J. Nonlinear Funct. Anal. 2023 (2023) 11 https://doi.org/10.23952/jnfa.2023.11

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

VARIATIONAL DISCRETIZATION COMBINED WITH FULLY DISCRETE SPLITTING POSITIVE DEFINITE MIXED FINITE ELEMENTS FOR PARABOLIC OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we consider a variational discretization combined with fully discrete splitting positive definite mixed finite element approximation of parabolic optimal control problems. For the state and co-state, Raviart-Thomas mixed finite element spaces and backward Euler scheme are used for space and time discretization, respectively. The variational discretization technique is used for the approximation of the control variable. We derive a priori error estimates for the control, state, and co-state. A numerical example is presented to demonstrate the theoretical results.

Keywords. Finite elements; Parabolic optimal control problems; Priori error estimates; Variational discretization.

1. INTRODUCTION

There are numerous research on various finite element methods (FEMs) used to solve optimal control problems (OCPs) with control constraints. A systematic introduction can be found in [4, 10, 16, 18, 19]. Due to the low regularity of the control variable, it is usually approximated by piecewise constant functions. Hence the convergence result is $\mathscr{O}(h)$ [1] and superconvergence result is $\mathscr{O}(h^{1.5})$ [24] or $\mathscr{O}(h^2)$ [21]. In 2005, Hinze improved the convergence order to $\mathscr{O}(h^2)$ by introducing a variational discretization (VD) concept in [9]. Then VD combined with standard FEMs [11, 25] or mixed finite element methods (MFEMs) [2] were used to solve parabolic OCPs. The convergence order of VD combined with standard FEM approxiamtion was improved to $\mathscr{O}(h^2)$. It is well known that MFEMs is a good choice for solving temperature or flow OCPs. Because their objective functional includes not only the primal state variable but also its gradient. There is a great deal of results on the Raviart-Thomas MFEM solving elliptic or parabolic OCPs [3, 5, 20]. However, the classical MFEM contains two discrete spaces, which have to satisfy the Ladyženskaja-Babuška-Brezzi (LBB) consistency condition. It brings very little available approximation spaces and expensive computing costs. To overcome these

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Received November 8, 2022; Accepted March 14, 2023.

difficulties, Pani in [22] presented an H^1-Galerkin MFEM. It can not only avoid the restrictions of LBB condition, but also obtain the H^1-norm error estimates of the numerical solution and its flux. Recently, Hou et al. extended the H^1-Galerkin MFEM to solve parabolic OCPs in [12]. Nevertheless, this method requires the solution of the original problem to have higher regularity. Splitting positive definite mixed finite element (SPDMFE) was first proposed in [27]. The advantages of this method are the LBB condition is not necessary and the original equations can be split into two independent symmetric positive definite sub-schemes. In recent years, convergence or superconvergence results of SPDMFE for hyperbolic equations and elliptic OCPs have been established in [17, 26, 28] and [8], respectively. Recently, a priori error estimates $\mathscr{O}(h)$ of semi-discrete SPDMFE for parabolic and hyperbolic OCPs were obtained in [7] and [13], respectively.

The goal of this paper is to investigate a VD combined with fully discrete SPDMFE approximation of parabolic OCPs, where the order $k \leq 1$ Raviart-Thomas mixed finite element spaces are used for the approximation of the state and co-state. A priori error estimates for the numerical solution of the control, state and dual state will be improved to $\mathscr{O}(h^{1+k}+\tau)$.

We are concerned with the following parabolic OCPs:

$$J(u) = \min_{u\in K\subset U} \frac{1}{2}\int_0^T \left(\|\boldsymbol{p}-\boldsymbol{p}_d\|^2 + \|y-y_d\|^2 + \nu\|u\|^2\right)dt \tag{1.1}$$

$$c(x)y_t(x,t) + \mathrm{div}\boldsymbol{p}(x,t) = f(x,t)+u(x,t), \quad x\in\Omega, \quad t\in J, \tag{1.2}$$

$$\boldsymbol{p}(x,t) = -A(x)\nabla y(x,t), \quad x\in\Omega,\ t\in J, \tag{1.3}$$

$$y(x,t) = 0, \quad x\in\partial\Omega, \quad t\in J, \tag{1.4}$$

$$y(x,0) = y_0(x), \quad x\in\Omega, \tag{1.5}$$

where $\Omega\subset\mathbf{R}^2$ is a rectangle, $J=[0,T]$, $\nu>0$ and $c(x)>0$. Let $U=L^2(J;L^2(\Omega))$, $f,y_d\in U$, $\boldsymbol{p}_d\in U^2$ and $y_0\in H^1(\Omega)$. We assume that the coefficient matrix $A(x)=(a_{ij}(x))_{2\times 2}\in W^{1,\infty}(\bar{\Omega};\mathbf{R}^{2\times 2})$ is a symmetric matrix and there are constants $c_1,c_2>0$ satisfying for $\forall\,\mathbf{X}\in\mathbf{R}^2$, $c_1\|\mathbf{X}\|^2_{\mathbf{R}^2}\leq \mathbf{X}^TA\mathbf{X}\leq c_2\|\mathbf{X}\|^2_{\mathbf{R}^2}$. K is a closed convex subset of U defined by

$$K=\{v\in U: v\geq 0\}.$$

We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|^p_{m,p}=\sum\limits_{|\alpha|\leq m}\|D^\alpha v\|^p_{L^p(\Omega)}$, a semi-norm $|\cdot|_{m,p}$ given by $|v|^p_{m,p}=\sum\limits_{|\alpha|=m}\|D^\alpha v\|^p_{L^p(\Omega)}$. For $p=2$, we denote $H^m(\Omega)=W^{m,2}(\Omega), H^m_0(\Omega)=\{v\in W^{m,p}(\Omega): v|_{\partial\Omega}=0\}$, and $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|\cdot\|=\|\cdot\|_{0,2}$. We denote by $L^s(J;W^{m,p}(\Omega))$ all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))}=\left(\int_0^T\|v\|^s_{W^{m,p}(\Omega)}dt\right)^{\frac{1}{s}}$ for $s\in[1,\infty)$, and the standard modification for $s=\infty$. For ease of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^l(J;W^{m,p}(\Omega))$. In addition, C denotes a general positive constant.

The plan of this paper is as follows. In Section 2, we construct a VD combined with fully discrete SPDMFE approximation scheme for the parabolic OCPs (1.1)-(1.5). In Section 3, we introduce some useful intermediate variables and important error estimates. In Section 4, we derive a priori error estimates for the control, state and co-state. In Section 5, the last section, we provide a numerical example to illustrate our theoretical results.

2. Variational Discretization Combined with SPDMFE Approximation

In this section, we consider a VD combined with fully discrete SPDMFE approximation of parabolic OCPs (1.1)-(1.5). For simplicity, we take the following state spaces $\boldsymbol{L} = H^1(J;\boldsymbol{V})$ and $Q = H^1(J;W)$, where $\boldsymbol{V}$ and W are defined as follows:

$$\boldsymbol{V} = H(\mathrm{div};\Omega) = \left\{\boldsymbol{v} \in (L^2(\Omega))^2, \mathrm{div}\boldsymbol{v} \in L^2(\Omega)\right\}, W = L^2(\Omega).$$

Furthermore, we define the following inner products

$$(f_1, f_2) = \int_\Omega f_1 f_2, \quad \forall f_1, f_2 \in L^2(\Omega),$$
$$(\boldsymbol{\psi}, \boldsymbol{v}) = \sum_{i=1}^{2} (\boldsymbol{\psi}_i, \boldsymbol{v}_i), \quad \forall \boldsymbol{\psi}, \boldsymbol{v} \in (L^2(\Omega))^2,$$

and the space

$$K' = \{v \in W : v \geq 0, \text{ a.e. in } \Omega\}.$$

Let $b = 1/c(x)$. Then state equations (1.2)-(1.3) can be rewritten as the following classical mixed weak form:

$$(y_t, w) + (b\mathrm{div}\boldsymbol{p}, w) = (bf, w) + (bu, w), \quad \forall w \in W, t \in J, \tag{2.1}$$
$$(A^{-1}\boldsymbol{p}, \boldsymbol{v}) = (y, \mathrm{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J. \tag{2.2}$$

We differentiate (2.2) with respect to t and take $w = \mathrm{div}\boldsymbol{v}, \boldsymbol{v} \in \boldsymbol{V}$ in (2.1). Substituting the two resulting equations, we derive the following splitting positive definite mixed weak form:

$$(A^{-1}\boldsymbol{p}_t, \boldsymbol{v}) + (b\mathrm{div}\boldsymbol{p}, \mathrm{div}\boldsymbol{v}) = (bf, \mathrm{div}\boldsymbol{v}) + (bu, \mathrm{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J,$$
$$(y_t, w) = -(b\mathrm{div}\boldsymbol{p}, w) + (bf, w) + (bu, w), \quad \forall w \in W, t \in J.$$

Hence, OCPs (1.1)-(1.5) can be recasted as the following weak form: find $(\boldsymbol{p}, y, u) \in \boldsymbol{L} \times Q \times K$ such that

$$J(u) = \min_{u \in K \in U} \frac{1}{2} \int_0^T \left(\|\boldsymbol{p} - \boldsymbol{p}_d\|^2 + \|y - y_d\|^2 + \nu \|u\|^2 \right) dt \tag{2.3}$$
$$(A^{-1}\boldsymbol{p}_t, \boldsymbol{v}) + (b\mathrm{div}\boldsymbol{p}, \mathrm{div}\boldsymbol{v}) = (bf, \mathrm{div}\boldsymbol{v}) + (bu, \mathrm{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.4}$$
$$\boldsymbol{p}(x, 0) = \boldsymbol{p}_0(x), \quad \forall x \in \Omega, \tag{2.5}$$
$$(y_t, w) = -(b\mathrm{div}\boldsymbol{p}, w) + (bf, w) + (bu, w), \quad \forall w \in W, t \in J, \tag{2.6}$$
$$y(x, 0) = y_0(x), \quad \forall x \in \Omega, \tag{2.7}$$

where $\boldsymbol{p}_0(x) = -A\nabla y_0(x)$.

It follows from [16] that OCPs (2.3)-(2.7) has a unique solution $(\boldsymbol{p}, y, u)$, and that a triplet $(\boldsymbol{p}, y, u)$ is a solution of (2.3)-(2.7) if and only if there is a co-state $(\boldsymbol{q}, z) \in \boldsymbol{L} \times Q$ such that

$(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ satisfies the following optimality conditions:

$$(A^{-1}\boldsymbol{p}_t, \boldsymbol{v}) + (b\text{div}\boldsymbol{p}, \text{div}\boldsymbol{v}) = (bf, \text{div}\boldsymbol{v}) + (bu, \text{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.8}$$

$$\boldsymbol{p}(x,0) = \boldsymbol{p}_0(x), \quad \forall x \in \Omega, \tag{2.9}$$

$$(y_t, w) = -(b\text{div}\boldsymbol{p}, w) + (bf, w) + (bu, w), \quad \forall w \in W, t \in J, \tag{2.10}$$

$$y(x,0) = y_0(x), \quad \forall x \in \Omega, \tag{2.11}$$

$$(z_t, w) = (y - y_d, w), \quad \forall w \in W, t \in J, \tag{2.12}$$

$$z(x,T) = 0, \quad \forall x \in \Omega, \tag{2.13}$$

$$-(A^{-1}\boldsymbol{q}_t, \boldsymbol{v}) + (b\text{div}\boldsymbol{q}, \text{div}\boldsymbol{v}) = -(\boldsymbol{p} - \boldsymbol{p}_d, \boldsymbol{v}) - (bz, \text{div}\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}, t \in J, \tag{2.14}$$

$$\boldsymbol{q}(x,T) = 0, \quad \forall x \in \Omega, \tag{2.15}$$

$$(\nu u - bz - b\text{div}\boldsymbol{q}, \tilde{u} - u) \geq 0, \quad \forall \tilde{u} \in K', t \in J. \tag{2.16}$$

We introduce a pointwise projection $P_K : W \to K'$, which satisfies, for any $u \in W$,

$$P_K u(x) = \max\left\{0, \frac{u(x)}{\nu}\right\}, \quad \forall x \in \Omega.$$

Then variational inequality (2.16) can be equivalently expressed as

$$u = P_K\left(bz + b\text{div}\boldsymbol{q}\right). \tag{2.17}$$

Let $\mathscr{T}_h$ be a regular triangulation of the domain Ω, and let h_e denote the diameter of e and $h = \max_{e\in\mathscr{T}_h}\{h_e\}$. Let $\boldsymbol{V}_h \times W_h \subset \boldsymbol{V} \times W$ denote the order $k \leq 1$ Raviart-Thomas mixed finite element spaces [6, 23], namely,

$$\boldsymbol{V}_h := \{\boldsymbol{v}_h \in \boldsymbol{V} : \boldsymbol{v}_h|_e \in (P_k(e))^2 + x \cdot P_k(e), \forall e \in \mathscr{T}_h, \},$$
$$W_h := \{w_h \in W : w_h|_e \in P_k(e), \forall e \in \mathscr{T}_h\},$$

where $P_k(e)$ indicates the space of polynomials of total degree no more than k on e.

Let $N \in \mathbb{Z}^+$, $\tau = T/N$, and $t_n = n\tau$ for $n = 0, 1, 2, \cdots, N$. Set $\psi^n = \psi^n(x) = \psi(x, t_n)$ and

$$d_t\psi^n = \frac{\psi^n - \psi^{n-1}}{\tau}, \quad \delta\psi^n = \psi^n - \psi^{n-1}, n = 1, 2, \cdots, N.$$

We define for $1 \leq s < \infty$ the discrete time dependent norms

$$|||\psi|||_{l^s(J;W^{m,p}(\Omega))} := \left(\sum_{n=1-l}^{N-l} \tau \|\psi^n\|_{m,p}^s\right)^{\frac{1}{s}},$$

where $l = 0$ for the control u and the state variables $y, \boldsymbol{p}$, and $l = 1$ for the co-state variables $z, \boldsymbol{q}$ with the standard modification for $s = \infty$. Just for simplicity, we denote $|||\psi|||_{l^s(J;W^{m,p}(\Omega))}$ and $|||\psi|||_{l^\infty(J;W^{m,p}(\Omega))}$ by $|||\psi|||_{l^s(W^{m,p})}$ and $|||\psi|||_{l^\infty(W^{m,p})}$, respectively.

Then a VD combined with fully discrete SPDMFE approximation of (2.3)-(2.7) is to find $(\boldsymbol{p}_h^n, y_h^n, u_h^n) \in \boldsymbol{V}_h \times W_h \times K', n = 1, 2, \cdots, N$ such that

$$\min_{u_h^n \in K'} \frac{1}{2} \sum_{n=1}^{N} \tau \left(\|\boldsymbol{p}_h^n - \boldsymbol{p}_d^n\|^2 + \|y_h^n - y_d^n\|^2 + \nu \|u_h^n\|^2 \right) \tag{2.18}$$

$$(A^{-1} d_t \boldsymbol{p}_h^n, \boldsymbol{v}_h) + (b\mathrm{div}\boldsymbol{p}_h^n, \mathrm{div}\boldsymbol{v}_h) = (bf^n + bu_h^n, \mathrm{div}\boldsymbol{v}_h), \ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.19}$$

$$\boldsymbol{p}_h^0(x) = \Pi_h \boldsymbol{p}_0(x), \ \forall\, x \in \Omega, \tag{2.20}$$

$$(d_t y_h^n, w_h) = -(b\mathrm{div}\boldsymbol{p}_h^n, w_h) + (bf^n + bu_h^n, w_h), \ \forall\, w_h \in W_h, \tag{2.21}$$

$$y_h^0(x) = P_h y_0(x), \ \forall\, x \in \Omega, \tag{2.22}$$

where Π_h and P_h are two projection operators, which will be specified later on.

It is well known [2] that problem (2.18)-(2.22) has a unique solution $(\boldsymbol{p}_h^n, y_h^n, u_h^n)$, $n = 1, 2, \cdots, N$, and that a triplet $(\boldsymbol{p}_h^n, y_h^n, u_h^n) \in \boldsymbol{V}_h \times W_h \times K', n = 1, 2, \cdots, N$, is the solution of (2.18)-(2.22) if and only if there is a co-state $(\boldsymbol{q}_h^{n-1}, z_h^{n-1}) \in \boldsymbol{V}_h \times W_h$ such that $(\boldsymbol{p}_h^n, y_h^n, \boldsymbol{q}_h^{n-1}, z_h^{n-1}, u_h^n) \in (\boldsymbol{V}_h \times W_h)^2 \times K'$ satisfies the following discrete optimality conditions:

$$(A^{-1} d_t \boldsymbol{p}_h^n, \boldsymbol{v}_h) + (b\mathrm{div}\boldsymbol{p}_h^n, \mathrm{div}\boldsymbol{v}_h) = (bf^n + bu_h^n, \mathrm{div}\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.23}$$

$$\boldsymbol{p}_h^0(x) = \Pi_h \boldsymbol{p}_0(x), \quad \forall x \in \Omega, \tag{2.24}$$

$$(d_t y_h^n, w_h) = -(b\mathrm{div}\boldsymbol{p}_h^n, w_h) + (bf^n + bu_h^n, w_h), \quad \forall w_h \in W_h, \tag{2.25}$$

$$y_h^0(x) = P_h y_0(x), \quad \forall x \in \Omega, \tag{2.26}$$

$$(d_t z_h^n, w_h) = (y_h^n - y_d^n, w_h), \quad \forall w_h \in W_h, \tag{2.27}$$

$$z_h^N(x) = 0, \quad \forall x \in \Omega, \tag{2.28}$$

$$-(A^{-1} d_t \boldsymbol{q}_h^n, \boldsymbol{v}_h) + (b\mathrm{div}\boldsymbol{q}_h^{n-1}, \mathrm{div}\boldsymbol{v}_h) = -(\boldsymbol{p}_h^n - \boldsymbol{p}_d^n, \boldsymbol{v}_h) - (bz_h^{n-1}, \mathrm{div}\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.29}$$

$$\boldsymbol{q}_h^N(x) = 0, \quad \forall x \in \Omega, \tag{2.30}$$

$$(\nu u_h^n - bz_h^{n-1} - b\mathrm{div}\boldsymbol{q}_h^{n-1}, \tilde{u} - u_h^n) \geq 0, \quad \forall \tilde{u} \in K'. \tag{2.31}$$

It should be pointed out that we minimize over the infinite dimensional set K' instead of minimizing over a finite dimensional subset of K'. Similar to (2.17), variational inequality (2.31) can be equivalently rewritten as

$$u_h^n = P_K \left(bz_h^{n-1} + b\mathrm{div}\boldsymbol{q}_h^{n-1} \right), \quad n = 1, 2, \cdots, N. \tag{2.32}$$

This means that we can obtain u_h^n from z_h^{n-1} and $\boldsymbol{q}_h^{n-1}$ by using relation (2.32).

3. Error Estimates of Intermediate Variables

In this section, we introduce some useful intermediate variables and give their corresponding error estimates. The following projection operators and intermediate variables are commonly used in error estimates of SPDMFE approximation. First, we define the standard $L^2(\Omega)$-projection [6] $P_h : W \to W_h$, which satisfies, for any $\phi \in W$,

$$(P_h \phi - \phi, w_h) = 0, \quad \forall w_h \in W_h,$$
$$\|\phi - P_h \phi\| \leq Ch^r \|\phi\|_r, \quad 0 \leq \rho \leq \infty, \quad \forall \phi \in W^{r,\rho}(\Omega), \quad 1 \leq r \leq 1 + k. \tag{3.1}$$

Second, we define the Fortin projection (see [6]) $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies, for any $\boldsymbol{q} \in \boldsymbol{V}$,

$$(\text{div}(\Pi_h \boldsymbol{q} - \boldsymbol{q}), w_h) = 0, \quad \forall w_h \in W_h, \tag{3.2}$$

$$\|\boldsymbol{q} - \Pi_h \boldsymbol{q}\| \le Ch^r \|\boldsymbol{q}\|_r, \quad \forall \boldsymbol{q} \in (H^r(\Omega))^2, \quad 1 \le r \le 1+k, \tag{3.3}$$

$$\|\text{div}(\boldsymbol{q} - \Pi_h \boldsymbol{q})\| \le Ch^r \|\text{div}\boldsymbol{q}\|_r, \quad \forall \text{div}\boldsymbol{q} \in H^r(\Omega), \quad 1 \le r \le 1+k. \tag{3.4}$$

Third, for any $\tilde{u} \in K$, we define variables $(\boldsymbol{p}_h^n(\tilde{u}), y_h^n(\tilde{u}), \boldsymbol{q}_h^{n-1}(\tilde{u}), z_h^{n-1}(\tilde{u})), n = 1, 2, \cdots, N$, associated with $\tilde{u}$, which satisfies

$$(A^{-1} d_t \boldsymbol{p}_h^n(\tilde{u}), \boldsymbol{v}_h) + (b\text{div}\boldsymbol{p}_h^n(\tilde{u}), \text{div}\boldsymbol{v}_h) = (bf^n + b\tilde{u}^n, \text{div}\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.5}$$

$$\boldsymbol{p}_h^0(\tilde{u})(x) = \Pi_h \boldsymbol{p}_0(x), \quad \forall x \in \Omega, \tag{3.6}$$

$$(d_t y_h^n(\tilde{u}), w_h) = -(b\text{div}\boldsymbol{p}_h^n(\tilde{u}), w_h) + (bf^n + b\tilde{u}^n, w_h), \quad \forall w_h \in W_h, \tag{3.7}$$

$$y_h^0(\tilde{u})(x) = P_h y_0(x), \quad \forall x \in \Omega, \tag{3.8}$$

$$(d_t z_h^n(\tilde{u}), w_h) = (y_h^n(\tilde{u}) - y_d^n, w_h), \quad \forall w_h \in W_h, \tag{3.9}$$

$$z_h^N(\tilde{u})(x) = 0, \quad \forall x \in \Omega, \tag{3.10}$$

$$\begin{aligned} -(A^{-1} d_t \boldsymbol{q}_h^n(\tilde{u}), \boldsymbol{v}_h) + (b\text{div}\boldsymbol{q}_h^{n-1}(\tilde{u}), \text{div}\boldsymbol{v}_h) &= -(\boldsymbol{p}_h^n(\tilde{u}) - \boldsymbol{p}_d^n, \boldsymbol{v}_h) \\ &\quad - (bz_h^{n-1}(\tilde{u}), \text{div}\boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \end{aligned} \tag{3.11}$$

$$\boldsymbol{q}_h^N(\tilde{u})(x) = 0, \quad \forall x \in \Omega. \tag{3.12}$$

Next, we derive some important error estimates on the above intermediate variables.

Lemma 3.1. *Let $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h)$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the discrete solutions of (3.5)-(3.12) with $\tilde{u} = u_h$ and $\tilde{u} = u$, respectively. Then*

$$|||y_h - y_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{p}_h - \boldsymbol{p}_h(u)|||_{l^\infty(L^2)} \le C|||u - u_h|||_{l^2(L^2)}, \tag{3.13}$$

$$|||z_h - z_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{q}_h - \boldsymbol{q}_h(u)|||_{l^\infty(L^2)} \le C|||u - u_h|||_{l^2(L^2)}, \tag{3.14}$$

$$|||div(\boldsymbol{p}_h - \boldsymbol{p}_h(u))|||_{l^2(L^2)} + |||div(\boldsymbol{q}_h - \boldsymbol{q}_h(u))|||_{l^2(L^2)} \le C|||u - u_h|||_{l^2(L^2)}. \tag{3.15}$$

Proof. Let $\boldsymbol{r}_1 = \boldsymbol{p}_h(u) - \boldsymbol{p}_h$, $r_2 = y_h(u) - y_h$, $r_3 = z_h(u) - z_h$, $\boldsymbol{r}_4 = \boldsymbol{q}_h(u) - \boldsymbol{q}_h$. From (3.5)-(3.12), for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, $n = 1, 2, \ldots, N$, we obtain

$$(A^{-1} d_t \boldsymbol{r}_1^n, \boldsymbol{v}_h) + (b\text{div}\boldsymbol{r}_1^n, \text{div}\boldsymbol{v}_h) = (bu^n - bu_h^n, \text{div}\boldsymbol{v}_h), \tag{3.16}$$

$$(d_t r_2^n, w_h) = -(b\text{div}\boldsymbol{r}_1^n, w_h) + (bu^n - bu_h^n, w_h), \tag{3.17}$$

$$(d_t r_3^n, w_h) = (r_2^n, w_h), \tag{3.18}$$

$$-(A^{-1} \boldsymbol{r}_4^n, \boldsymbol{v}_h) + (b\text{div}\boldsymbol{r}_4^{n-1}, \text{div}\boldsymbol{v}_h) = -(\boldsymbol{r}_1^n, \boldsymbol{v}_h) - (br_3^{n-1}, \text{div}\boldsymbol{v}_h). \tag{3.19}$$

Selecting $\boldsymbol{v}_h = \boldsymbol{r}_1^n$ in (3.16), we obtain

$$(A^{-1} d_t \boldsymbol{r}_1^n, \boldsymbol{r}_1^n) + \|b^{\frac{1}{2}} \text{div}\boldsymbol{r}_1^n\|^2 = (bu^n - bu_h^n, \text{div}\boldsymbol{r}_1^n). \tag{3.20}$$

According to the inequality $(a-b)a \ge \frac{a^2 - b^2}{2}$, we have

$$(A^{-1} d_t \boldsymbol{r}_1^n, \boldsymbol{r}_1^n) \ge \frac{1}{2\tau}(\|A^{-\frac{1}{2}} \boldsymbol{r}_1^n\|^2 - \|A^{-\frac{1}{2}} \boldsymbol{r}_1^{n-1}\|^2). \tag{3.21}$$

Multiplying both sides of (3.20) by 2τ and summing over n from 1 to M $(1 \le M \le N)$, using (3.21), $\boldsymbol{r}_1^0 = 0$, Cauchy inequality, and the assumptions on A and c, we obtain

$$\|\boldsymbol{r}_1^M\|^2 + \sum_{n=1}^{M} \tau \|\mathrm{div}\boldsymbol{r}_1^n\|^2 \le C \sum_{n=1}^{M} \tau \|u^n - u_h^n\|^2, \tag{3.22}$$

which yields to

$$|||\boldsymbol{r}_1|||_{l^\infty(L^2)} + |||\mathrm{div}\boldsymbol{r}_1|||_{l^2(L^2)} \le C|||u-u_h|||_{l^2(L^2)}. \tag{3.23}$$

Setting $w_h = r_2^n$ in (3.17), similar to (3.23), it is easy to obtain

$$|||r_2|||_{l^\infty(L^2)} \le C|||\mathrm{div}\boldsymbol{r}_1|||_{l^2(L^2)} + C|||u-u_h|||_{l^2(L^2)}. \tag{3.24}$$

Similarly, choosing $w_h = r_3^{n-1}$ in (3.18) and $\boldsymbol{v}_h = \boldsymbol{r}_4^{n-1}$ in (3.19), respectively, we see that

$$|||r_3|||_{l^\infty(L^2)} \le C|||r_2|||_{l^2(L^2)}, \tag{3.25}$$

$$|||\boldsymbol{r}_4|||_{l^\infty(L^2)} + |||\mathrm{div}\boldsymbol{r}_4|||_{l^2(L^2)} \le C|||\boldsymbol{r}_1|||_{l^2(L^2)} + C|||r_3|||_{l^2(L^2)}. \tag{3.26}$$

Combining (3.25)-(3.26), we complete the proof. □

Lemma 3.2. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{p}_h(u), y_h(u), \boldsymbol{q}_h(u), z_h(u))$ be the solutions of (2.8)-(2.16) and (3.5)-(3.12) with $\tilde{u} = u$ respectively. Assume that $y_t, (y_d)_t, y_{tt}, z_{tt} \in L^2(L^2)$, $y, z \in L^2(H^{1+k})$, $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{p}_t, \boldsymbol{q}_t, (\boldsymbol{p}_d)_t \in L^2((H^{1+k})^2)$, then*

$$|||y - y_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{p} - \boldsymbol{p}_h(u)|||_{l^\infty(L^2)} \le C(h^{1+k} + \tau), \tag{3.27}$$

$$|||z - z_h(u)|||_{l^\infty(L^2)} + |||\boldsymbol{q} - \boldsymbol{q}_h(u)|||_{l^\infty(L^2)} \le C(h^{1+k} + \tau), \tag{3.28}$$

$$|||div(\boldsymbol{p} - \boldsymbol{p}_h(u))|||_{l^2(L^2)} + |||div(\boldsymbol{q} - \boldsymbol{q}_h(u))|||_{l^2(L^2)} \le C(h^{1+k} + \tau). \tag{3.29}$$

Proof. Let

$$\boldsymbol{e}_1 = \Pi_h\boldsymbol{p} - \boldsymbol{p}_h(u),\ e_2 = P_h y - y_h(u),\ \boldsymbol{\rho}_1 = \boldsymbol{p} - \Pi_h\boldsymbol{p},\ \rho_2 = y - P_h y,$$
$$e_3 = P_h z - z_h(u),\ \boldsymbol{e}_4 = \Pi_h\boldsymbol{q} - \boldsymbol{q}_h(u),\ \rho_3 = z - P_h z,\ \boldsymbol{\rho}_4 = \boldsymbol{q} - \Pi_h\boldsymbol{q}.$$

From (2.8)-(2.16) and (3.5)-(3.12), we have

$$(A^{-1}d_t\boldsymbol{e}_1^n, \boldsymbol{v}_h) + (b\mathrm{div}\boldsymbol{e}_1^n, \mathrm{div}\boldsymbol{v}_h) = (A^{-1}\boldsymbol{\varepsilon}_1^n, \boldsymbol{v}_h) - (A^{-1}d_t\boldsymbol{\rho}_1^n, \boldsymbol{v}_h) - (b\mathrm{div}\boldsymbol{\rho}_1^n, \mathrm{div}\boldsymbol{v}_h), \tag{3.30}$$

$$(d_t e_2^n, w_h) = (\varepsilon_2^n, w_h) - (b\mathrm{div}\boldsymbol{\rho}_1^n, w_h) - (b\mathrm{div}\boldsymbol{e}_1^n, w_h), \tag{3.31}$$

$$-(d_t e_3^n, w_h) = (\varepsilon_3^{n-1}, w_h) - (\delta y_d^n - \delta y^n + e_2^n, w_h), \tag{3.32}$$

$$\begin{aligned} -(A^{-1}d_t\boldsymbol{e}_4^n, \boldsymbol{v}_h) + (b\mathrm{div}\boldsymbol{e}_4^{n-1}, \mathrm{div}\boldsymbol{v}_h) &= (A^{-1}\boldsymbol{\varepsilon}_4^{n-1}, w_h) + (A^{-1}d_t\boldsymbol{\rho}_4^{n-1}, \boldsymbol{v}_h) - (b\mathrm{div}\boldsymbol{\rho}_4^{n-1}, \mathrm{div}\boldsymbol{v}_h) \\ &\quad - (\delta\boldsymbol{p}_d^n - \delta\boldsymbol{p}^n + \boldsymbol{e}_1^n, \boldsymbol{v}_h) - (b\rho_3^{n-1} + be_3^{n-1}, \mathrm{div}\boldsymbol{v}_h), \end{aligned} \tag{3.33}$$

for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, where

$$\boldsymbol{\varepsilon}_1^n = d_t\boldsymbol{p}^n - \boldsymbol{p}_t^n,\ \varepsilon_2^n = d_t y^n - y_t^n,\ n = 1, 2, \cdots, N,$$
$$\varepsilon_3^n = z_t^{n-1} - d_t z^n,\ \boldsymbol{\varepsilon}_4^n = \boldsymbol{q}_t^{n-1} - d_t\boldsymbol{q}^n,\ n = 1, 2, \cdots, N.$$

By standard backward difference error analysis [12], we have

$$\|\boldsymbol{\varepsilon}_1^n\|^2 \le C\tau \left\|\frac{\partial^2 \boldsymbol{p}}{\partial s^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}, \quad \|\varepsilon_2^n\|^2 \le C\tau \left\|\frac{\partial^2 y}{\partial s^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}, \tag{3.34}$$

$$\|\varepsilon_3^n\|^2 \le C\tau \left\|\frac{\partial^2 z}{\partial s^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}, \quad \|\boldsymbol{\varepsilon}_4^n\|^2 \le C\tau \left\|\frac{\partial^2 \boldsymbol{q}}{\partial s^2}\right\|^2_{L^2(t_{n-1},t_n;L^2(\Omega))}. \tag{3.35}$$

Choosing $\boldsymbol{v}_h = \boldsymbol{e}_1^n$ in (3.30), we obtain

$$\begin{aligned}\frac{1}{2\tau}(\|A^{-\frac{1}{2}}\boldsymbol{e}_1^n\|^2 - \|A^{-\frac{1}{2}}\boldsymbol{e}_1^n\|^2) + \|b^{\frac{1}{2}}\mathrm{div}\boldsymbol{e}_1^n\|^2 \\ \le (A^{-1}\boldsymbol{\varepsilon}_1^n, \boldsymbol{e}_1^n) - (A^{-1}d_t\boldsymbol{\rho}_1^n, \boldsymbol{e}_1^n) - (b\mathrm{div}\boldsymbol{\rho}_1^n, \mathrm{div}\boldsymbol{e}_1^n).\end{aligned} \tag{3.36}$$

Using (3.1), (3.3) and Cauchy inequality, for $1 \le M \le N$, we have

$$\sum_{n=1}^{M} \tau(A^{-1}d_t\boldsymbol{\rho}_1^n, \boldsymbol{e}_1^n) \le Ch^{2+2k}\|\boldsymbol{p}_t\|_{L^2(H^{1+k})} + \sum_{n=1}^{M}\tau\|\boldsymbol{e}_1^n\|^2, \tag{3.37}$$

and

$$\begin{aligned}\sum_{n=1}^{M}\tau(b\mathrm{div}\boldsymbol{\rho}_1^n, \mathrm{div}\boldsymbol{e}_1^n) =& \sum_{n=1}^{M}\tau((b-P_hb)\mathrm{div}\boldsymbol{\rho}_1^n, \mathrm{div}\boldsymbol{e}_1^n) \\ &\le Ch^{2+2k}|||\boldsymbol{p}|||_{l^2(H^{1+k})} + \frac{1}{2}\sum_{n=1}^{M}\tau\|b^{\frac{1}{2}}\mathrm{div}\boldsymbol{e}_1^n\|^2.\end{aligned} \tag{3.38}$$

Multiplying both sides of (3.36) by 2τ and summing it over n from 1 to M $(1 \le M \le N)$, using (3.37)-(3.38) and $\boldsymbol{e}_1^0 = 0$, we find that

$$\begin{aligned}&\|A^{-\frac{1}{2}}\boldsymbol{e}_1^M\|^2 + \sum_{n=1}^{M}\tau\|b^{\frac{1}{2}}\mathrm{div}\boldsymbol{e}_1^n\|^2 \\ \le& Ch^{2+2k}|||\boldsymbol{p}|||^2_{l^2(H^{1+k})} + Ch^{2+2k}\|\boldsymbol{p}_t\|^2_{L^2(H^{1+k})} + C\tau^2\left\|\frac{\partial^2\boldsymbol{p}}{\partial s^2}\right\|^2_{L^2(L^2)} + C\sum_{n=1}^{M}\tau\|\boldsymbol{e}_1^n\|^2.\end{aligned} \tag{3.39}$$

Applying the discrete Gronwall's lemma to (3.39) and the assumptions on A and c, we obtain that

$$\begin{aligned}&\|\boldsymbol{e}_1^M\|^2 + \sum_{n=1}^{M}\tau\|\mathrm{div}\boldsymbol{e}_1^n\|^2 \\ \le& Ch^{2+2k}|||\boldsymbol{p}|||^2_{l^2(H^{1+k})} + Ch^{2+2k}\|\boldsymbol{p}_t\|^2_{L^2(H^{1+k})} + C\tau^2\left\|\frac{\partial^2 y}{\partial s^2}\right\|^2_{L^2(L^2)}.\end{aligned} \tag{3.40}$$

Setting $w_h = e_2^n$ in (3.31), similar to (3.40), we have

$$\|e_2^M\|^2 \le C\sum_{n=1}^{M}\tau\|\mathrm{div}\boldsymbol{e}_1^n\|^2 + Ch^{2+2k}|||\boldsymbol{p}|||^2_{l^2(H^{1+k})} + + C\tau^2\left\|\frac{\partial^2 y}{\partial s^2}\right\|^2_{L^2(L^2)}. \tag{3.41}$$

Choosing $w_h = e_3^{n-1}$ in (3.32), we arrive at

$$\frac{1}{2\tau}(\|e_3^{n-1}\|^2 - \|e_3^n\|^2) \le (\varepsilon_3^{n-1} - \delta y_d^n + \delta y^n - e_2^n, e_3^{n-1}). \tag{3.42}$$

Notice that

$$(\delta y^n, e_3^{n-1}) \le C\tau^{\frac{1}{2}} \|y_t\|_{L^2(t_{n-1},t_n;L^2(\Omega))} \|e_3^{n-1}\|, \tag{3.43}$$

$$(\delta y_d^n, e_3^{n-1}) \le C\tau^{\frac{1}{2}} \|(y_d)_t\|_{L^2(t_{n-1},t_n;L^2(\Omega))} \|e_3^{n-1}\|. \tag{3.44}$$

Multiplying both sides of (3.42) by 2τ and summing it over n from N to $M+1$ $(0 \le M \le N-1)$, using (3.43)-(3.44), Cauchy inequality, Gronwall's Lemma and $e_3^N = 0$, we have

$$\|e_3^M\|^2 \le C\tau^2 \left(\|y_t\|_{L^2(L^2)} + \|(y_d)_t\|_{L^2(L^2)} + \left\| \frac{\partial^2 z}{\partial s^2} \right\|^2_{L^2(L^2)} \right) + C \sum_{n=M+1}^{N} \tau \|\boldsymbol{e}_1^n\|^2. \tag{3.45}$$

Finally, selecting $\boldsymbol{v}_h = \boldsymbol{e}_4^{n-1}$ in (3.33), similar to (3.45), we find that

$$\begin{aligned} \|\boldsymbol{e}_4^M\|^2 + \sum_{n=M+1}^{N} \tau \|\mathrm{div}\boldsymbol{e}_4^{n-1}\|^2 \le & Ch^{2+2k} \left(|||\boldsymbol{q}|||^2_{l^2(H^{1+k})} + \|\boldsymbol{q}_t\|^2_{L^2(H^{1+k})} + |||z|||^2_{l^2(H^{1+k})} \right) \\ & + C\tau^2 \left(\left\| \frac{\partial^2 \boldsymbol{q}}{\partial s^2} \right\|^2_{L^2(L^2)} + \|\boldsymbol{p}_t\|^2_{L^2(L^2)} + \|(\boldsymbol{p}_d)_t\|^2_{L^2(L^2)} \right) \\ & + C \sum_{n=M+1}^{N} \tau \left(\|\boldsymbol{e}_1^n\|^2 + \|e_3^{n-1}\|^2 \right). \end{aligned} \tag{3.46}$$

Combining (3.40)-(3.41), (3.45)-(3.46) and the triangle inequality, we complete the proof. □

Lemma 3.3. *Let u be the solution of (2.8)-(2.16) and u_h be the solution of (2.23)-(2.31). Then*

$$\sum_{n=1}^{N} \tau(bu^n - bu_h^n, div\boldsymbol{r}_4^{n-1} + r_3^{n-1}) \le 0. \tag{3.47}$$

Proof. Take $\boldsymbol{v}_h = \boldsymbol{r}_4^{n-1}$ in (3.16), $w_h = r_3^{n-1}$ in (3.17), $w_h = r_2^n$ in (3.18), and $\boldsymbol{v}_h = \boldsymbol{r}_1^n$ in (3.19), respectively. Then multiplying the four resulting equations by τ and summing it over n from 1 to N, we can find that

$$\sum_{n=1}^{N} \tau(bu^n - bu_h^n, \mathrm{div}\boldsymbol{r}_4^{n-1} + r_3^{n-1}) = -|||\boldsymbol{r}_1|||^2_{l^2(L^2)} - |||r_2|||^2_{l^2(L^2)}, \tag{3.48}$$

which yields to (3.47). □

4. A Priori Error Estimates

In this section, we derive a priori error estimates of the VD combined with fully discrete SPDMFE approximation scheme (2.23)-(2.31).

Theorem 4.1. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ be the solution of (2.8)-(2.16) and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ be the solution of (2.23)-(2.31). Assume that all the conditions in Lemmas 3.1-3.3 are valid. Then,*

$$|||u - u_h|||_{l^2(L^2)} \le C(h^{1+k} + \tau). \tag{4.1}$$

Proof. It follows from (2.16) and (2.31) that

$$
\begin{aligned}
\nu|||u-u_h|||_{l^2(L^2)}^2 &= \sum_{n=1}^{N} k(u^n-u_h^n, u^n-u_h^n) \\
&\le \sum_{n=1}^{N} k(bz^n - bz_h^{n-1} + b\mathrm{div}\boldsymbol{q}^n - b\mathrm{div}\boldsymbol{q}_h^{n-1}, u^n-u_h^n) \\
&= \sum_{n=1}^{N} k(bz^n - bz^{n-1} + b\mathrm{div}(\boldsymbol{q}^n-\boldsymbol{q}^{n-1}), u^n-u_h^n) \\
&\quad + \sum_{n=1}^{N} k(bz^{n-1} - bz_h^{n-1}(u) + b\mathrm{div}(\boldsymbol{q}^{n-1}-\boldsymbol{q}_h^{n-1}(u)), u^n-u_h^n) \\
&\quad + \sum_{n=1}^{N} k(bz_h^{n-1}(u) - bz_h^{n-1} + b\mathrm{div}(\boldsymbol{q}_h^{n-1}(u)-\boldsymbol{q}_h^{n-1}), u^n-u_h^n) \\
&=: \sum_{i=1}^{3} I_i.
\end{aligned}
\tag{4.2}
$$

Now we estimate (4.2) term by term. For I_1, using Cauchy inequality and (3.43), we find that

$$
I_1 \le C\tau^2\left(\|z_t\|_{L^2(L^2)}^2 + \|\mathrm{div}\boldsymbol{q}_t\|_{L^2(L^2)}^2\right) + \frac{1}{3}|||u-u_h|||_{l^2(L^2)}^2. \tag{4.3}
$$

Using Lemma 3.1 and Lemma 3.2, (3.37), and Cauchy inequality, we have

$$
\begin{aligned}
I_2 \le& C\left(\|z-z_h(u)\|_{L^2(L^2)}^2 + \|\mathrm{div}(\boldsymbol{q}-\boldsymbol{q}_h(u))\|_{L^2(L^2)}^2\right) + \frac{1}{3}|||u-u_h|||_{l^2(L^2)}^2 \\
\le& C(h^{1+k}+\tau) + \frac{1}{3}|||u-u_h|||_{l^2(L^2)}^2.
\end{aligned}
\tag{4.4}
$$

From Lemma 3.3, we know that

$$
I_3 = \sum_{n=1}^{N} \tau(z_h^{n-1}(u) - z_h^{n-1} + \mathrm{div}(\boldsymbol{q}_h^{n-1}(u)-\boldsymbol{q}_h^{n-1}), bu^n - bu_h^n) \le 0. \tag{4.5}
$$

Substituting the estimates I_1-I_3 in (4.2), we derive (4.1). This completes the proof. □

Theorem 4.2. *Let $(\boldsymbol{p}, y, \boldsymbol{q}, z, u)$ and $(\boldsymbol{p}_h, y_h, \boldsymbol{q}_h, z_h, u_h)$ be the solution of (2.8)-(2.16) and the solution of (2.23)-(2.31), respectively. Assume that all the conditions in Theorem 4.1 are valid. Assume that $y_t, (y_d)_t, z_{tt} \in L^\infty(L^2)$, $\boldsymbol{p}, \boldsymbol{q} \in L^\infty((H^{1+k})^2)$, $\boldsymbol{p}_t, \boldsymbol{q}_t \in L^2((H^{1+k})^2)$ and $(\boldsymbol{p}_d)_t, \boldsymbol{p}_{tt}, (\boldsymbol{p}_d)_{tt} \in L^\infty((L^2)^2)$. Then*

$$
|||y-y_h|||_{l^\infty(L^2)} + |||\boldsymbol{p}-\boldsymbol{p}_h|||_{l^\infty(L^2)} \le C(h^{1+k}+\tau), \tag{4.6}
$$

$$
|||z-z_h|||_{L^\infty(l^2)} + |||\boldsymbol{q}-\boldsymbol{q}_h|||_{l^\infty(L^2)} \le C(h^{1+k}+\tau), \tag{4.7}
$$

$$
|||div(\boldsymbol{p}-\boldsymbol{p}_h)|||_{l^2(L^2)} + |||div(\boldsymbol{q}-\boldsymbol{q}_h)|||_{l^2(L^2)} \le C(h^{1+k}+\tau). \tag{4.8}
$$

Proof. By using (3.1), (3.3), (3.4), Lemma 3.1-3.3, Theorem 4.1, and the triangle inequality, one obtains (4.6)-(4.8) immediately. □

5. Numerical Experiments

For a constrained optimization problem:

$$\min_{u\in K\subset U} J(u),$$

where $J(u)$ is a convex functional on U and K is a close convex subset of U. The iterative scheme reads ($n=0,1,2,\cdots$):

$$\begin{cases} b(u_{n+\frac{1}{2}},v)=b(u_n,v)-\rho_n\left(J'(u_n),v\right), & \forall v\in U, \\ u_{n+1}=P_K^b(u_{n+\frac{1}{2}}), \end{cases} \tag{5.1}$$

where $b(\cdot,\cdot)=\int_0^T(\cdot,\cdot)$ is a symmetric and positive definite bilinear form, ρ_n is a step size of iteration, and the projection operator P_K^b can be computed by (2.32).

Similar to [15], for an acceptable error Tol, by applying (5.1) to discrete parabolic OCPs (2.18)-(2.22), we present the following projection gradient algorithm. For ease of exposition, we have omitted the subscript h.

Algorithm 5.1. Projection gradient algorithm

Step 1. Initialize $u^i_{(0)}, i=1,2,\cdots,N$.

Step 2. I. Solve $\boldsymbol{p}^i_{(n)}\in\boldsymbol{V}_h, i=1,2,\cdots,N$ such that

$$(A^{-1}d_t\boldsymbol{p}^i_{(n)},\boldsymbol{v})+(b\mathrm{div}\boldsymbol{p}^i_{(n)},\mathrm{div}\boldsymbol{v})=(bf^i+bu^i_{(n)},\mathrm{div}\boldsymbol{v}),\quad \boldsymbol{p}^0_{(n)}=\Pi_h\boldsymbol{p}_0.$$

II. Solve $y^i_{(n)}\in W_h, i=1,2,\cdots,N$ such that

$$(d_ty^i_{(n)},w)=-(b\mathrm{div}\boldsymbol{p}^i_{(n)},w)+(bf^i+bu^i_{(n-1)},w),\quad y^0_{(n)}=P_hy_0.$$

III. Solve $z^i_{(n)}\in W_h i=1,2,\cdots,N$ such that

$$(d_tz^i_{(n)},w)=(y^i_{(n)}-y^i_d,w),\quad z^N_{(n)}=0.$$

IV. Solve $\boldsymbol{q}^i_{(n)}\in\boldsymbol{V}_h i=1,2,\cdots,N$ such that

$$-(A^{-1}d_t\boldsymbol{q}^i_{(n)},\boldsymbol{v})+(b\mathrm{div}\boldsymbol{q}^{i-1}_{(n)},\mathrm{div}\boldsymbol{v})=-(\boldsymbol{p}^i_{(n)}-\boldsymbol{p}^i_d,\boldsymbol{v})-(bz^{i-1}_{(n)},\mathrm{div}\boldsymbol{v}),\quad \boldsymbol{q}^N_{(n)}=0.$$

V. Compute $u^i_{(n+1)}, i=1,2,\cdots,N$ by

$$\begin{cases} b(u^i_{(n+\frac{1}{2})},v)=b(u^i_{(n)},v)-\rho_{(n)}(J'(u^i_{(n)}),v), & \forall v\in U, \\ u^i_{(n+1)}=P_K^b(u^i_{(n+\frac{1}{2})}). \end{cases}$$

Step 3. Calculate the iterative error: $E_{n+1}=|||u_{(n+1)}-u_{(n)}|||_{l^2(L^2)}$.

Step 4. If $E_{n+1}\le Tol$, stop; else set $n:=n+1$ go to Step 2.

Let $\Omega=(0,1)\times(0,1)$, $T=1$, $\nu=1$, $c(x)=1$, and A be an unit matrix. We solve the following parabolic OCPs where codes are developed based on AFEPack. The package is freely available and the details can be found at [14]. The discretization scheme is already described in previous sections.

Example 1. The data are as follows:

$$
\begin{aligned}
&y = e^t \sin(2\pi x_1)\sin(2\pi x_2),\\
&\boldsymbol{p} = -\begin{pmatrix} 2\pi e^t \cos(2\pi x_1)\sin(2\pi x_2)\\ 2\pi e^t \sin(2\pi x_1)\cos(2\pi x_2)\end{pmatrix},\\
&\boldsymbol{q} = \begin{pmatrix} (1-t)^2 \cos(\pi x_1)\sin(\pi x_2)\\ (1-t)^2 \sin(\pi x_1)\cos(\pi x_2)\end{pmatrix},\\
&z = -\mathrm{div}\boldsymbol{q} + (1-t)\sin(\pi x_1)\sin(\pi x_2),\\
&\boldsymbol{p}_d = \boldsymbol{p} - \boldsymbol{q}_t + \begin{pmatrix} \pi(1-t)\cos(\pi x_1)\sin(\pi x_2)\\ \pi(1-t)\sin(\pi x_1)\cos(\pi x_2)\end{pmatrix},\\
&y_d = y - z_t,\\
&f = y_t + \mathrm{div}\boldsymbol{p} - u,\\
&u = \max\{0, z + \mathrm{div}\boldsymbol{q}\}.
\end{aligned}
$$

This example is solved by the Algorithm 5.1. Some numerical results based on a sequence uniformly refined meshes are demonstrated in Table 1 and Table 2. In Table 1 and Figure 1, we can see the convergence order $O(h+\tau)$ with $k=0$. If $k=1$, we also see that the convergence order $O(h^2+\tau)$ in Table 2 and Figure 2. They are consistent with our theoretical results.

TABLE 1. The errors with $k=0$, Example 1.

$h=\tau$	1/10	1/20	1/40	1/80
$\vert\vert\vert\boldsymbol{p}-\boldsymbol{p}_h\vert\vert\vert_{l^\infty(L^2)}$	1.0430e+00	5.2853e-01	2.5262e-01	1.3664e-01
$\vert\vert\vert y-y_h\vert\vert\vert_{l^\infty(L^2)}$	4.1573e-01	2.1174e-01	1.0793e-01	5.3782e-02
$\vert\vert\vert z-z_h\vert\vert\vert_{l^\infty(L^2)}$	6.3702e-01	3.4379e-01	1.6101e-01	8.3538e-02
$\vert\vert\vert\boldsymbol{q}-\boldsymbol{q}_h\vert\vert\vert_{l^\infty(L^2)}$	8.9619e-01	4.3653e-01	2.2054e-01	1.0454e-01
$\vert\vert\vert u-u_h\vert\vert\vert_{l^2(L^2)}$	6.3712e-01	3.1387e-01	1.6099e-01	8.3536e-02

TABLE 2. The errors with $k=1$, Example 1.

h	1/10	1/20	1/40	1/80
τ	1/10	1/40	1/160	1/640
$\vert\vert\vert\boldsymbol{p}-\boldsymbol{p}_h\vert\vert\vert_{l^\infty(L^2)}$	8.1547e-01	2.0387e-01	5.0971e-02	1.2743e-02
$\vert\vert\vert y-y_h\vert\vert\vert_{l^\infty(L^2)}$	2.5836e-01	6.4591e-02	1.6248e-02	4.0623e-03
$\vert\vert\vert z-z_h\vert\vert\vert_{l^\infty(L^2)}$	4.4627e-01	1.1257e-01	2.8243e-02	7.0608e-03
$\vert\vert\vert\boldsymbol{q}-\boldsymbol{q}_h\vert\vert\vert_{l^\infty(L^2)}$	6.3825e-01	1.5961e-01	3.9903e-02	9.9758e-03
$\vert\vert\vert u-u_h\vert\vert\vert_{l^2(L^2)}$	4.8523e-01	1.2231e-01	3.1068e-02	7.7670e-03

Funding

This work was supported by the Natural Science Foundation of Hunan Province (2020JJ4323), the Scientific Research Project of Hunan Provincial Department of Education (20A211,20C0854),

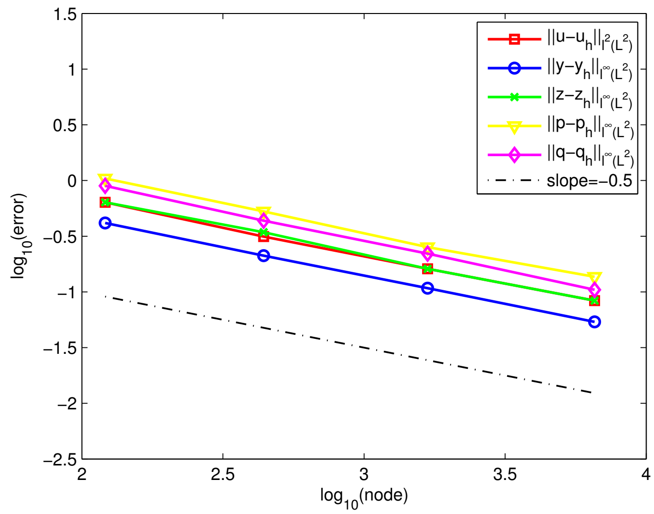

FIGURE 1. Convergence order with $k = 0$, Example 1.

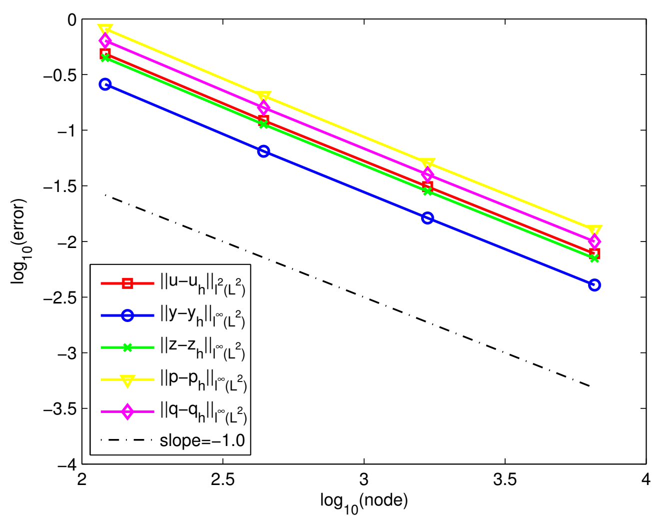

FIGURE 2. Convergence order with $k = 1$, Example 1.

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