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GLOBAL STABILITY OF MULTI-GROUP EPIDEMIC MODEL WITH DISTRIBUTED DELAYS AND INDIRECT TRANSMISSION

YUN ZHENG, XAMXINUR ABDURAHMAN*

College of Mathematics and Systems Science, Xinjiang University, Urumqi 830046, China

Abstract. In this paper, a multi-group epidemic model with distributed delays and indirect transmission is discussed. We identified the basic reproduction number R_0 and investigated the dynamical properties of the model with respect to R_0. It is proved that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$, and there exists a unique endemic equilibrium which is globally asymptotically stable if $R_0 > 1$. Specifically, we used the method of Lyapunov functionals and graph-theoretical approach to prove that the global stability of the endemic equilibrium.

Keywords. Basic reproduction number; Distributed delays; Global stability; Lyapunov functionals; Multi-group.

1. INTRODUCTION

Multi-group epidemic models are significant for the study of transmission mechanisms of diseases and they have been applied to some diseases, such as measles, mumps, gonorrhea, and HIV/AIDS; see, e.g., [20, 21, 26]. In reality, the host population can be divided into different groups based on their heterogeneity (i.e., species, age, position, social behaviors etc), and the individuals of each group are homogeneous. In most cases, there are different transmission rates between groups. For instance, the transmission rate of COVID-19 is different between cities because of the level of epidemic control capacity in each region. The transmission rate of gonorrhea is greater in the young adults than in the older adults.

For some diseases including malaria, cholera and brucellosis, indirect transmission plays an important role in the spread of these diseases, which means that outside pathogens can infect susceptible individuals to make them sick. Several epidemic models have been proposed to describe the diseases mentioned above that contain modes of indirect transmission [1, 4, 7, 18, 22, 25]. Aïnseba, C. Benosman, and P. Magal [1] proposed a brucellosis model with indirect transmission. Li et al. [18] presented a brucellosis multi-group model with indirect transmission. Sun and Zhang [25] investigated a brucellosis model with immigration.

*Corresponding author.
E-mail address: xamxinur@sina.com (X. Abdurahman).
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Eisenberg et al. [7] studied a cholera model in a patchy environment with water and human movement. In the case of brucellosis, it takes a period of time after an animal is infected with the disease before it becomes infectious and releases the pathogens into the environment [5]. Some single-group brucellosis models with latency time delays were proposed to describe this phenomenon; see, e.g., [13, 14, 15]. Hou and Wang [13] developed a discrete delayed SVIP epidemic model with indirect transmission. Hou and Qin [14] presented a multi-stage brucellosis model with distributed delays and indirect transmission. As mentioned before, multi-group models allow for a more explicit classification of the host population than single-group models. After considering indirect transmission and distributed delays, we build a multi-group epidemic model with distributed delays and indirect transmission, and determine the expressions for the basic reproduction number R_0.

Furthermore, we demonstrate that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$; and the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. In particular, the proof of global asymptotic stability of the endemic equilibrium of the multi-group model is not easy. Fortunately, Li and Guo indicated that Lyapunov functionals and graph-theoretical approach can prove the global asymptotic stability of the endemic equilibrium of the multi-group model [8, 19].

This paper is organized as follows. In Section 2, we establish a multi-group epidemic model with distributed delays and indirect transmission, and obtain the basic reproduction number R_0. In Section 3, the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium is proved. In Section 4, we give a short conclusion of the paper.

2. Modeling and preliminaries

The total population N is homogeneously classified into n groups. Further, group i contains the susceptible S_i and the infected I_i, where $i = 1, 2, \ldots, n$. In addition, B represents the number of pathogens in the common environment, which is released by infectious individuals. There are two patterns of the disease transmission: direct and indirect. Susceptible individuals become infected due to contact with infectious individuals, which is referred direct transmission. And indirect transmission means susceptible individuals are infected by pathogens in the common environment. Infected individuals do not become infectious until after a certain delay. From the illustration of [2], the kernel function $f_j(s)$ is used to describe the probability of an infected individual at time $t-s$, becomes an infectious individual at time t, and s is a parameter of variation. The transmission rate coefficient of disease between the susceptible S_i and the infected I_j is denoted by β_{ij}. The indirect transmission rate from pathogen B to the susceptible S_i is denoted by ϕ_i. Then new infection with distributed delays and indirect transmission in group i at time t is given by

$$\sum_{j=1}^{n} \beta_{ij} S_i \int_0^{\infty} f_j(s) g_j(I_j(t-s)) ds + \phi_i S_i r(B),$$

where the kernel function $f_j(s) \geq 0$ satisfies $\int_0^{\infty} f_j(s) ds = a_j > 0$, $g_j(I_j)$ denotes contact infection rate of I_j, and $r(B)$ is the indirect infection rate. Since only infectious individuals can discharge pathogens, the number of new pathogens in the common environment at time t is given by

$$\sum_{j=1}^{n} k_j \int_0^{\infty} f_j(s) h_j(I_j(t-s)) ds,$$

where k_j is the pathogen shedding rate of infectious individuals in group j. Therefore, a multi-group model with distributed delays and indirect transmission can be described by the following equations:

$$\begin{cases} \dfrac{dS_i}{dt} = A_i - \mu_i S_i - \sum\limits_{j=1}^{n} \beta_{ij} S_i \int_0^{\infty} f_j(s) g_j(I_j(t-s))ds - \phi_i S_i r(B), \\ \dfrac{dI_i}{dt} = \sum\limits_{j=1}^{n} \beta_{ij} S_i \int_0^{\infty} f_j(s) g_j(I_j(t-s))ds + \phi_i S_i r(B) - (\mu_i + \alpha_i) I_i, \\ \dfrac{dB}{dt} = \sum\limits_{j=1}^{n} k_j \int_0^{\infty} f_j(s) h_j(I_j(t-s))ds - \delta B, \ i = 1,2,\ldots,n. \end{cases} \tag{2.1}$$

All parameters of system (2.1) are nonnegative. In group i, A_i is the supplementary rate of the susceptible S_i, μ_i is natural mortality rate of individuals, and α_i is mortality rate due to the disease, δ is natural death rate of pathogens.

We assume that the kernel function $f_i(s)$ satisfies the following conditions:

$$\int_0^{\infty} f_i(s) e^{\lambda_i s} ds < \infty, \tag{2.2}$$

where $\lambda_i > 0$, $i = 1,2,\ldots,n$. Define the following Banach space of fading memory type [10]:

$$C_i = \left\{ \psi \in ((-\infty,0],\mathbb{R}) : \psi(s)e^{\lambda_i s} \text{ is uniformly continuous for } s \in (-\infty,0], \text{ and } \sup_{s\le 0} |\psi(s)| e^{\lambda_i s} < \infty \right\},$$

with norm

$$||\psi||_i = \sup_{s\le 0} |\psi(s)| e^{\lambda_i s} < \infty.$$

We analyze system (2.1) in the following phase space

$$X = \prod_{i=1}^{n} (\mathbb{R} \times C_i \times \mathbb{R}).$$

The initial conditions of system (2.1) is given as follows

$$(S_{1,0}, I_{1,0}, \ldots, S_{n,0}, I_{n,0}, B_0) = (S_1(0), \psi_1(t), \ldots, S_n(0), \psi_n(t), B(0)) \in X. \tag{2.3}$$

By the standard theory of functional differential equations in [11, 12], the solution

$$(S_1(t), I_1(t), \ldots, S_n(t), I_n(t), B(t))$$

of system (2.1) is existent and unique. In the epidemiological significance, we make the following assumptions

(H_1) $g_i(I_i), h_i(I_i)$ and $r(B)$ are sufficiently smooth; $g_i(0) = h_i(0) = r(B) = 0$, and $g_i(I_i) > 0, h_i(I_i) > 0$ for $I_i > 0$, and $r(B) > 0$ for $B > 0$, $i = 1,2,\ldots,n$.

(H_2) $g_i'(I_i), h_i'(I_i) > 0$ and $r'(B) > 0$ for $I_i, B > 0$, i=1,2,... n.

(H_3) $g_i''(I_i), h_i''(I_i) \le 0$ and $r''(B) \le 0$ for $I_i, B > 0$, i=1,2,... n.

(H_4) There exist finite numbers b_i and c_i such that $g_i'(0) = b_i, h_i'(0) = c_i$, $i = 1,2,\ldots n$.

From $(H_1) - (H_3)$, we obatin

$$\frac{g_i(I_i)}{I_i} = \frac{g_i(I_i) - 0}{I_i - 0} = g_i'(\xi) \ge g_i'(I), \ \ \xi \in (0,I),$$

and

$$\left(\frac{g_i(I_i)}{I_i}\right)' = \frac{g_i'(I_i)I_i - g_i(I_i)}{I_i^2} \leq 0.$$

Then the function $\frac{g_i(I_i)}{I_i}$ is monotonic decreasing for $I_i, B > 0$, $i = 1,2,\ldots,n$. Using the same method, it also can be proved that $\frac{h_i(I_i)}{I_i}, \frac{r(B)}{B}$ are monotonic decreasing.

Lemma 2.1. *The solution of system (2.1) is nonnegative and ultimately uniformly bounded in X, and it satisfies the initial conditions in (2.3), where $S_i(0) \in \mathbb{R}_+, B(0) \in \mathbb{R}_+$ and $\psi_i(s) \in C_i$ such that $\psi_i(s) \geq 0$ in $(-\infty, 0]$.*

Proof. Suppose that there exists $t_i^1 > 0$ such that $S_i(t_i^1) < 0$. Then $S_i(t_i^2) = 0$, where

$$t_i^2 = \inf\{0 < t < t_i^1 : S_i(t) < 0\}.$$

From the first equation in (2.1), we see that $S_i'(t_i^2) = A_i > 0$. By continuity of the solution, there exists a sufficiently small ε such that $S_i(t) < 0$ when $t \in (t_i^2 - \varepsilon, t_i^2)$. This contradicts $S_i(t) \geq 0$ for $t \in [0, t_i^2)$. So $S_i(t) \geq 0$ for $t \in [0,\infty)$, i=1,2,...,n.

From the second equation of model (2.1), we can solve that

$$I_i(t) = e^{-(\mu_i+\alpha_i)t}\left(I_i(0) + \int_0^t P_i(u)e^{(\mu_i+\alpha_i)u}du\right),$$

where

$$P_i(t) = \sum_{j=1}^{n} \beta_{ij}S_i \int_0^\infty f_j(s)g_j(I_j(t-s))ds + \phi_i S_i r(B),$$

and

$$B(t) = e^{-\delta t}\left(B(0) + \int_0^t Q(u)e^{\delta u}du\right),$$

where $Q(t) = \sum_{j=1}^n k_j \int_0^\infty f_j(s)h_j(I_j(t-s))ds$. Therefore, $I_i(t) \geq 0, B(t) \geq 0$ for $t \in [0,\infty]$, $i = 1,2,\ldots,n$. For each i, adding the first two equations of system (2.1) yields

$$\frac{d(S_i+I_i)}{dt} = A_i - \mu_i(S_i+I_i) - \alpha_i I_i \leq A_i - \mu_i(S_i+I_i).$$

This demonstrates that

$$\limsup_{t\to\infty}(S_i+I_i) \leq \frac{A_i}{\mu_i}, \quad \limsup_{t\to\infty} B(t) \leq \frac{\sum_{j=1}^n k_j a_j h_j(\frac{A_j}{\mu_j})}{\delta}.$$

Therefore, S_i, I_i and B are ultimately uniformly bounded in X. □

The set

$$\Gamma = \left\{(S_i,I_i,B) \in X : 0 \leq S_i + I_i \leq \frac{A_i}{\mu_i}, 0 \leq B \leq \frac{\sum_{j=1}^n k_j a_j h_j(\frac{A_j}{\mu_j})}{\delta}, i = 1,2,\ldots,n.\right\}.$$

is positively invariant for system (2.1).

The disease-free equilibrium of system (2.1) is $P_0 = (S_1^0, 0, \ldots, S_n, 0, 0)$, where $S_i^0 = \frac{A_i}{\mu_i}$, and $i = 1, 2, \ldots, n$. In epidemiology, the basic reproduction number R_0 is the threshold for disease prevalence or not. If $R_0 \leq 1$, the disease goes extinct; if $R_0 > 1$, the disease is perisitent. Let

$$M_0 = \left(\frac{\beta_{ij} S_i^0 a_j b_j}{\mu_i + \alpha_i} + \frac{\phi_i S_i^0 k_j a_j c_j}{(\mu_i + \alpha_i)\delta} \right)_{1 \leq i,j \leq n}. \tag{2.4}$$

According to the definition of the basic reproduction number in the epidemic models [6], one has $R_0 = \rho(M_0)$, where $\rho(\cdot)$ is the spectral radius of the matrix.

3. Main Results

In this section, we mainly prove the global stability of the disease-free and the endemic equilibrium.

3.1. Global stability of the disease-free equilibrium.

In this subsection, using the same approach as Theorem 4.1 in [23], we prove the global stability of the disease-free equilibrium and discuss the uniform persistence of system (2.1). Furthermore, we can obtain the existence of endemic equilibrium of system (2.1).

Theorem 3.1. *The disease-free equilibrium P_0 of system (2.1) is globally asymptotically stable in Γ, if $R_0 \leq 1$.*

Proof. The matrix M_0 has a positive left eigenvector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ corresponding to the spectral radius $R_0 = \rho(M_0) > 0$, and $\omega M_0 = \omega \rho(M_0)$. Let

$$d_i = \frac{1}{\mu_i + \alpha_i}, \quad i = 1, 2, \ldots, n.$$

Define a Lyapunov functional

$$\begin{aligned} L = \sum_{i=1}^{n} d_i \Big(S_i - S_i^0 - S_i^0 \ln \frac{S_i}{S_i^0} + I + \frac{\phi_i S_i^0}{\delta} B + \sum_{j=1}^{n} \beta_{ij} S_i^0 \int_0^{\infty} f_j(s) \int_{t-s}^{t} g_j(I_j(u)) du ds \\ + \frac{\phi_i S_i^0}{\delta} \sum_{j=1}^{n} k_j \int_0^{\infty} f_j(s) \int_{t-s}^{t} h_j(I_j(u)) du ds \Big). \end{aligned}$$

The derivative for L along system (2.1) is

$$\begin{aligned} \frac{dL}{dt} = \sum_{i=1}^{n} d_i \Big(\mu_i S_i^0 \big(2 - \frac{S_i}{S_i^0} - \frac{S_i^0}{S_i}\big) + \sum_{j=1}^{n} \beta_{ij} S_i^0 \int_0^{\infty} f_j(s) g_j(I_j(t)) ds + \phi_i S_i^0 r(B) \\ + \frac{\phi_i S_i^0}{\delta} \sum_{j=1}^{n} k_j \int_0^{\infty} f_j(s) h_j(I_j(t)) ds - \phi_i S_i^0 B - (\mu_i + \alpha_i) I_i \Big). \end{aligned} \tag{3.1}$$

According to assumption $(H_1), (H_4)$, and the expression of matrix M_0 in (2.4), we have

$$\begin{aligned} L' &\leq \sum_{i=1}^{n} \omega_i \left(\Big(\sum_{j=1}^{n} \beta_{ij} S_i^0 a_j b_j + \frac{\phi_i S_i^0}{\delta} \sum_{j=1}^{n} k_j a_j c_j \Big) \frac{1}{\mu_i + \alpha_i} I_j - I_i \right) \\ &= (\omega_1, \omega_2, \ldots, \omega_n)[M_0 I - I] \\ &= (\rho(M_0) - 1)(\omega_1, \omega_2, \ldots, \omega_n) I, \end{aligned}$$

where $I(t)=(I_1(t),I_2(t),\dots,I_n(t))^T$. Denote

$$Z=\{(S_1,I_1,\dots,S_n,I_n,B)\in\Sigma : L'=0\}.$$

Then $L'\le 0$ for $P\in\Gamma$, and $L'=0$ implies that either $I=0$ or $R_0=1$ and $B=0,S_i=S_i^0,i=1,2,\dots,n$. Therefore, the isolated set $\{P_0\}$ is the largest invariant set of Z. By LaSalle invariance principle [16], the disease-free equilibrium P_0 is globally asymptotically stable in Γ if $R_0\le 1$. □

If $R_0>1$, then $L'>0$ for $P\in\Gamma$ by (3.1). So P_0 is unstable when $R_0>1$. Using the same approach as [17, Proposition 3.3], we can demonstrate that system (2.1) is uniformly persistent when $R_0>1$. Further, system (2.1) has the endemic equilibrium $P^*=(S_i^*,I_i^*,B^*)$(see [3, Theorem 2.8.6] or in [24, Theorem D.3]), where $i=1,2,\dots,n$.

3.2. **Global stability of the endemic equilibrium.** In this subsection, we show that the endemic equilibrium is globally asymptotically stable when $R_0>1$, which indicates that there exists a unique endemic equilibrium. Before proving above fact, we show the following result.

Lemma 3.2. *If $(H_1)-(H_3)$ holds, then*

$$\left(\frac{g_i(I_i)}{g_i(I_i^*)}-1\right)\left(1-\frac{g_i(I_i^*)I_i}{g_i(I_i)I_i^*}\right)\le 0,$$

$$\left(\frac{h_i(I_i)}{h_i(I_i^*)}-1\right)\left(1-\frac{h_i(I_i^*)I_i}{h_i(I_i)I_i^*}\right)\le 0,$$

and

$$\left(\frac{r(B)}{r(B^*)}-1\right)\left(1-\frac{r(B^*)B}{r(B)B^*}\right)\le 0,$$

where $i=1,2,\dots,n$.

Proof.

$$\left(\frac{g_i(I_i)}{g_i(I_i^*)}-1\right)\left(1-\frac{g_i(I_i^*)I_i}{g_i(I_i)I_i^*}\right)=\frac{I_i}{g_i(I_i)g_i(I_i^*)}(g_i(I_i)-g_i(I_i^*))\left(\frac{g_i(I_i)}{I_i}-\frac{g_i(I_i^*)}{I_i^*}\right).$$

Since $\frac{g_i(I_i)}{I_i}$ is decreasing and $g_i(I_i)$ is increasing for $I_i>0$, one has

$$\left(\frac{g_i(I_i)}{g_i(I_i^*)}-1\right)\left(1-\frac{g_i(I_i^*)I_i}{g_i(I_i)I_i^*}\right)\le 0.$$

Similarly, we can prove the rest of the results. □

Theorem 3.3. *Assume that matrix $(\beta_{ij})_{1\le i,j\le n}$ is irreducible. The only endemic equilibrium P^* of system (2.1) is globally asymptotically stable in $\mathring{\Gamma}$, if $R_0>1$.*

Proof. If $n=1$, then system (2.1) become a single-group model. Define a Lyapunov functional

$$\begin{aligned}V=&S-S^*-S^*\ln\frac{S}{S^*}+I-I^*-I^*\ln\frac{I}{I^*}+\frac{\phi S^* r(B^*)}{nh(I^*)}\left(B-B^*-B^*\ln\frac{B}{B^*}\right)\\&-\beta S^*\int_0^\infty f(s)\int_{-s}^0 g(I^*)\varphi\Big(\frac{g(I_t(u))}{g(I^*)}\Big)duds\\&-\frac{\phi S^* r(B^*)}{a}\int_0^\infty f(s)\int_{-s}^0 h(I^*)\varphi\Big(\frac{h(I_t(u))}{h(I^*)}\Big)duds,\end{aligned}$$

where $a=\int_0^\infty f(s)ds$. The function $\varphi(x)=1-x+\ln x$ is non-positive for $x\geq 0$, and $\varphi(x)=0$ if and only if $x=1$. Denote $I(t-u)=I_t(u)$. Then the derivative of V along a positive solution of system (2.1) is

$$\begin{aligned}
\frac{dV}{dt}=&\left(1-\frac{S^*}{S}\right)\left(A-\mu S-\beta S\int_0^\infty f(s)g(I(t-s))ds-\phi Sr(B)\right)\\
&+\left(1-\frac{I^*}{I}\right)\left(\beta S\int_0^\infty f(s)g(I(t-s))ds+\phi Sr(B)-(\mu+\alpha)I\right)\\
&+\left(1-\frac{B^*}{B}\right)\left(\int_0^\infty f(s)h(I(t-s))ds-\delta B\right)\\
=&\mu S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\beta S^*g(I^*)a\left(\varphi\left(\frac{S^*}{S}\right)+\varphi\left(\frac{Sg(I)I^*}{S^*g(I^*)I}\right)+\varphi\left(\frac{g(I^*)I}{g(I)I^*}\right)\right)\\
&+\beta S^*g(I^*)a\left(\frac{g(I)}{g(I^*)}-1\right)\left(1-\frac{g(I^*)I}{g(I)I^*}\right)+\beta S^*g(I^*)\int_0^\infty f(s)\left(\varphi\left(\frac{I^*Sg(I_t(s))}{IS^*g(I)}\right)\right.\\
&\left.-\varphi\left(\frac{I^*Sg(I)}{IS^*g(I^*)}\right)\right)ds+\phi S^*r(B^*)\left(\varphi\left(\frac{S^*}{S}\right)+\varphi\left(\frac{I^*Sr(B)}{IS^*r(B^*)}\right)+\varphi\left(\frac{r(B^*)B}{r(B)B^*}\right)\right.\\
&\left.+\varphi\left(\frac{Ih(I^*)}{I^*h(I)}\right)\right)+\frac{\phi S^*r(B^*)}{a}\int_0^\infty f(s)\varphi\left(\frac{B^*h(I_t(s))}{Bh(I^*)}\right)ds\\
&+\phi S^*r(B^*)\left(\left(\frac{r(B)}{r(B^*)}-1\right)\left(1-\frac{r(B^*)B}{r(B)B^*}\right)+\left(\frac{h(I)}{h(I^*)}-1\right)\left(1-\frac{h(I^*)I}{h(I)I^*}\right)\right).
\end{aligned}$$

Therefore, $V'\leq 0$ for $(S,I,B)\in\mathring{\Gamma}$, and $V'=0$ if and only if $(S,I,B)=(S^*,I^*,B^*)$. By LaSalle invariance principle [16], endemic equilibrium P^* is globally asymptotically stable in $\mathring{\Gamma}$ if $R_0>1$. If $n\geq 2$, then system (2.1) is a multi-group model. In order to prove the global asymptotic stability of the endemic equilibrium, we establish a appropriate Lyapunov functional for the multi-group model. And the approach of constructing the Lyapunov functional is mainly referred to literature [8, 9, 19, 23]. Define a Lyapunov functional

$$\begin{aligned}
U=&\sum_{i=1}^n d_i\left(S_i-S_i^*-S_i^*\ln\frac{S_i}{S_i^*}+I_i-I_i^*-I_i^*\ln\frac{I_i}{I_i^*}+\frac{\phi_iS_i^*r(B^*)}{\sum_{j=1}^n k_jh_j(I_j^*)a_j}\left(B-B^*-B^*\ln\frac{B}{B^*}\right)\right)\\
&-\sum_{i,j=1}^n d_i\beta_{ij}S_i^*\int_0^\infty f_j(s)\int_{-s}^0 g_j(I_j^*)\varphi\left(\frac{g_j(I_{jt}(u))}{g_j(I_j^*)}\right)duds\\
&-\sum_{i=1}^n d_i\frac{\phi_iS_i^*r(B^*)}{\sum_{j=1}^n k_ja_j}\sum_{j=1}^n k_j\int_0^\infty f_j(s)\int_{-s}^0 h_j(I_j^*)\varphi\left(\frac{h_j(I_{jt}(u))}{h_j(I_j^*)}\right)duds.
\end{aligned}$$

The derivative of V along a positive solution of system (2.1) is

$$
\begin{aligned}
&\frac{dU}{dt} = \\
&\sum_{i=1}^{n} d_i \Bigg(\left(1-\frac{S_i^*}{S_i}\right)\left(A_i-\mu_i S_i-\sum_{j=1}^{n}\beta_{ij}S_i\int_0^\infty f_j(s)g_j(I_j(t-s))ds-\phi_i S_i r(B)\right) \\
&+\left(1-\frac{I_i^*}{I_i}\right)\left(\sum_{j=1}^{n}\beta_{ij}S_i\int_0^\infty f_j(s)g_j(I_j(t-s))ds+\phi_i S_i r(B)-(\mu_i+\alpha_i)I_i\right) \\
&+\frac{\phi_i S_i^* r(B^*)}{\sum_{j=1}^{n}k_j h_j(I_j^*)a_j}\left(1-\frac{B^*}{B}\right)\left(\sum_{j=1}^{n}k_j\int_0^\infty f_j(s)h_j(I_j(t-s))ds-\delta B\right)\Bigg) \\
&+\sum_{i,j=1}^{n} d_i\beta_{ij}S_i^* g_j(I_j^*)\int_0^\infty f_j(s)\left(\frac{g(I_j)}{g_j(I_j^*)}-\frac{g_j(I_j(t-s))}{g_j(I_j^*)}+\ln\frac{g_j(I_j(t-s))}{g_j(I_j)}\right)ds \\
&+\sum_{i=1}^{n} d_i\frac{\phi_i S_i^* r(B^*)}{\sum_{j=1}^{n}k_j a_j}\sum_{j=1}^{n}k_j\int_0^\infty f_j(s)\left(\frac{h_j(I_j)}{h_j(I_j^*)}-\frac{h_j(I_j(t-s))}{h_j(I_j^*)}+\ln\frac{h_j(I_j(t-s))}{h_j(I_j)}\right)ds \\
=&\sum_{i=1}^{n} d_i\Bigg(\mu_i S_i^*\left(2-\frac{S_i^*}{S_i}-\frac{S_i}{S_i^*}\right)+\sum_{j=1}^{n}\beta_{ij}S_i^* g_j(I_j^*)a_j\left(\varphi\left(\frac{I_i^* S_i g_j(I_j)}{I_i S_i^* g_j(I_j^*)}\right)+\varphi\left(\frac{I_j g_j(I_j^*)}{I_j^* g_j(I_j)}\right)+\varphi\left(\frac{S_i^*}{S_i}\right)\right) \\
&+\sum_{j=1}^{n}\beta_{ij}S_i^* g_j(I_j^*)a_j\left(\frac{g_j(I_j)}{g_j(I_j^*)}-1\right)\left(1-\frac{g_j(I_j^*)I_j}{g_j(I_j)I_j^*}\right)\Bigg)+\sum_{i,j=1}^{n} d_i\beta_{ij}S_i^* g_j(I_j^*)a_j\left(\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}-\ln\frac{I_j I_i^*}{I_j^* I_i}\right) \\
&+\sum_{i,j=1}^{n} d_i\beta_{i,j}S_i^* g_j(I_j^*)\int_0^\infty f_j(s)\left(\varphi\left(\frac{g_j(I_j(t-s))I_i^* S_i}{g_j(I_j^*)I_i S_i^*}\right)\right)ds \\
&+\sum_{i=1}^{n} d_i\frac{\phi_i S_i^* r(B^*)}{\sum_{j=1}^{n}k_j a_j}\sum_{j=1}^{n}k_j\int_0^\infty f_j(s)\left(\varphi\left(\frac{B^* h_j(I_j(t-s))}{B h_j(I_j^*)}\right)\right)ds \\
&+\sum_{i,j=1}^{n} d_i\phi_i S_i^* r(B^*)\left(\varphi\left(\frac{S_i^*}{S_i}\right)+\varphi\left(\frac{I_i^* S_i r(B)}{I_i S_i^* r(B^*)}\right)+\varphi\left(\frac{r(B^*)B}{r(B)B^*}\right)+\varphi\left(\frac{I_j h_j(I_j^*)}{I_j^* h_j(I_j)}\right)\right) \\
&+\sum_{i,j=1}^{n} d_i\phi_i S_i^* r(B^*)\left(\frac{r(B)}{r(B^*}-1\right)\left(1-\frac{r(B^*)B}{r(B)B^*}\right)+\sum_{i,j=1}^{n} d_i\phi_i S_i^* r(B^*)\left(\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}-\ln\frac{I_j I_i^*}{I_j^* I_i}\right) \\
\le&\sum_{i,j=1}^{n} d_i\left(\phi_i S_i^* r(B^*)+\beta_{ij}S_i^* g_j(I_j^*)a_j\right)\left(\frac{I_j}{I_j^*}-\frac{I_i}{I_i^*}-\ln\frac{I_j I_i^*}{I_j^* I_i}\right).
\end{aligned}
$$

Denote $p_{ij}=\phi_i S_i^* r(B^*)+\beta_{ij}S_i^* g_j(I_j^*)a_j$. It is easy to see matrix $(p_{ij})_{1\le i,j\le n}$ is irreducible. Using the Theorem 3.1 and Corollary 3.3 of [19], we can prove that $U'\le 0$ for $(S_i,I_i,B)\in\mathring{\Gamma}$, and $U'=0$ if and only if $(S_i,I_i,B)=(S_i^*,I_i^*,B^*)$, $i=1,2,\ldots,n$. By LaSalle invariance principle [16], the endemic equilibrium P^* is globally asymptotically stable in $\mathring{\Gamma}$. □

4. Conclusion

In the background of diseases (malaria, cholera, brucellosis, etc) with indirect transmission route, this paper established a multi-group epidemic model with distributed delays and indirect transmission. We obtained the basic reproduction number R_0 of system (2.1) and proved that the disease-free equilibrium is globally asymptotically stable if $R_0 \leq 1$. Using the method of Lyapunov functionals and graph-theoretical approach [8, 19], we also proved the global asymptotic stability of the endemic equilibrium when $R_0 > 1$.

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