J. Nonlinear Funct. Anal. 2023 (2023) 12 https://doi.org/10.23952/jnfa.2023.12

Journal of Nonlinear Functional Analysis
Available online at http://jnfa.mathres.org

GENERALIZATIONS OF FRACTIONAL q-INTEGRALS INVOLVING CAUCHY POLYNOMIALS AND SOME APPLICATIONS

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Abstract. In this paper, we demonstrate the technique of iterations and generalize Riemann–Liouville fractional q-integrals involving Cauchy polynomials. We obtain the generalizations of Srivastava–Agarwal type generating functions by generalized fractional q-integrals involving Cauchy polynomials. Moreover, we also derive generating functions for Rajković–Marinković–Stanković polynomials involving Cauchy polynomial by fractional q-integrals. At last, we deduce a generalization of Jackson's transformation formula by fractional q-integrals involving Cauchy polynomials.

Keywords. Cauchy polynomials; Fractional q-integrals; Fractional q-polynomials; Generating functions; Transformation formulas.

1. Introduction

Since fractional q-integrals have been widely used in numerous fields of sciences, such as mathematics, physics, acoustics, electrochemistry, and material science. Its related theoretical and applied research has become a hot issue in the world. The operators of fractional calculus provide very suitable tools in describing and solving a number of problems in many areas of science and engineering; see, e.g., [6, 19]. Their treatment from the viewpoint of the fractional q-calculus can additionally open up new perspectives as it did, for example, in optimal control problems [7]. For further information about fractional q-integrals, we refer to [1, 2, 4, 6, 7, 9, 10, 15, 18, 19, 20, 21, 22, 23] and the references therein.

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Received February 9, 2023; Accepted March 17, 2023.

In this paper, we follow the notations and terminology in [14] and suppose that $0<q<1$. The q-series and its compact factorials are defined respectively by

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k)$$

and $(a_1,a_2,...,a_m;q)_n = (a_1;q)_n(a_2;q)_n...(a_m;q)_n$, where $m \in \mathbb{N} := 1,2,3,...$ and $n \in \mathbb{N}_0 := \mathbb{N} \cup 0$.

The basic hypergeometric series ${}_r\Phi_s$ [1] was given by

$${}_r\Phi_s \left[\begin{array}{c} a_1,a_2,\ldots,a_r; \\ \\ b_1,b_2,\ldots,b_s; \end{array} q,z \right] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(q,b_1,b_2,\ldots,b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n,$$

which is convergent for either $|q|<1$ and $|z|<\infty$ when $r \le s$ or $|q|<1$ and $|z|<1$ when $r=s+1$, provided that no zero appears in the denominator.

The Thomae–Jackson q-integral was defined by [14]

$$\int_a^b f(x)\mathrm{d_q}x = (1-q)\sum_{n=0}^{\infty}\Big[bf(bq^n)-af(aq^n)\Big]q^n.$$

For more information about q-series, we refer to [3, 5, 8, 12, 13, 16, 17, 25] and the references therein.

The Riemann–Liouville fractional q-integral operator was introduced in [14]

$$\left(I_{q,a}^{\alpha}f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}f(t)\mathrm{d_q}t,$$

where the q-gamma function was defined by [14]

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad x \in \mathbb{R}\backslash\{0,-1,-2,...\}.$$

The generalized Riemann–Liouville fractional q-integral operator was given by [7]

$$\left(I_{q,a}^{\alpha}f\right)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1}f(t)\mathrm{d_q}t, \quad \alpha \in \mathbb{R}^+.$$

Rajković, Marinković and Stanković [20] obtained the following fractional q-identities.

Proposition 1.1 ([20, Corollary 4.1])**.** *For $\alpha \in \mathbb{R}^+$ and $0<a<x<1$, the following fractional q-integrals are valid*

$$I_{q,a}^{\alpha}\left\{\frac{1}{(x;q)_\infty}\right\} = \frac{(1-q)^\alpha}{(a;q)_\infty}\sum_{n=0}^{\infty}\frac{x^{\alpha+n}(a/x;q)_{\alpha+n}}{(q;q)_{\alpha+n}},$$

$$I_{q,a}^{\alpha}\Big\{(-x;q)_\infty\Big\} = (1-q)^\alpha(-a;q)_\infty\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}x^{\alpha+n}(a/x;q)_{\alpha+n}}{(-a;q)_n(q;q)_{\alpha+n}}.$$

In [26], Zhou, Cao, and Arjika built the relations between the following fractional q-integrals and certain generating functions for q-polynomials.

Proposition 1.2 ([26, Theorem 3]). *For $\alpha \in \mathbb{R}^+$ and $0 < a < x < 1$, if*

$$\max\{|at|, |az|, |ars_3|, |ars_4|, ..., |ars_k|\} < 1,$$

then

$$\begin{aligned}
&I_{q,a}^{\alpha}\left\{\frac{(bxz, xt, xr_3u_3, ..., xr_ku_k; q)_\infty}{(xs, xz, xu_3, ..., xu_k; q)_\infty}\right\} \\
&= \frac{(1-q)^\alpha (abz, at, ar_3u_3, ..., ar_ku_k; q)_\infty}{(as, az, au_3, ..., au_k; q)_\infty} \\
&\times \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q; q)_{\alpha+k}} {}_{k+1}\Phi_k \left[\begin{matrix} q^{-k}, as, az, au_3, ..., au_k; \\ \\ abz, at, ar_3u_3, ..., ar_ku_k; \end{matrix} \; q, q\right]. \qquad (1.1)
\end{aligned}$$

Remark 1.3. For $b = t = r_3 = ... = r_k = 0$ in equation (1.1), it reduces to

$$\begin{aligned}
&I_{q,a}^{\alpha}\left\{\frac{1}{(xs, xz, xu_3, ..., xu_k; q)_\infty}\right\} \\
&= \frac{(1-q)^\alpha}{(as, az, au_3, ..., au_k; q)_\infty} \\
&\times \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q; q)_{\alpha+k}} {}_{k+1}\Phi_k \left[\begin{matrix} q^{-k}, as, az, au_3, ..., au_k; \\ \\ 0, 0, ..., 0; \end{matrix} \; q, q\right].
\end{aligned}$$

Our present investigation is essentially motivated by the earlier works by Rajković, Marinković and Stanković [20]. It is natural to ask whether some general fractional q-integrals exist or not, which involvs certain q-polynomials. The novelty of this paper is to find these generalized fractional q-integrals.

Here, we consider the fractional q-integrals involving Cauchy polynomials as follows.

Theorem 1.4. *For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$ and $n_1, n_2, ..., n_s \in \mathbb{N}$, if $\max\{|at_1|, |at_2|, ..., |at_s|\} < 1$, we have*

$$\begin{aligned}
&I_{q,a}^{\alpha}\left\{\frac{P_{n_1}(x, c_1)..., P_{n_s}(x, c_s)}{(xt_1, xt_2, ..., xt_s; q)_\infty}\right\} \\
&= \frac{(1-q)^\alpha \prod_{i=1}^{s}[(t_1c_i; q)_{n_i}]}{t_1^{\sum_{i=1}^{s} n_i} \prod_{i=1}^{s}[(at_i; q)_\infty]} \sum_{\substack{0 \le k_i \le n_i \\ 1 < i < s}} \prod_{i=1}^{s} \left\{\frac{(q^{-n_i}; q)_{k_i} q^{k_i}}{(q; q)_{k_i} q^{\sum_{j=i+1}^{s} k_i n_j}} \frac{(t_1c_iq^{n_i}; q)_{\sum_{j=0}^{i-1} k_j}}{(t_1c_i; q)_{\sum_{j=1}^{i} k_j}}\right\} \\
&\quad \times (at_1; q)_{\sum_{j=1}^{s} k_j} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x; q)_{\alpha+k}}{a^k (q; q)_{\alpha+k}} {}_{s+1}\Phi_s \left[\begin{matrix} q^{-k}, at_2, at_3, ..., at_s, at_1q^{\sum_{j=1}^{s} k_j}; \\ \\ 0, 0, ..., 0; \end{matrix} \; q, q\right], \quad (1.2)
\end{aligned}$$

where Cauchy polynomials $P_n(a, b) = (a - b)(a - bq)...(a - bq^{n-1})$.

Corollary 1.5. *For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$ and $n, m \in \mathbb{N}$, if $max\{|at|, |as|\} < 1$, we have*

$$\begin{aligned} & I_{q,a}^{\alpha}\left\{\frac{P_n(x,c)P_m(x,d)}{(xt,xs;q)_\infty}\right\} \\ &= \frac{(1-q)^{\alpha}(tc;q)_n}{t^{m+n}(as,at;q)_\infty}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}(td;q)_{m+k_1}q^{k_1}}{(tc,q;q)_{k_1}q^{k_1 m}}\sum_{k_2=0}^{m}\frac{(q^{-m};q)_{k_2}(at;q)_{k_2+k_1}q^{k_2}}{(q;q)_{k_2}(td;q)_{k_2+k_1}} \\ &\times \sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_3\Phi_2\left[\begin{matrix} q^{-k}, as, atq^{k_1+k_2}; \\ 0,0; \end{matrix} \; q, q\right]. \end{aligned} \tag{1.3}$$

Remark 1.6. For $n_1 = n_2 = n_3 = \ldots = n_s = 0$ in Theorem 1.4, equation (1.2) reduces to equation (1.3).

The rest of the paper is organized as follows. In Section 2, we give the proof of the main Theorem. In Section 3, we generalize Srivastava–Agarwal type generating functions by generalized fractional q-integrals involving Cauchy polynomials. In Section 4, we generalize generating functions for Rajković–Marinković–Stanković polynomials involving Cauchy polynomial by generalized fractional q-integrals. In Section 5, the last section, we give a generalization of Jackson's transformation formula by generalized fractional q-integrals.

2. Proof of the Main Theorem

In this section, the following Lemmas are necessary for the proof of our results.

Lemma 2.1 (q-Chu–Vandermonde [14])**.** *For $n \in \mathbb{N}$, we have*

$${}_2\Phi_1\left[\begin{matrix} q^{-n}, a; \\ c; \end{matrix} \; q, q\right] = \frac{a^n(c/a;q)_n}{(c;q)_n}, \tag{2.1}$$

and

$${}_2\Phi_1\left[\begin{matrix} q^{-n}, a; \\ c; \end{matrix} \; q, \frac{cq^n}{a}\right] = \frac{(c/a;q)_n}{(c;q)_n}.$$

Lemma 2.2 ([9, Eq. (1.17)])**.** *For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$ and $|as| < 1$, we have*

$$I_{q,a}^{\alpha}\left\{\frac{(xt;q)_\infty}{(xs;q)_\infty}\right\} = \frac{(1-q)^{\alpha}(at;q)_\infty}{(as;q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}(t/s;q)_k s^k}{(q;q)_{\alpha+k}(at;q)_k}. \tag{2.2}$$

For $s = 0$, one has the following Corollary.

Corollary 2.3. *For $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$ and $|as| < 1$, we have*

$$I_{q,a}^{\alpha}\left\{(xt;q)_\infty\right\} = (1-q)^{\alpha}(at;q)_\infty\sum_{k=0}^{\infty}\frac{(-1)^k q^{\binom{k}{2}} t^k x^{\alpha+k}(a/x;q)_{\alpha+k}}{(q;q)_{\alpha+k}(at;q)_k}.$$

Proof of Lemma 2.2. The left-hand side (LHS) of equation (2.2) is equal to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(t/s;q)_n s^n}{(q;q)_n} I_{q,a}^{\alpha}\left\{x^n\right\}\\
&=\sum_{n=0}^{\infty}\frac{(t/s;q)_n s^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q;q)_n a^{n-k}}{(q;q)_k(q;q)_{n-k}}\frac{(q;q)_k(1-q)^{-k}}{(q;q)_{\alpha+k}(1-q)^{-\alpha-k}}x^{\alpha+k}(a/x;q)_{\alpha+k}\\
&=(1-q)^{\alpha}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k(t/s;q)_k s^k}}{(q;q)_{\alpha+k}}\sum_{n=0}^{\infty}\frac{(tq^k/s;q)_n(as)^n}{(q;q)_n},
\end{aligned}$$

which equals the right-hand side (RHS) of equation (2.2) after simplification. The proof is complete. □

Now we are in a position to prove our main theorem.

Proof of Theorem 1.4. We may rewrite the q-Chu–Vandermonde formula (2.1) equivalently by

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,tc;q)_k}\frac{1}{(xtq^k;q)_\infty}=\frac{t^n}{(tc;q)_n}\frac{P_n(x,c)}{(xt;q)_\infty}. \tag{2.3}$$

Multiplying both sides of equation (2.3) by $\frac{1}{(xs;q)_\infty}$, we have

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,tc;q)_k}\frac{1}{(xtq^k,xs;q)_\infty}=\frac{t^n}{(tc;q)_n}\frac{P_n(x,c)}{(xt,xs;q)_\infty}. \tag{2.4}$$

Applying the fractional integral $I_{q,a}^{\alpha}$ on both sides of equation (2.4), we have

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,tc;q)_k} I_{q,a}^{\alpha}\left\{\frac{1}{(xtq^k,xs;q)_\infty}\right\}=\frac{t^n}{(tc;q)_n}I_{q,a}^{\alpha}\left\{\frac{P_n(x,c)}{(xt,xs;q)_\infty}\right\},$$

that is,

$$\begin{aligned}
&I_{q,a}^{\alpha}\left\{\frac{P_n(x,c)}{(xt,xs;q)_\infty}\right\}\\
&=\frac{(tc;q)_n}{t^n}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}q^{k_1}}{(tc,q;q)_{k_1}}I_{q,a}^{\alpha}\left\{\frac{1}{(xtq^{k_1},xs;q)_\infty}\right\}\\
&=\frac{(tc;q)_n}{t^n}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}q^{k_1}}{(tc,q;q)_{k_1}}\frac{(1-q)^{\alpha}}{(as,atq^{k_1};q)_\infty}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_3\Phi_2\left[\begin{matrix}q^{-k},as,atq^k;\\ 0,0;\end{matrix}\quad q,q\right]\\
&=\frac{(1-q)^{\alpha}(tc;q)_n}{t^n(as,at;q)_\infty}\sum_{k_1=0}^{n}\frac{(q^{-n},at;q)_{k_1}q^{k_1}}{(tc,q;q)_{k_1}}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}{}_3\Phi_2\left[\begin{matrix}q^{-k},as,atq^{k_1};\\ 0,0;\end{matrix}\quad q,q\right].
\end{aligned}$$

If we rewrite equation (2.3) equivalently by

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_\infty q^k}{(q,tc;q)_\infty}\frac{P_m(x,d)}{(xtq^k,xs;q)_\infty}=\frac{t^n}{(tc;q)_n}\frac{P_n(x,c)P_m(x,d)}{(xt,xs;q)_\infty},$$

and use the fractional integral $I_{q,a}^{\alpha}$ on both sides of equation, then

$$\sum_{k=0}^{n}\frac{(q^{-n};q)_k q^k}{(q,tc;q)_k} I_{q,a}^{\alpha}\left\{\frac{P_m(x,d)}{(xtq^k,xs;q)_\infty}\right\}=\frac{t^n}{(tc;q)_n} I_{q,a}^{\alpha}\left\{\frac{P_n(x,c)P_m(x,d)}{(xt,xs;q)_\infty}\right\},$$

that is,

$$
\begin{aligned}
& I_{q,a}^{\alpha}\left\{\frac{P_n(x,c)P_m(x,d)}{(xt,xs;q)_\infty}\right\} \\
&= \frac{(tc;q)_n}{t^n}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}q^{k_1}}{(tc,q;q)_{k_1}}\, I_{q,a}^{\alpha}\left\{\frac{P_m(x,d)}{(xtq^k,xs;q)_\infty}\right\} \\
&= \frac{(tc;q)_n}{t^n}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}q^{k_1}(tdq^{k_1};q)_m(1-q)^{\alpha}}{(tc,q;q)_{k_1}(as,atq^{k_1};q)_\infty t^m q^{k_1 m}}\sum_{k_2=0}^{m}\frac{(q^{-m},atq^{k_1};q)_{k_2}q^{k_2}}{(tdq^{k_1},q;q)_{k_2}} \\
&\quad\times\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{(q;q)_{\alpha+k}}\,{}_3\Phi_2\left[\begin{matrix} q^{-k},as,atq^{k_1+k_2}; \\ 0,0; \end{matrix}\, q,q\right] \\
&= \frac{(1-q)^{\alpha}(tc;q)_n}{t^{m+n}(as,at;q)_\infty}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}(td;q)_{m+k_1}q^{k_1}}{(tc,q;q)_{k_1}q^{k_1 m}}\sum_{k_2=0}^{m}\frac{(q^{-m};q)_{k_2}(at;q)_{k_2+k_1}q^{k_2}}{(q;q)_{k_2}(td;q)_{k_2+k_1}} \\
&\quad\times\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_3\Phi_2\left[\begin{matrix} q^{-k},as,atq^{k_1+k_2}; \\ 0,0; \end{matrix}\, q,q\right],
\end{aligned}
$$

which is equation (1.2) after iterations. The proof is complete. □

3. A Generalization of Srivastava–Agarwal Type Generating Functions

The Al-Salam–Carlitz polynomial [11, P. 92] is given by

$$\phi_n^{(a)}(x|q) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_q (a;q)_k x^k.$$

Srivastava and Agarwal deduced the following generating function for Al-Salam–Carlitz polynomial.

Lemma 3.1 ([24, Eq. (3.20)])**.** *It is asserted that*

$$\sum_{n=0}^{\infty}\phi_n^{(\sigma)}(x|q)(\lambda;q)_n\frac{t^n}{(q;q)_n} = \frac{(\lambda t;q)_\infty}{(t;q)_\infty}\,{}_2\Phi_1\left[\begin{matrix}\lambda,\sigma; \\ \lambda t;\end{matrix}\, q;xt\right], \qquad \big(\max\{|t|,|xt|\}<1\big). \tag{3.1}$$

In this section, we generalize Srivastava–Agarwal type generating functions by generalized fractional q-integrals involving Cauchy polynomials.

Theorem 3.2. *For $\alpha\in\mathbb{R}^+$, $0<a<x<1$ and $n,m\in\mathbb{N}$, if $\max\{|ax|,|as|\}<1$, we have*

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\phi_n^{(\sigma)}(x|q)\frac{(x\lambda;q)_n s^n}{(q;q)_n x^n}\sum_{k_1=0}^{n}\frac{(q^{-n},axt;q)_{k_1}q^{k_1}}{(x\lambda,q;q)_{k_1}}\sum_{k=0}^{\infty}\frac{t^{\alpha+k}(a/t;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_2\Phi_1\left[\begin{matrix} q^{-k},axq^{k_1}; \\ 0; \end{matrix}\, q,q\right] \\
&=\frac{(\lambda s;q)_\infty}{(as;q)_\infty}\sum_{m=0}^{\infty}\frac{(\sigma,x\lambda;q)_m s^m}{(\lambda s,q;q)_m}\sum_{k_1=0}^{m}\frac{(q^{-m},ax;q)_{k_1}q^{k_1}}{(x\lambda,q;q)_{k_1}}\sum_{k=0}^{\infty}\frac{t^{\alpha+k}(a/t;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\,{}_3\Phi_2\left[\begin{matrix} q^{-k},as,axq^{k_1}; \\ 0,0; \end{matrix}\, q,q\right].
\end{aligned}
$$

Proof of Theorem 3.2. The equation (3.1) can be rewrite equivalently by

$$\sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q) \frac{P_n(t,\lambda)}{(tx;q)_\infty} \frac{s^n}{(q;q)_n} = (\lambda s;q)_\infty \sum_{k=0}^{\infty} \frac{P_k(t,\lambda)}{(tx,ts;q)_\infty} \frac{(xs)^k(\sigma;q)_k}{(\lambda s,q;q)_k}.$$

Applying the operator $I_{q,a}^{\alpha}$ with respect to the variable t, we obtain

$$\sum_{n=0}^{\infty} \phi_n^{(\sigma)}(x|q)\, I_{q,a}^{\alpha} \left\{ \frac{P_n(t,\lambda)}{(tx;q)_\infty} \right\} \frac{s^n}{(q;q)_n} = (\lambda s;q)_\infty \sum_{k=0}^{\infty} \frac{(xs)^k(\sigma;q)_k}{(\lambda s,q;q)_k}\, I_{q,a}^{\alpha} \left\{ \frac{P_k(t,\lambda)}{(tx,ts;q)_\infty} \right\}. \tag{3.2}$$

Taking $(x,t_1,c_1) = (t,x,\lambda)$ and $n_2 = \ldots = n_s = t_2 = \ldots = t_s = 0$ in Theorem 1.4, we have

$$\begin{aligned} &I_{q,a}^{\alpha} \left\{ \frac{P_n(t,\lambda)}{(tx;q)_\infty} \right\} \\ &= \frac{(1-q)^\alpha (x\lambda;q)_n}{t^n (ax;q)_\infty} \sum_{k_1=0}^{n} \frac{(q^{-n},ax;q)_{k_1} q^{k_1}}{(x\lambda,q;q)_{k_1}} \sum_{k=0}^{\infty} \frac{t^{\alpha+k}(a/t;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\, {}_2\Phi_1 \left[\begin{matrix} q^{-k}, axq^{k_1}; \\ 0; \end{matrix} \; q,q \right] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &I_{q,a}^{\alpha} \left\{ \frac{P_m(t,\lambda)}{(tx,ts;q)_\infty} \right\} \\ &= \frac{(1-q)^\alpha (x\lambda;q)_m}{x^m (ax,as;q)_\infty} \sum_{k_1=0}^{m} \frac{(q^{-m},ax;q)_{k_1} q^{k_1}}{(x\lambda,q;q)_{k_1}} \sum_{k=0}^{\infty} \frac{t^{\alpha+k}(a/t;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}}\, {}_3\Phi_2 \left[\begin{matrix} q^{-k}, as, axq^{k_1}; \\ 0,0; \end{matrix} \; q,q \right]. \end{aligned}$$

Combining the above two equations into equation (3.2), we achieve the proof of Theorem 3.2. □

4. Generating Functions for Rajković–Marinković–Stanković Polynomials Involving Cauchy Polynomial

Recall that the Rajković–Marinković–Stanković polynomials are defined [9, Eq. (1.16)]

$$\mathscr{P}_n(\alpha,a,x|q) = I_{q,a}^{\alpha}\ \{x^n\} = \sum_{k=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{[k]_q!\, a^{n-k}}{\Gamma_q(\alpha+k+1)} x^{\alpha+k}(a/x;q)_{\alpha+k}.$$

Lemma 4.1. *For $\alpha \in \mathbb{R}^+$, $0<a<x<1$, we have*

$$\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha,a,x|q) \frac{w^n}{(q;q)_n} = \frac{(1-q)^\alpha}{(aw;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k} w^k}{(q;q)_{\alpha+k}}. \tag{4.1}$$

Proof of Lemma 4.1. The LHS of equation (4.1) is equal to

$$I_{q,a}^{\alpha} \left\{ \sum_{n=0}^{\infty} \frac{(xw)^n}{(q;q)_n} \right\} = I_{q,a}^{\alpha} \left\{ \frac{1}{(xw;q)_\infty} \right\},$$

which equals the RHS of equation (4.1) after using equation (2.2). The proof is complete. □

In this section, we generalize generating functions for Rajković–Marinković–Stanković polynomials involving Cauchy polynomial by generalized fractional q-integrals as follows.

Theorem 4.2. *For* $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, *and max*$\{|awv|, |awu|\} < 1$, *we have*

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha, a, x|q) P_n(u, v) \frac{w^n}{(q;q)_n} \\ &= \frac{(1-q)^{\alpha}(awv;q)_\infty}{(awu;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{(q;q)_{\alpha+k}} {}_2\Phi_1 \left[\begin{matrix} q^{-k}, awu; \\ awv; \end{matrix} \; q, q \right] \\ &= \frac{(1-q)^{\alpha}(awv;q)_\infty}{(awu;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}(v/u;q)_k (wu)^k}{(q;q)_{\alpha+k}(awv;q)_k}.\end{aligned}$$

Remark 4.3. When $v = 0$ in Theorem 4.2, equation (4.2) reduces to equation (4.1).

Proof of Theorem 4.2. The LHS of equation (4.2) is equal to

$$I_{q,a}^{\alpha} \left\{ \sum_{n=0}^{\infty} P_n(u, v) \frac{(xw)^n}{(q;q)_n} \right\} = I_{q,a}^{\alpha} \left\{ \sum_{n=0}^{\infty} \frac{(xwu)^n (v/u;q)_n}{(q;q)_n} \right\} = I_{q,a}^{\alpha} \left\{ \frac{(xwv;q)_\infty}{(xwu;q)_\infty} \right\}, \tag{4.2}$$

which equals the RHS of equation (4.2) after using equation (2.2). The proof is complete. □

Theorem 4.4. *For* $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, *and max*$\{|awv|, |awu|, |awvz|, |awuz|\} < 1$, *we have*

$$\begin{aligned}&\sum_{n=0}^{\infty} \mathscr{P}_n(\alpha, a, x|q) P_n(u, v) P_n(u, z) \frac{w^n}{(q;q)_n (xwvz;q)_n} \\ &= \frac{(1-q)^{\alpha}(awvu, awuz;q)_\infty}{(awu^2;q)_\infty} \sum_{k=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{(q;q)_{\alpha+k}} \times {}_3\Phi_2 \left[\begin{matrix} q^{-k}, awu^2, awvz; \\ awuv, awuz; \end{matrix} \; q, q \right].\end{aligned} \tag{4.3}$$

Proof of Theorem 4.4. The LHS of equation (4.3) is equal to

$$\begin{aligned}I_{q,a}^{\alpha} \left\{ \sum_{n=0}^{\infty} P_n(u, v) P_n(u, z) \frac{(xw)^n}{(xwvz, q;q)_n} \right\} &= I_{q,a}^{\alpha} \left\{ \sum_{n=0}^{\infty} \frac{(xwu^2)^n (v/u;q)_n (z/u;q)_n}{(xwvz, q;q)_n} \right\} \\ &= I_{q,a}^{\alpha} \left\{ \frac{(xwuz, xwuv;q)_\infty}{(xwu^2, xwvz;q)_\infty} \right\},\end{aligned}$$

which equals the RHS of equation (4.3) after using equation (1.1). The proof is complete. □

Theorem 4.5. *For* $\alpha \in \mathbb{R}^+$, $0 < a < x < 1$, *and* $\max\{|aw|, |xwc|\} < 1$, *we have*

$$\begin{aligned}\sum_{n=0}^{\infty} \mathscr{P}_{n+k}(\alpha, a, x|q) \frac{(c;q)_k w^n}{(q;q)_n} &= \frac{(1-q)^{\alpha}(xwc;q)_k}{w^k (aw;q)_\infty} \sum_{n_1=0}^{k} \frac{(q^{-k}, aw;q)_{n_1} q^{n_1}}{(xwc, q;q)_{n_1}} \\ &\times \sum_{n=0}^{\infty} \frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k (q;q)_{\alpha+k}} \, {}_2\Phi_1 \left[\begin{matrix} q^{-k}, awq^{n_1}; \\ 0; \end{matrix} \; q, q \right].\end{aligned} \tag{4.4}$$

Proof of Theorem 4.5. The LHS of equation (4.4) is equal to

$$\begin{aligned}\sum_{n=0}^{\infty}\mathscr{P}_{n+k}(\alpha,a,x|q)\frac{(c;q)_k w^n}{x^{n+k}(q;q)_n} &= \sum_{n=0}^{\infty} I_{q,a}^{\alpha}\left\{x^{n+k}\right\}\frac{(c;q)_k w^n}{(q;q)_n}\\ &= I_{q,a}^{\alpha}\left\{\sum_{n=0}^{\infty}\frac{(xw)^n x^k (c;q)_k}{(q;q)_n}\right\}\\ &= I_{q,a}^{\alpha}\left\{\frac{P_k(x,cx)}{(xw;q)_\infty}\right\},\end{aligned}$$

which equals the RHS of equation (4.4) after using equation (3.3). The proof is complete. □

5. Generalization of Jackson's Transformation Formula

Jackson's transformation formula is

$${}_2\Phi_1\left[\begin{matrix}a,b;\\ c;\end{matrix}\ q;z\right] = \frac{(bz;q)_\infty}{(z;q)_\infty}{}_2\Phi_2\left[\begin{matrix}b,c/a;\\ bz,c;\end{matrix}\ q;az\right]. \tag{5.1}$$

Taking (b,z,a,c) *by* $(b/x,xz,x/z,c/z)$ in equation (5.1) and performing some calculation, we have

$$\sum_{n=0}^{\infty}\frac{(x/z,b/x;q)_n}{(c/z,q;q)_n}(xz)^n = \frac{(bz;q)_\infty}{(xz;q)_\infty}\sum_{n=0}^{\infty}\frac{(b/x,c/x;q)_n}{(bz,c/z,q;q)_n}(-1)^n q^{\binom{n}{2}}(x^2)^n. \tag{5.2}$$

In this section, we give a generalization of Jackson's transformation formula (5.2) by generalized fractional q-integrals as follows

Theorem 5.1. *For $\alpha\in\mathbb{R}^+$, $0<a<x<1$, and $\max\{|az|,|bz|\}<1$, we have*

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{z^{2n}(bz^{-1};q)_{2n}}{q^{n^2}(c/z,q;q)_n}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}(az^{-1};q)_{n+k_1}q^{k_1}}{(bz^{-1};q)_{n+k_1}(q;q)_{k_1}}\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(a;a)_{\alpha+k}}\,{}_2\Phi_1\left[\begin{matrix}q^{-k},aq^{n+k_1}z^{-1};\\ 0;\end{matrix}\ q;q\right]\\ &=\frac{(bz;q)_\infty}{(az;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{\binom{n}{2}}(-1)^n}{z^{2n}(c/z,q;q)_n}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}(zc;q)_{n+k_1}q^{k_1}}{(bz;q)_{k_1}(q;q)_{k_1}q^{k_1 n}}\sum_{k_2=0}^{n}\frac{(q^{-n};q)_{k_2}(az;q)_{k_1+k_2}q^{k_2}}{(zc;q)_{k_1+k_2}(q;q)_{k_2}}\\ &\quad\times\sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(a;q)_{\alpha+k}}\,{}_3\Phi_2\left[\begin{matrix}q^{-k},az^{-1},azq^{k_1+k_2};\\ 0,0;\end{matrix}\ q;q\right].\end{aligned}$$

Proof of Theorem 5.1. The equation (5.2) can be rewrite equivalently by

$$\sum_{n=0}^{\infty}\frac{z^n}{(c/z,q;q)_n}\frac{P_n(x,b)}{(xq^n/z;q)_\infty} = (bz;q)_\infty\sum_{n=0}^{\infty}\frac{P_n(x,b)P_n(x,c)}{(xz,x/z;q)_\infty}\frac{(-1)^n q^{\binom{n}{2}}}{(bz,c/z,q;q)_n}. \tag{5.3}$$

Taking $(n_1, c_1, s) = (n, b, q^n z^{-1})$ and $n_2 = \ldots = n_s = t_2 = \ldots = t_s = z = 0$ in Theorem 1.4, we have

$$\begin{aligned}
&I_{q,a}^{\alpha}\left\{\frac{P_n(x,b)}{(xq^n z^{-1};q)_\infty}\right\} \\
&= \frac{(1-q)^{\alpha}(bq^n z^{-1};q)_n}{q^n z^{-1^n}(aq^n z^{-1};q)_\infty}\sum_{k_1=0}^{n}\frac{(q^{-n}, aq^n z^{-1};q)_{k_1} q^{k_1}}{(bq^n z^{-1}, q;q)_{k_1}}\sum_{k=0}^{\infty}\frac{xt^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(q;q)_{\alpha+k}} \\
&\times {}_2\Phi_1\left[\begin{matrix} q^{-k}, aq^{n+k_1}z^{-1}; & \\ & q, q \\ 0; & \end{matrix}\right]
\end{aligned} \tag{5.4}$$

and

$$\begin{aligned}
&I_{q,a}^{\alpha}\left\{\frac{P_n(x,b)P_n(x,c)}{(xz, x/z;q)_\infty}\right\} \\
&= \frac{(1-q)^{\alpha}(bz;q)_n}{z^{2n}(az, a/z;q)_\infty}\sum_{k_1=0}^{n}\frac{(q^{-n};q)_{k_1}(zc;q)_{n+k_1} q^{k_1}}{(bz,q;q)_{k_1} q^{k_1 n}}\sum_{k_2=0}^{n}\frac{(q^{-n};q)_{k_2}(az;q)_{k_1+k_2} q^{k_2}}{(zc;q)_{k_1+k_2}(q;q)_{k_2}} \\
&\times \sum_{k=0}^{\infty}\frac{x^{\alpha+k}(a/x;q)_{\alpha+k}}{a^k(a;q)_{\alpha+k}}\, {}_3\Phi_2\left[\begin{matrix} q^{-k}, az^{-1}, azq^{k_1+k_2}; & \\ & q;q \\ 0,0; & \end{matrix}\right].
\end{aligned} \tag{5.5}$$

Combining the above two equations (5.4) and (5.5) into equation (5.3), we achieve the proof of Theorem 5.1. □

6. Concluding Remarks and Observations

Our present investigation is motivated by the fact that fractional q-integrals play important roles in many scientific fields in mathematical, physical and engineering sciences. It stems from a natural question to whether the potentially useful fractional q-integrals involving Cauchy polynomials exists and is worthy of further investigation. Here, in this paper, we demonstrated the method of iteration and introduced a class of fractional q-integrals involving Cauchy polynomials. As their applications, we generalized Srivastava–Agarwal type generating functions, generating functions for Rajković–Marinković–Stanković polynomials, and the generalization of Jackson's transformation formula by generalized fractional q-integrals involving Cauchy polynomials. We also briefly considered relevant connections of our result with other known results.

Funding

This work was supported by the Zhejiang Provincial Natural Science Foundation of China (No. LY21A010019) and the National Natural Science Foundation of China (No. 12071421).

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