J. Nonlinear Funct. Anal. 2023 (2023) 14 https://doi.org/10.23952/jnfa.2023.14

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

LINEAR CONVERGENCE OF AN ITERATIVE ALGORITHM FOR SOLVING THE SPLIT EQUALITY MIXED EQUILIBRIUM PROBLEM

YUJING WU[1], SHIJIE SUN[2], TONGXIN XU[3], MEIYING WANG[4], LUOYI SHI[5,*]

[1]Tianjin Vocational Institute, Tianjin 300410, China
[2]Faculty of Medical Device, Shenyang Pharmaceutical University, Shenyang 110016, China
[3]School of Mathematics and Statistics, Xidian University, Xi'an 710126, China
[4]School of Mathematical Sciences, Tiangong University, Tianjin 300387, China
[5]School of Software, Tiangong University, Tianjin 300387, China

Abstract. This paper investigates the linear convergence of a projection algorithm for solving the split equality mixed equilibrium problem (SEMEP). We introduce the notion of bounded linear regularity property for the SEMEP and construct several sufficient conditions to prove its linear convergence. Furthermore, the result of the linear convergence of the SEMEP is applied to split equality equilibrium problems, split equality convex minimization problems, split equality mixed variational inequality problems, and split equality variational inequality problems. Finally, numerical results are provided to verify the effectiveness of our proposed algorithm.

Keywords. Bounded linearly regularity property; Intensity-modulated radiation therapy; Linear convergence; Numerical results; Split equality mixed equilibrium problem.

1. Introduction

Let H_1, H_2, and H_3 be three real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. $domA$ and $F(A)$ are borrowed to denote the domain and the fixed point set of the mapping A, respectively.

In 2013, Moudafi [8] first proposed the following split equality problem (SEP for short), which can be represented as

$$\text{finding } x \in C,\ y \in Q \text{ such that } Ax = By,$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators.

This kind of problem has attracted the attention of numerous authors because it has wide real applications, such as intensity-modulated radiation therapy [1]. In order to solve the split equality problem, various algorithms were introduced; see, e.g., [13, 17, 19]). One of the

*Corresponding author.
E-mail address: shiluoyi@tiangong.edu.cn (L. Shi).
Received October 31, 2022; Accepted March 17, 2023.

most significant algorithms is the alternating CQ-algorithm (ACQA), which was proposed by Moudafi [8]. The iterative form of the ACQA is as below:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases}$$

where $\{\gamma_k\}$ is a nondecreasing sequence, P_C and P_Q are metric projections on C and Q from H_1 and H_2, A^* and B^* represent dual mappings of A and B, respectively. Moudafi proved that this algorithm converges weakly to a solution of the SEP.

In order to obtain strong convergence results, Shi et al. [13] introduced a modification of Moudafi's *ACQA* algorithm:

$$\begin{cases} x_{k+1} = P_C\{(1-\mu_k)[x_k - \gamma A^*(Ax_k - By_k)]\}, \; n \geq 0, \\ y_{k+1} = P_Q\{(1-\mu_k)[y_k + \gamma B^*(Ax_k - By_k)]\}, \; n \geq 0, \end{cases}$$

where $\{\mu_k\}$ is a positive sequence in $(0,1)$. They proved that the algorithm converges strongly to a solution of the SEP.

In 2018, Shi et al. [14] proposed a varying step-size gradient-projection algorithm to solve SEP and obtained linear convergence results. For the results on linear convergence, we refer to [15, 16, 20] and the references therein.

Let $F : C \times C \to R$ be a nonlinear bifunction. The equilibrium problem (EP for short) is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0, \; \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(F)$. Let $T : C \to C$ be a mapping and set $F(x,y) = \langle Tx, y-x \rangle$, for all $x, y \in C$. One has $x^* \in EP(F)$ if and only if $x^* \in C$ is a solution of the variational inequality $\langle Tx, y-x \rangle \geq 0$ for all $y \in C$. The equilibrium problem was extensively investigated numerically in Hilbert spaces and Banach spaces; see, e.g., [4, 5, 11, 12, 18] and the references therein.

Recall that the mixed equilibrium problem (MEP for short) is to

$$\text{find } x^* \in C \text{ such that } F(x^*, y) + \phi(y) - \phi(x^*) \geq 0, \; \forall y \in C,$$

where $\phi : C \to R \cup \{+\infty\}$ is a function. We use $MEP(F, \; \phi)$ to denote the set of solutions of the MEP. The MEP includes several important problems arising in physics, engineering, transportation, economics, structural analysis, and network.

In [7], Moudafi introduced the following split equilibrium problems (SE for short). Let $F : C \times C \to R$ and $J : Q \times Q \to R$ be nonlinear bifunctions, and let $A : H_1 \to H_2$ be a bounded linear operator. The SE is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \forall x \in C,$$

and

$$y^* = Ax^* \in Q \text{ solves } J(y^*, y) \geq 0, \; \forall y \in Q.$$

Remark 1.1. (1) If $\phi = 0$, then the MEP reduces to the EP.

(2) If $F = 0$, then the MEP reduces to the following convex minimization problem (CMP for short): find $x^* \in C$ such that

$$\phi(y) \geq \phi(x^*), \; \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $CMP(\phi)$.

In 2015, Ma et al. [9] introduced the following split equality mixed equilibrium problems (SEMEP for short).

Definition 1.2. [9] Let C and Q be nonempty, closed, and convex subsets of H_1 and H_2, respectively. Let $F : C \times C \to R$ and $J : Q \times Q \to R$ be nonlinear bifunctions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap dom\phi \neq \emptyset$ and $Q \cap dom\varphi \neq \emptyset$. Then the split equality mixed equilibrium problem (SEMEP for short) is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} F(x^*, x) + \phi(x) - \phi(x^*) \geq 0, \forall x \in C, \\ J(y^*, y) + \varphi(y) - \varphi(y^*) \geq 0, \forall y \in Q, \\ Ax^* = By^*. \end{cases} \tag{1.3}$$

The set of solutions of (1.3) is denoted by $SEMEP(F, J, \phi, \varphi)$.

Remark 1.3. (1) In (1.3), if $\phi = 0$ and $\varphi = 0$, then the split equality mixed equilibrium problem reduces to the split equality equilibrium problem.

(2) If $F = 0$ and $J = 0$, then the split equality mixed equilibrium problem reduces to the following split equality convex minimization problem: find $x^* \in C$ and $y^* \in Q$, such that

$$\begin{cases} \phi(x) \geq \phi(x^*), \forall x \in C, \\ \varphi(y) \geq \varphi(y^*), \forall y \in Q, \\ Ax^* = By^*. \end{cases} \tag{1.4}$$

The set of solutions of (1.4) is denoted by $SECMP(\phi, \varphi)$.

(3) If $F = 0$, $J = 0$, $B = I$, and $y^* = Ax^*$, then the split equality mixed equilibrium problem reduces to the following split convex minimization problem: find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} \phi(x) \geq \phi(x^*), \forall x \in C, \\ y^* = Ax^* \in Q, \ \varphi(y) \geq \varphi(y^*), \forall y \in Q, \\ Ax^* = By^*. \end{cases} \tag{1.5}$$

The set of solutions of (1.5) is denoted by $SCMP(\phi, \varphi)$.

According to the definition of the MEP, the SEMEP can be formulated as

$$\text{finding } x^* \in D, \ y^* \in E \text{ such that } Ax^* = By^*,$$

where $D = MEP(F, \phi)$ and $E = MEP(J, \varphi)$.

Let $S = D \times E \subseteq H_1 \times H_2 := H$ and $G = [A, -B] : H \to H_3$, then original problem (1.3) can now be reformulated as

$$\text{finding } z^* = (x^*, y^*) \text{ such that } Gz^* = 0.$$

The strong and weak convergence algorithms of the split equality mixed equilibrium problem have been analyzed by Ma in [9], but the convergence rate of this problem has not been studied. Therefore, we, in this paper, introduce the notion of bounded linear regularity property for the SEMEP and provide some sufficient conditions to guarantee this regularity property, and the linear convergence of the proposed algorithm is finally proved.

The rest of this paper is organized as follows. In Section 2, we recall some definitions and lemmas that are useful for the convergence analysis in the sequel, and introduce the notion of bounded linear regularity property of the SEMEP. Some conditions that guarantee this property are provided. In Section 3, under the assumption of the bounded linear regularity, we research the linear convergence of the proposed algorithm. In Section 4, in terms of applications, the results are applied to the split equality equilibrium problem, the split equality convex minimization problem, the split equality mixed variational inequality problem, and the split equality variational inequality problem. In Section 5, the lat section, the effectiveness of the algorithm is verified by numerical experiments.

2. Preliminaries

In this section, we introduce some notations and results that can be used in the sequel for a better understanding of this paper.

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Recall that an operator T on H is nonexpansive if, for each x and y in H, $\|Tx-Ty\|\leq\|x-y\|$. Recall that an operator T on H is firmly nonexpansive if, for each x and y in H, $\langle x-y,Tx-Ty\rangle\geq\|Tx-Ty\|^2$.

The definition of the linear convergence is as follows.

Definition 2.1. A sequence $\{x_n\}$ in H is said to converge linearly to its limit x^* (with rate $\sigma\in[0,1)$) if there exists $\omega>0$ and a positive integer N such that

$$\|x_n-x^*\|\leq\omega\sigma^n \text{ for all } n\geq N.$$

Let I denote the identity operator on H. For a set $S\subseteq H$, we denote the closure, interior, relative interior, and conical hull of S by clS, $intS$, riS, and $coneS$, respectively. For $x\in H$, we use B and $\bar{B}$ to denote the unit open ball and unit closed ball with centre at the origin, respectively. For a point x and a set $S\subseteq H$, the classical metric projection of x onto S and the distance of x from S, denoted by $P_S(x)$ and $d_S(x)$, respectively, and defined by

$$P_S(x):=\arg\min\{\|x-y\|:y\in S\} \text{ and } d_S(x):=\inf\{\|x-y\|:y\in S\}.$$

The following proposition is useful for our convergence analysis.

Proposition 2.2. [2] *Let $C\subseteq H$ be a convex, closed, and nonempty set. For any $x,y\in H$ and $z\in C$, the following assertions hold:*
(1) $\langle x-P_Cx,z-P_Cx\rangle\leq 0$;
(2) $\|P_Cx-P_Cy\|\leq\|x-y\|$;
(3) $\|P_Cx-P_Cy\|^2\leq\langle P_Cx-P_Cy,x-y\rangle$;
(4) $\|P_Cx-z\|^2\leq\|x-z\|^2-\|P_Cx-x\|^2$;
(5) $\langle(I-P_C)x-(I-P_C)y,x-y\rangle\geq\|(I-P_C)x-(I-P_C)y\|^2$.

Let an operator $G:H\to H_3$ be bounded and linear. We utilize $kerG=\{x\in H:Gx=0\}$ to denote the kernel of G. The orthogonal complement of $kerG$ is represented by $(kerG)^\perp=\{y\in H:\langle x,y\rangle=0 \text{ for all } x\in kerG\}$. It is known that $kerG$ and $(kerG)^\perp$ are closed subspaces of H. In this paper, we denote the set of solutions of the SEMEP by Γ, which is defined by $\Gamma:=S\cap kerG=\{z\in S,Gz=0\}$. We assume that the SEMEP is consistent. Γ is a closed, convex, and nonempty set.

For solving the split equality mixed equilibrium problem, let us give the following assumptions for the bifunction F, ϕ, and the set C. We use the following conditions of bifunctions (F and J) in the sequel.
(A1) $F(x,x) = 0$, $\forall x \in C$;
(A2) $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;
(A3) $\forall x,y,z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) $\forall x \in C$, $y \mapsto F(x,y)$ is a lower semicontinuous convex function.
From conditions (A1)-(A4), we have the following lemma.

Lemma 2.3. [10] *Let C be a nonempty, closed, and convex subset of H. Let F be a bifunction from $C \times C$ to R, and let $\phi : C \to R \cup \{+\infty\}$ be a proper, lower semi-continuous, and convex function such that $C \cap dom\varphi \neq \emptyset$. For $r > 0$ and $x \in H$, define a mapping $T_r^F : H_1 \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z,y) + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\},$$

for all $x \in H$. Then, the following conclusions hold:
(1) for each $x \in H$, $T_r^F(x) \neq \emptyset$;
(2) T_r^F is single-valued;
(3) for any $x,y \in H_1$, $\|T_r^F(x) - T_r^F(y)\|^2 \leq \langle T_r^F(x) - T_r^F(y), x - y\rangle$, that is, T_r^F is firmly nonexpansive. (4) $MEP(F,\phi) = F(T_r^F)$;
(5) $MEP(F,\phi)$ is convex and closed.

In order to obtain the linear convergence property of projection-based algorithms for solving convex feasibility problems, Zhao et al. [20] introduced the following linear regularity for a family of closed convex subsets of a real Hilbert space.

Definition 2.4. [20] Let $\{E_i\}_{i\in I}$ be a family of closed convex subsets of a real Hilbert space H and $E = \cap_{i\in I} E_i \neq \emptyset$. The family $\{E_i\}_{i\in I}$ is said to be bounded linearly regular if, for each $r > 0$, there exists a constant $\gamma_r > 0$ such that $d_E(w) \leq \gamma_r \sup\{d_{E_i}(w) : i \in I\}$ for all $w \in rB$.

Next, we introduce the concept of bounded linear regularity property of the SEMEP.

Definition 2.5. The SEMEP is said to satisfy the bounded linear regularity property if, for each $r > 0$, there exists $\gamma_r > 0$ such that $\gamma_r d_\Gamma(x) \leq \|Gx\|$ for all $x \in S \cap r\bar{B}$.

In order to obtain the sufficient condition of bounded linear regularity of the SEMEP, we also need the following Lemma.

Lemma 2.6. [3] *Let H be a real Hilbert space and let G be a bounded linear operator. Then G is injective and has closed range iff G is bounded below, i.e., there exists a positive constant γ such that $\|Gz\| \geq \gamma\|z\|$ for all $z \in H$.*

By Lemma 2.6, the sufficient condition of bounded linear regularity of the SEMEP is given below.

Lemma 2.7. *Let $\{S, kerG\}$ be bounded linearly regular and G has closed range. Then the SEMEP satisfies the bounded linear regularity property.*

Proof. Since $\{S, \ker G\}$ has a bounded linearly regular intersection, one finds that, for any $t > 0$, there exists $\gamma_t > 0$ such that

$$d_\Gamma(x) = d_{S \cap \ker G}(x) \le \gamma_t \max\{d_S(x), d_{\ker G}(x)\},\ \forall x \in t\bar{B},$$

that is,

$$d_\Gamma(x) \le \gamma_t d_{\ker G}(x), \forall x \in S \cap t\bar{B}. \tag{2.1}$$

Since G is restricted to $(\ker G)^\perp$, which is injective and its range is closed, by Lemma 2.6, we know that there exists $\mu > 0$ such that $\|Gx_1\| \ge \mu \|x_1\|$, for all $x_1 \in (\ker G)^\perp$. Hence,

$$d_{\ker G}(x) \le \frac{1}{\mu}\|Gx\|, \text{ for all } x \in H. \tag{2.2}$$

Combining (2.1) and (2.2), we have $d_\Gamma(x) \le \frac{\gamma_t}{\mu}\|Gx\|,\ \forall\, x \in S \cap t\bar{B}$. Then, the proof is complete. □

For completing the linear convergence analysis of the proposed algorithm, the following definition is also an essential tool.

Definition 2.8. [2] Let C be a nonempty subset of H, and let $\{x_k\}$ be a sequence in H. $\{x_k\}$ is called Fejér monotone with respect to C if $\|x_{k+1} - z\| \le \|x_k - z\|$ for all $z \in C$. Clearly, a Fejér monotone sequence $\{x_k\}$ is bounded and $\lim_{k\to\infty} \|x_k - z\|$ exists.

The following lemma provides sufficient conditions for bounded linear regularity property for two closed convex subsets of H.

Lemma 2.9. [20] *Let E and F be closed convex subsets of H. Then $\{E, F\}$ is bounded linearly regular provided that at least one of the following conditions holds:*
(a) $riE \cap F \ne \emptyset$ and F is a polyhedron.
(b) $riE \cap riF \ne \emptyset$ and E is finite dimensional.
(c) $riE \cap riF \ne \emptyset$ and E is finite codimensional.

Note that $kerG$ is a subspace of H and $rikerG = kerG$. Furthermore, it is well known that if $kerG$ is finite dimensional or finite codimensional, then the range of G is closed. By the Lemma 2.9, we have the following corollary which establishes sufficient conditions for bounded linear regularity property for the SEMEP.

Lemma 2.10. [14] *The SEMEP satisfies the bounded linear regularity property if one of the following conditions holds:*
(1) C and Q are polyhedrons, and G has closed range.
(2) $riS \cap kerG \ne \emptyset$, $kerG$ is finite dimensional.
(3) $riS \cap kerG \ne \emptyset$, $kerG$ is finite codimensional.
(4) $riS \cap kerG \ne \emptyset$, G has closed range and $S = C \times Q$ is finite dimensional.
(5) $riS \cap kerG \ne \emptyset$, G has closed range and $S = C \times Q$ is finite codimensional.

3. Main Results

In this section, the simultaneous iterative algorithm of the SEMEP is proposed and we prove its linear convergence.

Theorem 3.1. *Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty, convex, and closed subsets of H_1 and H_2, respectively. Assume that $F : C \times C \to R$ and $J : Q \times Q \to R$ are bifunctions, and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper, lower semi-continuous, and convex functions such that $C \cap dom\phi \neq \emptyset$ and $Q \cap dom\varphi \neq \emptyset$. Let $D = MEP(F, \ \phi)$ and $E = MEP(J, \ \varphi)$, let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be bounded linear operators. Let A^* and B^* be the adjoint operators of A and B. The iteration scheme $\{z_n\}$ is defined as follows: For each initial point $z_0 = (x_0, y_0) \in S$,*

$$\begin{cases} U(w_n, w) + \psi(w) - \psi(w_n) + \frac{1}{r_n}\langle w - w_n, w_n - z_n\rangle \geq 0, \forall w \in V; \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)P_S(w_n - \rho_n G^* G w_n), \end{cases}$$

or component-wise

$$\begin{cases} F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \ \forall u \in C; \\ J(v_n, v) + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, \ \forall v \in Q; \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)P_D(u_n - \rho_n A^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n)P_E(v_n + \rho_n B^*(Au_n - Bv_n)), \end{cases} \tag{3.1}$$

where $S = D \times E$, $V = C \times Q$, $U = F \times J$, $\psi = \phi \times \varphi$, $G = [A, -B]$, $z_{n+1} = (x_{n+1}, y_{n+1})$, $w_n = (u_n, v_n)$, $w = (u, v)$, $\alpha_n \in [0, 1)$, $r_n > 0$, $\rho_n > 0$, $\lim_{n\to\infty} \rho_n = 0$, and $\sum_{n=1}^{\infty} \rho_n = \infty$. Assume that the SEMEP satisfies the bounded linear regularity property. Then the sequence $\{z_n\}$ generated by the iteration (3.1) with $n \in [L, \infty)$ and integer $L > 0$ converges to a solution z^* of the SEMEP such that $\|z_n - z^*\| \leq \delta\sigma^n$, for $\delta \geq 1$ and $0 < \sigma < 1$, provided that one of the following conditions holds:
(a) $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < \frac{2}{\|G\|^2}$,
(b) $\rho_n = \begin{cases} 0, \ z_n \in \Gamma, \\ \frac{\gamma_n \|Gz_n\|^2}{\|G^* G z_n\|^2}, \ z_n \notin \Gamma, \end{cases}$ *, and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2$. Consequently, $\{z_n\}$ converges to $\bar{z}$ linearly in the case that (a) or (b) is assumed.*

Proof. Without loss of generality, one assumes that $z_n \notin \Gamma$, $\forall n \geq 0$. Otherwise, iteration (3.1) terminates in finite number of iteration, and then the conclusions follow immediately.

For the first assertion, we first prove that sequence $\{z_n\}$ is Fejér monotone with respect to Γ. Let $z^* \in \Gamma$. By the definition of Γ, i.e., $z^* \in S$ and $Gz^* = 0$, we have

$$\begin{aligned} \|z_{n+1} - z^*\|^2 &= \alpha_n^2 \|z_n - z^*\|^2 + 2\alpha_n(1-\alpha_n)\langle z_n - z^*, P_S(w_n - \rho_n G^* G w_n) - z^*\rangle \\ &\quad + (1-\alpha_n)^2 \|P_S(w_n - \rho_n G^* G w_n) - z^*\|^2 \\ &\leq \alpha_n^2 \|z_n - z^*\|^2 + \alpha_n(1-\alpha_n)(\|z_n - z^*\|^2 + \|P_S(w_n - \rho_n G^* G w_n) - z^*\|^2) \\ &\quad + (1-\alpha_n)^2 \|P_S(w_n - \rho_n G^* G w_n) - z^*\|^2 \\ &= \alpha_n \|z_n - z^*\|^2 + (1-\alpha_n)\|P_S(w_n - \rho_n G^* G w_n) - z^*\|^2. \end{aligned} \tag{3.2}$$

We obtain the following formulas by using Proposition 2.2, the definition of the adjoint operator, and $Gz^*=0$,

$$\begin{aligned}\|P_S(w_n-\rho_n G^*Gw_n)-z^*\|^2 &\le \|w_n-\rho_n G^*Gw_n-z^*\|^2\\ &=\|w_n-z^*\|^2+\rho_n^2\|G^*Gw_n\|^2-2\rho_n\langle w_n-z^*,G^*Gw_n\rangle\\ &=\|w_n-z^*\|^2+\rho_n^2\|G^*Gw_n\|^2-2\rho_n\langle Gw_n-Gz^*,Gw_n\rangle\\ &=\|w_n-z^*\|^2+\rho_n^2\|G^*Gw_n\|^2-2\rho_n\|Gw_n\|^2. \qquad (3.3)\end{aligned}$$

Let $T_{r_n}^U=T_{r_n}^F\times T_{r_n}^J:H_1\times H_2\to V$, i.e.,

$$T_{r_n}^U(a)=\{c\in V:U(c,b)+\psi(b)-\psi(c)+\frac{1}{r_n}\langle b-c,c-a\rangle\ge 0,\forall b\in V\}.$$

According to iteration (3.1), we have

$$T_{r_n}^U(z_n)=\{w_n\in V:U(w_n,w)+\psi(w)-\psi(w_n)+\frac{1}{r_n}\langle w-w_n,w_n-z_n\rangle\ge 0,\forall w\in V\},$$

which yields that $w_n=T_{r_n}^U(z_n)$. Since $z_n\in S=D\times E=MEP(F,\,\phi)\times MEP(J,\,\varphi)=MEP(U,\,\psi)$ for all $n\ge 0$ (see (5) in Lemma 2.3. By (4) in Lemma 2.3, $z_n\in F(T_{r_n}^U)$ (i.e., $T_{r_n}^U(z_n)=z_n$). It follows that

$$w_n=T_{r_n}^U(z_n)=z_n. \qquad (3.4)$$

Substituting inequality (3.3) and (3.4) into (3.2), we obtain

$$\|z_{n+1}-z^*\|^2\le\|z_n-z^*\|^2-(1-\alpha_n)\rho_n(2-\rho_n\frac{\|G^*Gz_n\|^2}{\|Gz_n\|^2})\|Gz_n\|^2. \qquad (3.5)$$

Note that $\frac{\|G^*Gz_n\|^2}{\|Gz_n\|^2}\le\|G\|^2$ holds. According to the assumptions of (a) and (b), it follows from (3.5) that $\|z_{n+1}-z^*\|^2\le\|z_n-z^*\|^2$. That is, $\{z_n\}$ is Fejér monotone with respect to Γ. Hence, $\{z_n\}$ is bounded and $\lim_{n\to\infty}\|z_n-z^*\|$ exists.

Then, we prove that $\{z_n\}$ converges linearly to a solution of SEMEP (1.3). Since SEMEP (1.3) satisfies the bounded linear regularity property and $z_n\in S$ for all $n\ge 0$, there exists $\beta>0$ such that $\beta d_\Gamma(z_n)\le\|Gz_n\|$. By inequality (3.5), it follows that

$$\|z_{n+1}-z^*\|^2\le\|z_n-z^*\|^2-(1-\alpha_n)\rho_n(2-\rho_n\frac{\|G^*Gz_n\|^2}{\|Gz_n\|^2})\beta^2 d_\Gamma^2(z_n).$$

Taking $z^*=\{z\in\Gamma|\min\|z_n-z\|\}$, one has

$$d_\Gamma^2(z_{n+1})\le\|z_{n+1}-z^*\|^2\le d_\Gamma^2(z_n)-(1-\alpha_n)\rho_n(2-\rho_n\frac{\|G^*Gz_n\|^2}{\|Gz_n\|^2})\beta^2 d_\Gamma^2(z_n). \qquad (3.6)$$

Note that if (a) or (b) holds, then $\liminf_{n\to\infty}(2-\rho_n\frac{\|G^*Gz_n\|^2}{\|Gz_n\|^2})>0$, Hence, there exists integer $M>0$ such that

$$\omega:=\inf_{n\ge M}(1-\alpha_n)\beta^2(2-\rho_n\frac{\|G^*Gz_n\|^2}{\|Gz_n\|^2})>0,$$

Then inequality (3.6) reduces to

$$d_\Gamma^2(z_{n+1})\le d_\Gamma^2(z_n)-\rho_n\omega d_\Gamma^2(z_n)=(1-\rho_n\omega)d_\Gamma^2(z_n),\ \forall n\ge M.$$

This implies that

$$d_\Gamma^2(z_{n+1}) \le d_\Gamma^2(z_M)\Pi_{i=M}^{n}(1-\rho_i\omega),\ \forall n \ge M. \tag{3.7}$$

Note that $\{z_n\}$ is Fejér monotone with respect to Γ, i.e., for all $z \in \Gamma$ and for all $m > k$, $\|z_m - z\| \le \|z_k - z\|$. It follows that

$$\|z_m - z_k\| \le \|z_m - P_\Gamma(z_k)\| + \|z_k - P_\Gamma(z_k)\| \le 2\|z_k - P_\Gamma(z_k)\| = 2d_\Gamma(z_k). \tag{3.8}$$

By (3.7) and (3.8), we have

$$\|z_m - z_{k+1}\| \le 2d_\Gamma(z_{k+1}) \le 2d_\Gamma(z_M)\Pi_{i=M}^{k}\sqrt{1-\rho_i\omega},\ \forall m > k \ge M.$$

Let $p := e^{-\frac{\omega}{2}} \in (0,1)$. Since $\ln(1-t) \le -t, \forall t \in [0,1)$, then

$$\Pi_{i=M}^{k}\sqrt{1-\rho_i\omega} = \exp\left\{\frac{1}{2}\Sigma_{i=M}^{k} ln(1-\rho_i\omega)\right\} \le \exp\left\{-\frac{\omega}{2}\Sigma_{i=M}^{k}\rho_i\right\} = p^{\Sigma_{i=M}^{k}\rho_i}. \tag{3.9}$$

Therefore,$\|z_m - z_{k+1}\| \le 2d_\Gamma(z_M)p^{\Sigma_{i=M}^{k}\rho_i}$ for all $m > k \ge M$. Since $\lim_{n\to\infty}\rho_n = 0$ and $\Sigma_{i=1}^{\infty}\rho_n = \infty$, it follows that $\{z_n\}$ is a Cauchy sequence and converges to $\bar{z}$, which satisfies $\|z_{k+1} - \bar{z}\| \le 2d_\Gamma(z_M)p^{\Sigma_{i=M}^{k}\rho_i}$, $k \ge M$. Take $\hat{z} \in \Gamma$ such that $\lim_{n\to\infty} d_\Gamma(z_{n+1}) = \lim_{n\to\infty}\|z_{n+1} - \hat{z}\|$. From (3.7), we have

$$0 \le \|\bar{z} - \hat{z}\| = \lim_{n\to\infty}\|z_{n+1} - \hat{z}\| = \lim_{n\to\infty} d_\Gamma(z_{n+1}) \le \lim_{n\to\infty} d_\Gamma(z_M)\Pi_{i=M}^{n}\sqrt{1-\rho_i\omega} = 0,\ \forall n \ge M,$$

so $\bar{z} = \hat{z} \in \Gamma$. Let

$$\lambda = 2\max\left\{d_\Gamma(z_M)p^{-\Sigma_{i=1}^{M-1}\rho_i},\ \max\left\{\|z_i - z^*\|p^{-\Sigma_{j=1}^{i}\rho_j - \rho_M}, i = 1,2,\cdots,M.\right\}\right\}.$$

It follows from inequations (3.7) and (3.9) that $\|z_n - \bar{z}\| \le \lambda p^{\Sigma_{i=1}^{n}\rho_i}$ for all $n \ge M$. Moreover, if (a) or (b) is assumed, then $\liminf_{n\to\infty}\rho_n > 0$. Let $\rho := \liminf_{n\to\infty}\rho_n$. Then there exists integer $L > 0$ such that $\rho_n > \rho$ for all $n \ge L$. Taking $N = \max\{L, M\}$, it follows that

$$\|z_n - \bar{z}\| \le \lambda p^{\Sigma_{i=1}^{L}\rho_i}p^{(n-L)\rho} = \lambda p^{\Sigma_{i=1}^{L}(\rho_i-\rho)}p^{n\rho} = \delta\sigma^n,\ \forall n \ge N,$$

where $\delta = \lambda p^{\Sigma_{i=1}^{L}(\rho_i-\rho)}$ and $\sigma = p^\rho \in (0,1)$. Hence, $\{z_n\}$ converges to $\bar{z}$ linearly. The proof is complete. □

4. Applications

In this section, we turn our attention to provide some applications relying on the result of the linear convergence of the split equality mixed equilibrium problem, such as the split equality equilibrium problem, split equality convex minimization problem, split equality mixed variational inequality problem, and split equality variational inequality problem.

4.1. Split equality equilibrium problem.

Definition 4.1. Let $F : C \times C \to R$ and $J : Q \times Q \to R$ be nonlinear bifunctions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Then the split equality equilibrium problem (SEEP for short) is to find $x^* \in C,\ y^* \in Q$ such that

$$\begin{cases} F(x^*, x) \ge 0,\ \forall x \in C, \\ J(y^*, y) \ge 0,\ \forall y \in Q, \\ Ax^* = By^*. \end{cases} \tag{4.1}$$

The set of solution of (4.1) is denoted by $SEEP(F, J)$.

According to the definition of EP (1.1), the SEEP can be formulated as:

$$\text{finding } x^* \in EP(F) \text{ and } y^* \in EP(J) \text{ such that } Ax^* = By^*. \tag{4.2}$$

Let $S = EP(F) \times EP(J) \subseteq H_1 \times H_2 := H$ and $G = [A, -B] : H \to H_3$. Then (4.2) can now be reformulated as:

$$\text{finding } z^* = (x^*, y^*) \in S \text{ such that } Gz^* = 0.$$

By Remark1.3, the following result can be directly deduced from Theorem 3.1.

Corollary 4.2. *Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty, convex and closed sets in H_1 and H_2, respectively. Let $F : C \times C \to R$ and $J : Q \times Q \to R$ be two bifunctions. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators. Let A^* and B^* be the self-adjoint operators of A and B. Let Γ be the set of solutions of the SEEP (4.1). For an initial point $z_0 = (x_0, y_0) \in S$, define*

$$\begin{cases} U(w_n, w) + \frac{1}{r_n}\langle w - w_n, w_n - z_n\rangle \geq 0, \ \forall w \in V; \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n) P_S(w_n - \rho_n G^* G w_n), \end{cases}$$

where $S = EP(F) \times EP(J)$, $U = F \times J$, $G = [A, -B]$, $z_{n+1} = (x_{n+1}, y_{n+1})$, $w_n = (u_n, v_n)$, $w = (u, v)$, $\alpha_n \in [0, 1)$, $r_n > 0$, $\rho_n > 0$, $\lim_{n\to\infty} \rho_n = 0$, and $\sum_{n=1}^{\infty} \rho_n = \infty$. Assume that the SEEP (4.1) satisfies the bounded linear regularity property. Then $\{z_n\}$ converges to a solution z^* of SEEP (4.1) provided that one of the following conditions holds:
(a) $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < \frac{2}{\|G\|^2}$*;*
(b) $\rho_n = \begin{cases} 0, \ z_n \in \Gamma, \\ \frac{\gamma_n \|Gz_n\|^2}{\|G^* G z_n\|^2}, \ z_n \notin \Gamma, \end{cases}$ *, and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2$*. Consequently, $\{z_n\}$ converges to z^* linearly when (a) or (b) is assumed.*

Proof. Let $T_{r_n}^U = T_{r_n}^F \times T_{r_n}^J : H_1 \times H_2 \to V$, i.e.,

$$T_{r_n}^U(a) = \{c \in V : U(c, b) + \frac{1}{r_n}\langle b - c, c - a\rangle \geq 0, \ \forall b \in V\}.$$

Then mapping $T_{r_n}^U$ satisfies Lemma 2.3, and $w_n = z_n$. According to Theorem 3.1, the proof is complete. □

4.2. **Split equality convex minimization problem.** According to the definition of CMP (1.2), the SECMP (1.4) can be formulated as:

$$\text{finding } x^* \in CMP(\phi) \text{ and } y^* \in CMP(\varphi) \text{ such that } Ax^* = By^*. \tag{4.3}$$

Let $S = CMP(\phi) \times CMP(\varphi) \subseteq H_1 \times H_2 := H$, $G = [A, -B] : H \to H_3$, then (4.3) can now be reformulated as:

$$\text{finding } z^* = (x^*, y^*) \in S \text{ such that } Gz^* = 0.$$

If $F = 0$ and $J = 0$, then the SEMEP (1.3) reduces to the SECMP (1.4). Therefore, Theorem 3.1 can be used to solve SECMP (1.4), and the following result can be directly deduced from Theorem 3.1.

Corollary 4.3. *Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty, convex and closed sets in H_1 and H_2, respectively. Let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper, lower semi-continuous, and convex functions such that $C \cap dom\phi \neq \emptyset$ and $Q \cap dom\varphi \neq \emptyset$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators and A^*, B^* being the self-adjoint operators of the operators A, B. Let Γ be the set of solutions of the SECMP (1.4). For an initial point $z_0 = (x_0, y_0) \in S$, define*

$$\begin{cases} \psi(w) - \psi(w_n) + \frac{1}{r_n}\langle w - w_n, w_n - z_n\rangle \geq 0, \ \forall w \in V; \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n)P_S(w_n - \rho_n G^* G w_n), \end{cases}$$

where $S = CMP(\phi) \times CMP(\varphi)$, $V = C \times Q$, $\psi = \phi \times \varphi$, $z_{n+1} = (x_{n+1}, y_{n+1})$, $w_n = (u_n, v_n)$, $w = (u, v)$, $\alpha_n \in [0,1)$, $r_n > 0$, $\rho_n > 0$, $\lim_{n\to\infty} \rho_n = 0$, and $\sum_{n=1}^{\infty} \rho_n = \infty$. Assume that the SECMP (1.4) satisfies the bounded linear regularity property. Then the sequence $\{z_n\}$ converges to a solution z^* of SECMP (1.4) provided that one of the following conditions holds:
(a) $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < \frac{2}{\|G\|^2}$;
(b) $\rho_n = \begin{cases} 0, \ z_n \in \Gamma, \\ \frac{\gamma_n \|Gz_n\|^2}{\|G^* G z_n\|^2}, \ z_n \notin \Gamma, \end{cases}$ *, and* $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2$. *Consequently,* $\{z_n\}$ *converges to z^* linearly when (a) or (b) is assumed.*

Proof. Let $T_{r_n} = T_{r_n} \times T_{r_n} : H_1 \times H_2 \to V$, i.e.,

$$T_{r_n}(a) = \{c \in V : \psi(b) - \psi(c) + \frac{1}{r_n}\langle b - c, c - a\rangle \geq 0, \ \forall b \in V\}.$$

Then the mapping T_{r_n} satisfies Lemma 2.3, and $w_n = z_n$. According to Theorem 3.1, the proof is complete. □

4.3. **Split equality mixed variational inequality problem.** Mixed variational inequality problems (MVIP for short) which can be formulated as

$$\text{finding } x^* \in C \text{ such that } \langle Kx^*, z - x^*\rangle + \phi(z) - \phi(x^*) \geq 0, \ \forall z \in C, \tag{4.4}$$

where $\phi : C \to R \cup \{+\infty\}$ is a function. We use $MVIP(K, C, \phi)$ to denote the set of solutions of the MVIP (4.4).

If $\phi = 0$, then the MVIP (4.4) reduces to the variational inequality problems (VIP for short), which can be formulated as

$$\text{finding } x^* \in C, \text{ such that } \langle Kx^*, z - x^*\rangle \geq 0, \ \forall z \in C. \tag{4.5}$$

We use $VIP(K, C)$ to denote the set of solutions of the VIP (4.5).

Recently, some scholars introduced the split equality mixed variational inequality problems (SEMVIP for short). The definition of SEMVIP can be expressed as finding $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} \langle K_1 x^*, x - x^*\rangle + \phi(x) - \phi(x^*) \geq 0, \ \forall x \in C, \\ \langle K_2 y^*, y - y^*\rangle + \varphi(y) - \varphi(y^*) \geq 0, \ \forall y \in Q, \\ Ax^* = By^*. \end{cases} \tag{4.6}$$

According to the definition of MVIP, (4.6) can be formulated as:

$$\text{finding } x^* \in MVIP(K_1, C, \phi) \text{ and } y^* \in MVIP(K_2, Q, \varphi) \text{ such that } Ax^* = By^*. \tag{4.7}$$

Let $S = MVIP(K_1, C, \phi) \times MVIP(K_2, Q, \varphi) \subseteq H_1 \times H_2 := H$, $G = [A, -B] : H \to H_3$, then (4.7) can now be reformulated as:

$$\text{finding } z^* = (x^*, y^*) \in S \text{ such that } Gz^* = 0.$$

An operator K is κ-inverse strongly monotone (κ-ism) with $\kappa > 0$ if $\langle Kx - Ky, x - y\rangle \geq \kappa\|Kx - Ky\|^2$ for all $x, y \in C$. Let $F(x, y) = \langle Kx, y - x\rangle$ and K be κ-ism, where F is a bifunction. Then the following result holds.

Corollary 4.4. *Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty convex, and closed sets in H_1 and H_2, respectively. Let K_1 and K_2 be two κ-ism mappings and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper, lower semi-continuous, and convex functions such that $C \cap dom\phi \neq \emptyset$ and $Q \cap dom\varphi \neq \emptyset$. Let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators with A^*, B^* being the self-adjoint operators of the operators A, B. Let Γ be the set of solution of the SEMVIP (4.6). For an initial point $z_0 = (x_0, y_0) \in S$, define*

$$\begin{cases} \langle Kw_n, w - w_n\rangle + \psi(w) - \psi(w_n) + \frac{1}{r_n}\langle w - w_n, w_n - z_n\rangle \geq 0, \forall w \in V; \\ z_{n+1} = \alpha_n z_n + (1 - \alpha_n)P_S(w_n - \rho_n G^* G w_n), \end{cases}$$

where $S = MVIP(K_1, C, \phi) \times MVIP(K_2, Q, \varphi)$, $V = C \times Q$, $K = K_1 \times K_2$, $G = [A, -B]$, $\psi = \phi \times \varphi$, $z_{n+1} = (x_{n+1}, y_{n+1})$, $w_n = (u_n, v_n)$, $w = (u, v)$, $\alpha_n \in [0, 1)$, $r_n > 0$, $\rho_n > 0$, $\lim_{n\to\infty} \rho_n = 0$, and $\sum_{n=1}^{\infty} \rho_n = \infty$. Assume that the SEMVIP (4.6) satisfies the bounded linear regularity property. Then $\{z_n\}$ converges to a solution z^* of SEMVIP (4.6) provided that one of the following conditions holds:
(a) $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < \frac{2}{\|G\|^2}$;
(b) $\rho_n = \begin{cases} 0, \ z_n \in \Gamma, \\ \frac{\gamma_n \|Gz_n\|^2}{\|G^* G z_n\|^2}, \ z_n \notin \Gamma, \end{cases}$, and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2$. Consequently, $\{z_n\}$ converges to z^* linearly in the case when (a) or (b) is assumed.

Proof. Let $T_{r_n}^K = T_{r_n}^{K_1} \times T_{r_n}^{K_2} : H_1 \times H_2 \to V$, i.e.,

$$T_{r_n}^K(a) = \{c \in V : \langle Kc, b - c\rangle + \psi(b) - \psi(c) + \frac{1}{r_n}\langle b - c, c - a\rangle \geq 0, \forall b \in V\}.$$

Then the mapping $T_{r_n}^K$ satisfies the Lemma 2.3, and $w_n = z_n$. According to Theorem 3.1, the proof is complete. □

4.4. **Split equality variational inequality problem.** If $\phi = \varphi = 0$, i.e., $\psi = 0$, then the SEMVIP (4.6) reduces to the following split equality variational inequality problem (SEVIP for short):

$$\text{finding } z^* = (x^*, y^*) \in S, \text{ such that } Gz^* = 0, \tag{4.8}$$

where $S = VIP(K_1, C) \times VIP(K_2, Q) \subseteq H_1 \times H_2 := H$ and $G = [A, -B] : H \to H_3$.

Then we have the following result.

Corollary 4.5. *Let H_1, H_2 and H_3 be three real Hilbert spaces. Let C and Q be nonempty, convex, and closed sets in H_1 and H_2, respectively. Assume that K_1 and K_2 are two κ-ism mappings and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper, lower semi-continuous, and convex function such that $C \cap dom\phi \neq \emptyset$ and $Q \cap dom\varphi \neq \emptyset$. Let $A : H_1 \to H_3$ and $B : H_2 \to$*

H_3 be bounded linear operators with A^*, B^* being the self-adjoint operators of the operators A, B. Let Γ be the set of solutions of the SEVIP (4.8). For initial point $z_0 = (x_0, y_0) \in S$, define

$$\begin{cases} \langle Kw_n, w - w_n \rangle + \frac{1}{r_n}\langle w - w_n, w_n - z_n \rangle \geq 0, \forall w \in V; \\ z_{n+1} = \alpha_n z_n + (1-\alpha_n) P_S(w_n - \rho_n G^* G w_n), \end{cases}$$

where $S = VIP(K_1, C) \times VIP(K_2, Q)$, $V = C \times Q$, $K = K_1 \times K_2$, $G = [A, -B]$, $\psi = \phi \times \varphi$, $z_{n+1} = (x_{n+1}, y_{n+1})$, $w_n = (u_n, v_n)$, $w = (u, v)$, $\alpha_n \in [0,1)$, $r_n > 0$, $\rho_n > 0$, $\lim_{n\to\infty} \rho_n = 0$, and $\sum_{n=1}^{\infty} \rho_n = \infty$. Assume that the SEVIP (4.8) satisfies the bounded linear regularity property. Then $\{z_n\}$ converges to a solution z^* of SEVIP (4.8) provided that one of the following conditions holds:
(a) $0 < \liminf_{n\to\infty} \rho_n \leq \limsup_{n\to\infty} \rho_n < \frac{2}{\|G\|^2}$;
(b) $\rho_n = \begin{cases} 0, \ z_n \in \Gamma, \\ \frac{\gamma_n \|Gz_n\|^2}{\|G^*Gz_n\|^2}, \ z_n \notin \Gamma, \end{cases}$, and $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 2$. Consequently, $\{z_n\}$ converges to z^* linearly in the case when (a) or (b) is assumed.

Proof. Let $T_{r_n}^K = T_{r_n}^{K_1} \times T_{r_n}^{K_2} : H_1 \times H_2 \to V$, i.e.,

$$T_{r_n}^K(a) = \{c \in V : \langle Kc, b - c \rangle + \frac{1}{r_n}\langle b - c, c - a \rangle \geq 0, \forall b \in V\}.$$

Then the mapping $T_{r_n}^K$ satisfies the Lemma 2.3, and $w_n = z_n$. According to Theorem 3.1, the proof is complete. □

5. Numerical Examples

In this section, we provide some numerical examples to demonstrate the numerical behavior of our proposed algorithm, namely Algorithm 3.1, and compare it with the Algorithm (3.1) of [9]. All codes were written in MATLAB2015B. The numerical results were carried out on a personal Lenovo computer with Intel Core(TM) i5-7200 CPU @ 3.1GHz.

Let $H_1 = H_2 = H_3 = \mathbb{R}^3$ and $C = Q = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$. Define $F(x,y) : C \times C \to \mathbb{R}^3$ by $F(x,y) = y - x$ and $\phi : C \to \mathbb{R}^3$ by $\phi(x) = -x$. It is easy to see that $T_r^F(x_n) = \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle$. Also, let $J(x,y) : Q \times Q \to \mathbb{R}^3$ by $J(x,y) = x - y$, and $\varphi : Q \to \mathbb{R}^3$ by $\varphi(x) = x$, for all $x \in Q$. Therefore, $T_r^J(y_n) = \frac{1}{r_n}\langle v - v_n, v_n - y_n \rangle$. Then the SEMEP satisfies the bounded linear regularity property.

The two operators $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are defined by

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Take $\alpha_n = \frac{1}{3}$ and $z_0 = (x_0, y_0) \in S$. In consideration of Algorithm 3.1, we have

$$\begin{cases} \frac{1}{r_n}\langle u - u_n, u_n - x_n \rangle \geq 0, \ \forall u \in C; \\ \frac{1}{r_n}\langle v - v_n, v_n - y_n \rangle \geq 0, \ \forall v \in Q; \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) P_D(u_n - \rho_n A^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n y_n + (1-\alpha_n) P_E(v_n + \rho_n B^*(Au_n - Bv_n)). \end{cases} \tag{5.1}$$

TABLE 1. The iterations of x_n and y_n at $\alpha_n = 1/3$

number of iteration	x_n	y_n
1	(1,1,1)	(0.9,0.9,0.9)
2	(0.718234,0.718234,0.718234)	(0.6849,0.6849,0.6849)
3	(0.624311,0.624311,0.624311)	(0.6132,0.6132,0.6132)
4	(0.593004,0.593004,0.593004)	(0.5893,0.5893,0.5893)
5	(0.582568,0.582568,0.582568)	(0.581334,0.581334,0.581334)
⋮	⋮	⋮
13	(0.577351,0.577351,0.577351)	(0.577351,0.577351,0.577351)
14	(0.577351,0.577351,0.577351)	(0.57735,0.57735,0.57735)
15	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
16	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
17	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
⋮	⋮	⋮

In Algorithm 3.1, we take $\rho_n = 1/n$, $\alpha_n = 1/3,\ 1/(1+n)$, respectively. In addition, we set $T = (1-\alpha_n)P_D$ and $S = (1-\alpha_n)P_E$ in [9]. Under other same conditions, we compared it with Ma's Algorithm (3.1) to verify the effectiveness of our proposed algorithm.

We choose initial value $x = (1,1,1)$ and $y = (0.9,0.9,0.9)$. Let error be 10^{-13}. Then we have the following numerical results (the x-coordinate denotes the number of iterations, and the y-coordinate denotes the logarithm of the error).

TABLE 2. The iterations of x_n and y_n at $\alpha_n = 1/(1+n)$

number of iteration	x_n	y_n
1	(1,1,1)	(0.9,0.9,0.9)
2	(0.788675,0.788675,0.788675)	(0.738675,0.738675,0.738675)
3	(0.647792,0.647792,0.647792)	(0.631125,0.631125,0.631125)
4	(0.594961,0.594961,0.594961)	(0.590794,0.590794,0.590794)
5	(0.580872,0.580872,0.580872)	(0.580039,0.580039,0.580039)
⋮	⋮	⋮
9	(0.577351,0.577351,0.577351)	(0.577351,0.577351,0.577351)
10	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
11	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
12	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
13	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
14	(0.57735,0.57735,0.57735)	(0.57735,0.57735,0.57735)
⋮	⋮	⋮

From the TABLE 1, TABLE 2, and FIGURE 1, it is easy to see that our iterative method converges to the point $x_0 = (0.57735, 0.57735, 0.57735)$ and $y_0 = (0.57735, 0.57735, 0.57735)$. Next, we compare our Algorithm (3.1) with the Ma's algorithm in [9], the FIGURE 2 of convergent rate is given as follows.

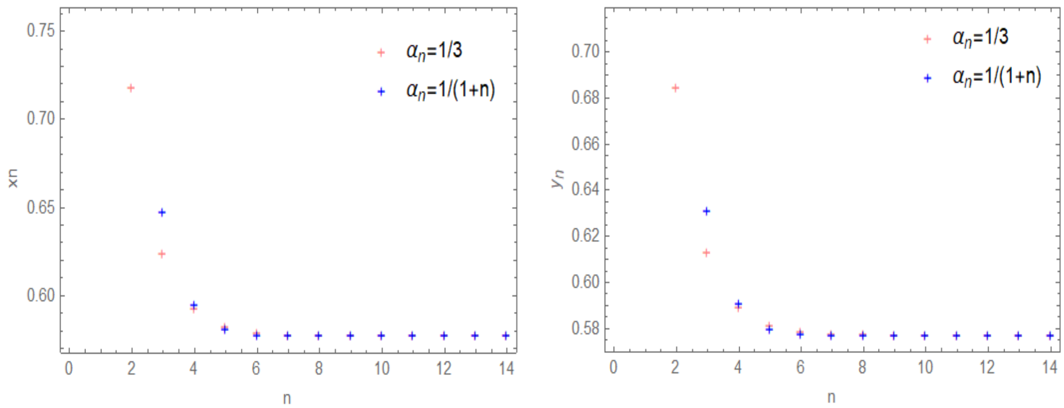

FIGURE 1. $x = (1,1,1)$ and $y = (0.9,0.9,0.9)$, the process of iteration x_n and y_n

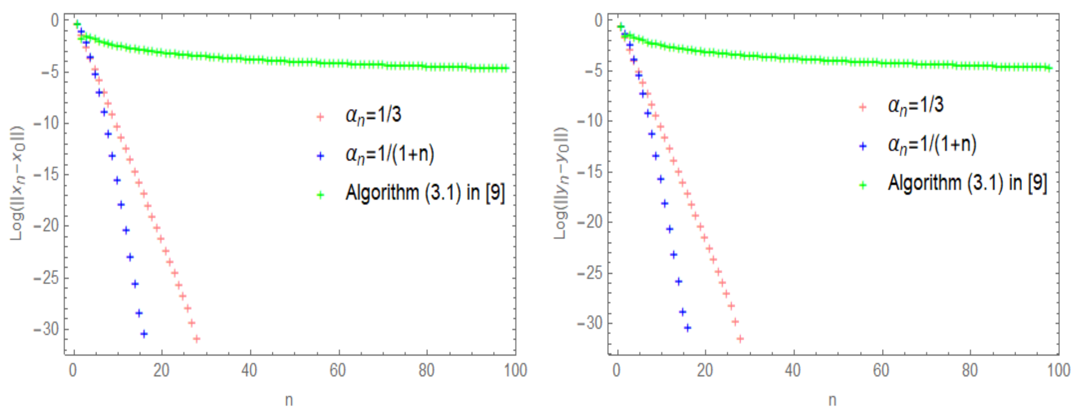

FIGURE 2. Comparison of algorithm (3.1) and Ma's algorithm in [9]

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