \end{aligned}$$

which demonstrates that (3.4) holds with $n+1$ in place of n. This establishes (3.2), and the proof is complete. □

Theorem 3.3. *Assume that assumption (H) hold. Then there exists a number $\lambda > 0$ such that problem (1.2) has a solution u satisfying $u(t) \in X_s$ for all $t \in [0,(b-s)/\lambda)$, $s \in [a,b)$.*

Proof. Let λ be chosen sufficiently large such that $L2^{\beta+2} < \lambda^2\beta(\beta+1)$ and $\lambda > (b-a)T^{-1}$. We prove that the operator F which is defined in Lemma (3.1) is condensing with respect to the MNC Φ. In view of Lemma (3.1)-(3.2) and Theorem (2.3), F has a fixed point in the space E. This fixed point is indeed a solution to (1.2) with desired properties. Assume that $\Phi(\Omega) \leq \Phi(F(\Omega))$ for some $\Omega \in \mathscr{M}$. From Proposition (2.4) and hypothesis (H), we obtain

$$\begin{aligned}
\Phi(\Omega)(t,s) \leq \Phi(F(\Omega))(t,s) &= \alpha_s\left(\left\{\int_0^t d\tau\int_0^\tau f(r,u(r))dr : u\in\Omega\right\}\right)\\
&\leq \int_0^t \alpha_s\left(\left\{\int_0^\tau f(r,u(r))dr : u\in\Omega\right\}\right)d\tau\\
&\leq \int_0^t d\tau\int_0^\tau \alpha_s\left(\{f(r,u(r)) : u\in\Omega\}\right)dr\\
&\leq \int_0^t d\tau\int_0^\tau \frac{L\alpha_{s'}(\Omega(r))}{(s'-s)^2}dr,
\end{aligned}$$

where $s' \in (s, b-\lambda r)$. By setting $s' = (b+s-\lambda r)/2$ and $g = \Phi(\Omega)$, we see that $g(t,s) \leq B(g)(t,s)$ for all $(t,s) \in \Delta$. Since g is increasing, we have $g(t,s) \leq B^n(g)(t,s)$ for all $(t,s) \in \Delta$. Therefore $\|g\| \leq \|B^n g\|$ and $g = 0_Q$ by (3.2) and $L2^{\beta+2} < \lambda^2\beta(\beta+1)$. We conclude from the regularity of Φ that $\overline{\Omega}$ is compact, and the proof is complete. □

4. Kirchhoff Equations in Gevrey Class

For $s > 0$, let E_s be the space of all function $u \in C^\infty(\Omega)$ such that (see [10])

$$|u|_s := \sum_{\alpha\in\mathbb{N}^n} \|D^\alpha u\| \frac{s^{|\alpha|}}{\alpha!} < \infty,$$

where $\Omega \subset \mathbb{R}^n$ is an open subset, $\|v\| = \sup\{|v(x)| : x \in \Omega\}$, and $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{N}^n, \alpha! = \alpha_1!\alpha_2!\ldots\alpha_n!, |\alpha| = \alpha_1+\ldots+\alpha_n$. Then $(E_s, |.|_s)_{s>0}$ forms a scale of Banach space with the properties (2.1). It is well known that the class Gevrey $\mathscr{G}(\Omega)$ consist all real functions $u \in C^\infty(\Omega)$ satisfying

$$\exists K > 0, c > 0 : \|D^\alpha u\| \leq K\frac{\alpha!}{c^{|\alpha|}}, \forall \alpha \in \mathbb{N}^n.$$

Moreover, if $u \in \mathscr{G}(\Omega)$, then, for $s < c$,

$$|u|_s = \sum_{\alpha\in\mathbb{N}^n} \|D^\alpha u\| \frac{c^{|\alpha|}}{\alpha!}\left(\frac{s}{c}\right)^{|\alpha|} \leq K\sum_{i=0}^\infty (i+1)\left(\frac{s}{c}\right)^i < \infty.$$

We thus obtain $\mathscr{G}(\Omega) = \bigcup_{s>0} E_s$.

Lemma 4.1. *(see [10, lemma 1]) The scale $(X_s, |.|_s), s \in [a,b]$ has the following properties:*

(i) If $u,v \in X_s$, then $uv \in X_s$ and one has $|uv|_s \leq |u|_s|v|_s$.

(ii) There exists a constant $M > 0$, depending only on a, b, such that, for $a \le s < s' \le b$,

$$|\Delta u|_s \le \frac{M|u|_{s'}}{(s'-s)^2}, \, u \in X_{s'},$$

where Δ is the Laplacian.

Following [8, 10], we consider the Cauchy problem

$$\begin{aligned} D_t^2 u(t,x) &= f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\Delta_x u(t,x), \; (t,x) \in \Omega_T = [0,T]\times\Omega, \\ u(0,x) &= u_0(x), u_t(0,x) = u_1(x), x \in \Omega, \end{aligned} \tag{4.1}$$

where P, Ω are open subsets in $\mathbb{R}^n$ and $P \subset \Omega$ is bounded. The problem has form (1.2) if we additionally assume that $u_0, u_1 \in X_b$ with $b > 0$ is fixed.

The assumptions for function $f : \Omega_T \times \mathbb{R}^+ \to \mathbb{R}$ are that:

(A1) $f(t,.,u) \in C^\infty(\Omega)$ for all $(t,u) \in [0,T]\times\mathbb{R}^+$ and for all $\alpha \in \mathbb{N}^n$ the operator $u \mapsto D_x^\alpha f(.,.,u)$ belongs to $C\big(\mathbb{R}^+, C(\Omega_T)\big)$.

(A2) There are $c > 0, K > 0$ such that

$$|D_x^\alpha f(t,x,u)| \le K\frac{\alpha!}{c^{|\alpha|}},$$

for all $(t,x,u) \in \Omega_T \times \mathbb{R}^+$ and $\alpha \in \mathbb{N}^n$.

Lemma 4.2. *Let assumptions (A1) and (A2) be hold, and let $C \subset X_s$ be a bounded subset. Then the subset*

$$F(C) := \left\{ f\left(t,x,\int_P |\nabla_x u|^2 dx\right) : u \in C \right\}$$

is compact in X_s for all $t \in [0,T]$ and $0 < s < c$.

Proof. Setting $|u|_s \le r$ for all $u \in C$, we have

$$\left(\frac{\partial u}{\partial x_i}\right)^2 \le \left(\frac{|u|_s}{s}\right)^2 \le \frac{r^2}{s^2}.$$

Therefore, the subset $\{\int_P |\nabla_x u|^2 dx : u \in C\}$ is compact in $\mathbb{R}$ as it is bounded. Consequently, for any sequence $\{u_n\}_n \subset C$, there exists a subsequence $\{v_k\}_k \subset \{u_n\}_n$ such that

$$\lim_{k\to\infty} \int_P |\nabla_x v_k|^2 dx = \int_P |\nabla_x v|^2 dx, \, v \in X_s.$$

We next prove that

$$\left| f\left(t,x,\int_P |\nabla_x v_k|^2 dx\right) - f\left(t,x,\int_P |\nabla_x v|^2 dx\right)\right|_s \to 0$$

as $k \to \infty$ and the proof will be complete. Given $\varepsilon > 0$, since the series $\sum_i (s/c)^i$ is convergent, we choose n_0 large so that

$$\begin{aligned} &\sum_{|\alpha|>n_0} \left\| D^\alpha f\left(t,x,\int_P |\nabla_x v_k|^2 dx\right) - D^\alpha f\left(t,x,\int_P |\nabla_x v|^2 dx\right)\right\| \frac{s^{|\alpha|}}{\alpha!} \\ &\le 2K \sum_{|\alpha|>n_0} \frac{s^{|\alpha|}\alpha!}{\alpha! c^{|\alpha|}} \le 2K \sum_{|\alpha|>n_0} \left(\frac{s}{c}\right)^{|\alpha|} < \frac{\varepsilon}{2}, \forall k. \end{aligned}$$

By (A1), the operators $u \mapsto D^\alpha f(.,.,u)$ are uniformly continuous on closed interval in $\mathbb{R}$. We thus obtain with any sufficiently large index m that

$$\sum_{|\alpha|\le n_0} \left\| D^\alpha f\left(t,x,\int_P |\nabla_x v_m|^2 dx\right) - D^\alpha f\left(t,x,\int_P |\nabla_x v|^2 dx\right)\right\| \frac{s^{|\alpha|}}{\alpha!} < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned}&\left| f\left(t,x,\int_P |\nabla_x v_k|^2 dx\right) - f\left(t,x,\int_P |\nabla_x v|^2 dx\right)\right|_s \le \\ &\sum_{\alpha\in\mathbb{N}^n} \left\| D^\alpha f\left(t,x,\int_P |\nabla_x v_m|^2 dx\right) - D^\alpha f\left(t,x,\int_P |\nabla_x v|^2 dx\right)\right\| \frac{s^{|\alpha|}}{\alpha!} < \varepsilon.\end{aligned}$$

□

Theorem 4.3. *Assume that (A1) and (A2) are satisfied and $u_0, u_1 \in X_b$, where $b < c$. Then there exists a number λ such that problem (4.1) has a solution $u(t) \in X_s$ for all $t \in [0,(b-s)/\lambda)$, $s < b$.*

Proof. We prove that the function

$$g(t,u(t))(x) = f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\Delta_x u(t,x)$$

satisfies assumption (H). In view of Theorem (3.3), problem (4.1) (in form (1.2)) has at least a solution with desired properties. In what follows, let $s < s' \le b$. By (A2) and lemma (4.1), we have

$$|g(t,u)|_s \le \left| f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\right|_s |\Delta_x u|_s \le K\frac{M|u|_{s'}}{(s'-s)^2}.$$

Let C be bounded subset in $X_{s'}$. We denote by $B_s(u,r)$ the ball centered at u with radius r. Let $\gamma > \alpha_{s'}(C)$ and set $C \subset \bigcup_{i=1}^n B_{s'}(u_i,\gamma)$. For each $u_i, i = 1,2,...,n$, the subset

$$F_i(C) := \left\{ f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\Delta_x u_i : u \in C\right\}$$

is compact in X_s by lemma (4.2). It follows that, for any $\varepsilon > 0$, there exists a finite covering with radius ε of $F_i(C)$. Set $F_i(C) \subset \bigcup_{j=1}^m B_s(v_j^i,\varepsilon)$. We now fix $u \in C$ and choose u_i, v_j^i such that

$$|u-u_i|_{s'} \le \gamma, \quad \left| f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\Delta_x u_i - v_j^i\right|_s \le \varepsilon.$$

The result is

$$\begin{aligned}|g(t,u)-v_j^i|_s &\le \left| f\left(t,x,\int_P |\nabla_x u|^2 dx\right)(\Delta_x u - \Delta_x u_i)\right|_s \\ &\quad + \left| f\left(t,x,\int_P |\nabla_x u|^2 dx\right)\Delta_x u_i - v_j^i\right|_s \\ &\le K\frac{M|u-u_i|_{s'}}{(s'-s)^2} + \varepsilon \\ &\le K\frac{M\gamma}{(s'-s)^2} + \varepsilon.\end{aligned}$$

we conclude from the arbitrariness of $\gamma > \alpha_{s'}(C)$ and $\varepsilon > 0$ that

$$\alpha_s(g(t,C)) \leq \frac{L\alpha_{s'}(C)}{(s'-s)^2}.$$

The proof is completed. $\square$

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