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FIXED POINT THEOREMS FOR $(a,b,\theta)-$ENRICHED CONTRACTIONS

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Abstract. This paper establishes a new concept for contractive mappings in Banach spaces, called the (a,b,θ)-enriched contractive mapping. Based upon this new contractive mapping, we obtain three classes of new enriched contractive mappings: (a,b,k)-enriched Kannan mappings, (a,b,p,q)-enriched Ćirić-Reich-Rus mappings, and (a,b,l)-enriched Chatterjea mappings, which extend the corresponding concepts in the literature. We also obtain some new fixed point theorems and Maia type fixed point theorems for these contractive mappings.

Keywords. (a,b,θ)-enriched contraction; Ćirić-Reich-Rus contraction; Chatterjea contraction; Kannan contraction.

1. Introduction

Throughout this paper, let X be a nonempty set and $T: X \to X$ a self mapping. We denote the set of fixed points of T by $Fix(T)$, i.e., $Fix(T) = \{x^* \in X : Tx^* = x^*\}$. The following famous theorem is referred to as the Banach contraction principle, which is the simplest and one of the most versatile elementary results in fixed point theory.

Theorem 1.1. [2] *Let (X,d) be a complete metric space and $T: X \to X$ be a contraction, i.e., there exists $r \in [0,1)$ such that $d(Tx,Ty) \leqslant rd(x,y)$ for all $x,y \in X$, Then T has a unique fixed point.*

Due to its importance and applications, the Banach contraction principle has been extensively investigated and generalized by several authors; see, e.g., [3, 4, 5, 6, 7, 16, 18, 23]. In 1968, Maia [23] extended the Banach contraction principle in the spaces equipped with two metrics. It was proved that a continuous mapping admits an unique fixed point, despite the contractive condition and the completeness of the space are not satisfied in the same metric. To be more precise, the theorem is presented below.

Theorem 1.2. [23] *Let X be a set endowed with two metrics d and ρ satisfying $d(x,y) \leqslant \rho(x,y)$ for all $x,y \in X$. Suppose*

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(i) (X,d) is a complete metric space;

(ii) $T:X\to X$ is continuous with respect to d;

(iii) T is a contraction with respect to ρ, that is, $\rho(Tx,Ty)\leqslant a\rho(x,y)$ for all $x,y\in X$, where $0\leqslant a<1$ is constant.

Then T has a unique fixed point in X.

In the past decades, numerous authors obtained various fixed point theorems by modifying the contractive conditions; see, e.g., [8, 9, 11, 12, 13, 14, 17, 18, 22, 27, 28, 32] and the references therein. In addition, researchers was also attracted to study the Maia type fixed point theorem and found it useful for studying data dependence problems, well-posedness problems, and certain classes of differential equations; see, e.g., [21, 26, 29, 30, 31, 33] and the references therein.

Recently, Berinde and Păcurar [3] introduced the concept of an enriched contraction. They established fixed point theorems, which extended and unified the results presented in [2, 16, 23]. Further improvements were presented later in [4, 5, 6, 7].

Inspired by the above results, we first introduce a new concept of (a,b,θ)-enriched contraction, which contains the enriched contractions. Based upon this new contractive mapping, we further obtain three classes of new enriched contractive mappings: (a,b,k)-enriched Kannan mappings, $(a,b,p,q)-$enriched Ćirić$-$Reich-Rus mappings, and (a,b,l)-enriched Chatterjea mappings, which extend the corresponding concepts in literature. And then we obtain some new fixed point theorems and Maia type fixed point theorems for these kinds of contractive mappings. The presented theorems extend, generalize, and improve the associated results in the literatures.

2. (a,b,θ)-ENRICHED CONTRACTIONS

To prove our main results, we list some basic concepts and theorems in the following.

Definition 2.1. [3] Let $(X,\|\cdot\|)$ be a linear normal space. A mapping $T:X\to X$ is said to be a (b,θ)-enriched contraction if there exist $b\in[0,+\infty)$ and $\theta\in[0,b+1)$ such that $\|b(x-y)+Tx-Ty\|\leqslant\theta\|x-y\|$ for all $x,y\in X$.

Theorem 2.2. [3] *Let $(X,\|\cdot\|)$ be a Banach space and $T:X\to X$ be a (b,θ)-enriched contraction. Then*

(i) Fix $(T)=p$ for some $p\in X$;

(ii) there exists $\lambda\in(0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, given by $x_{n+1}=(1-\lambda)x_n+\lambda Tx_n$, $n\geqslant 0$, converges to p, for any $x_0\in X$;

(iii) the following estimate holds $\|x_{n+i-1}-p\|\leqslant\frac{c^i}{1-c}\cdot\|x_n-x_{n-1}\|$, $n=1,2,\ldots;i=1,2,\ldots$, where $c=\frac{\theta}{b+1}$.

Now, a new concept which extends Definition 2.1 is presented as follows

Definition 2.3. Let $(X,\|\cdot\|)$ be a linear normal space and $T:X\to X$ be a given mapping. We say that T is a (a,b,θ)-*enriched contraction* if there exist $a,b\in(0,+\infty)$ and $\theta\in[0,a+b)$ such that

$$\|a(x-y)+b(Tx-Ty)\|\leqslant\theta\|x-y\|,\ \forall x,y\in X. \tag{2.1}$$

Remark 2.4. (1) If T is a (a,b,θ)-enriched contraction with $a=0$ and $b\neq0$, then we say that T is a Banach contraction. (2) If only $b=1$, then T is a (b,θ)-enriched contraction.

According to the definitions and theorem above, we give two new fixed point theorems which generalize Theorem 2.2.

Theorem 2.5. *Let $(X,\|\cdot\|)$ be a Banach space and $T : X \to X$ be a (a,b,θ)-enriched contraction. Then*

(i) Fix $(T) = x^*$ for some $x^* \in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ give by

$$x_{n+1} = \frac{a}{a+b}x_n + \frac{b}{a+b}Tx_n,\ n \geqslant 0, \tag{2.2}$$

converges to x^*, for any $x_0 \in X$;

(iii) the following estimate holds

$$\|x_{n+i-1} - x^*\| \leqslant \frac{\delta^i}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n = 1,2,\dots; i = 1,2,\dots, \tag{2.3}$$

where $\delta = \frac{\theta}{a+b}$.

Proof. Since T is a (a,b,θ)-enriched contraction, we find from (2.1) that

$$\|\frac{a}{a+b}(x-y) + \frac{b}{a+b}(Tx - Ty)\| \leqslant \frac{\theta}{a+b}\|x-y\|,\ \forall x,y \in X,$$

which can be written in an equivalent form as

$$\|T_{ab}x - T_{ab}y\| \leqslant \delta\|x-y\|,\ \forall x,y \in X, \tag{2.4}$$

where $\delta = \frac{\theta}{a+b}$, and T_{ab} is defined by

$$T_{ab}x = \frac{a}{a+b}x + \frac{b}{a+b}Tx,\ \forall x \in X. \tag{2.5}$$

If $\theta = 0$, it follows from (2.4) that $T_{ab}x = T_{ab}y$ for all $x,y \in X$, that is, there exists a constant c such that $T_{ab}x = c$ for all $x \in X$. Then Fix$(T_{ab}) = \{c\}$. Again, taking $x = c$ in (2.5), one has

$$c = T_{ab}c = \frac{a}{a+b}c + \frac{b}{a+b}Tc,$$

which yields $Tc = c$. Consequently, Fix(T)=Fix(T_{ab})=$\{c\}$. If $\theta \neq 0$, i.e., $\theta \in (0,a+b)$, then $\delta \in (0,1)$. According to (2.5), the iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.2) is the Picard iteration associated to T_{ab}, that is, $x_{n+1} = T_{ab}x_n$, $n \geqslant 0$. Letting $x = x_n$ and $y = x_{n-1}$ in (2.4), one has $\|x_{n+1} - x_n\| \leqslant \delta\|x_n - x_{n-1}\|$ for $n \geqslant 1$. By iteration, we obtain $\|x_{n+1} - x_n\| \leqslant \delta^n\|x_1 - x_0\|$ for $n \geqslant 0$. Hence,

$$\|x_{n+m} - x_n\| \leqslant \delta \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n \geqslant 1, m \geqslant 1, \tag{2.6}$$

and

$$\|x_{n+m} - x_n\| \leqslant \delta^n \cdot \frac{1-\delta^m}{1-\delta} \cdot \|x_1 - x_0\|,\ n \geqslant 0, m \geqslant 1. \tag{2.7}$$

Since $\delta \in (0,1)$, we deduce from (2.7) that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and thus it converges to some x^* in $(X,\|\cdot\|)$, that is, $\lim_{n\to\infty} x_n = x^*$. Letting $n \to \infty$ in the iteration $x_{n+1} = T_{ab}x_n$ and using the continuity of T_{ab}, we have $x^* = T_{ab}x^*$, that is, $x^* \in$Fix(T_{ab}).

Next, we prove that x^* is the unique fixed point of T_{ab}. Assume that $y^* \neq x^*$ is another fixed point of T_{ab}. Then, by (2.4) we have $0 < \|x^* - y^*\| \leqslant \delta \|x^* - y^*\| < \|x^* - y^*\|$. This is a contradiction. Hence $\mathrm{Fix}(T_{ab}) = \{x^*\}$. In view of Fix$(T)$=Fix$(T_{ab})$, conclusion (i) is proven. Conclusion (ii) is obvious due to $\lim_{n\to\infty} x_n = x^*$. To prove (iii), we let $m \to \infty$ in (2.6) and (2.7) to obtain

$$\|x_n - x^*\| \leqslant \frac{\delta}{1-\delta} \cdot \|x_n - x_{n-1}\|,\ n \geqslant 1$$

and

$$\|x_n - x^*\| \leqslant \frac{\delta^n}{1-\delta} \cdot \|x_1 - x_0\|,\ n \geqslant 1,$$

respectively, where $\delta = \frac{\theta}{a+b}$. From the two inequalities above, we obtain unifying error estimate (2.3). □

Remark 2.6. In the particular case $a = 0$ and $b \neq 0$, then by Theorem 2.5 we obtain Theorem 1.1; while, for $b = 1$, by Theorem 2.5 we obtain Theorem 2.2.

Example 2.7. Let $X = [\frac{1}{2}, 2]$ be endowed with the usual norm, and let $T : X \to X$ be defined by $Tx = \frac{1}{x}$ for all $x \in [\frac{1}{2}, 2]$. Then $\mathrm{Fix}(T) = 1$, and

(1) T is not a Banach contraction;

(2) T is a (a,b,θ)-enriched contraction.

Proof. (1) If T is a Banach contraction, then there would exist $c \in [0,1)$ such that $|Tx - Ty| = |\frac{y-x}{xy}| \leqslant c \cdot |x-y|$ for all $x, y \in [\frac{1}{2}, 2]$, so, for any $x \neq y$, $\frac{1}{xy} \leqslant c$, that is, $4 \leqslant c$, this is a contradiction.

(2) It follows from (2.1) that $|a(x-y) + b(\frac{1}{x} - \frac{1}{y})| = |a - \frac{b}{xy}| \cdot |x-y| \leqslant \theta \cdot |x-y|$ with $\theta \in [0, a+b)$.

If $a \geqslant \frac{b}{xy}$, it is obvious that there exist $a < \theta < a+b$ such that the above formula holds.

If $a < \frac{b}{xy}$, for any $a > \frac{3}{2}b$, we have $|a - \frac{b}{xy}| = \frac{b}{xy} - a \leqslant 4b - a < \frac{5}{2}b < a+b$. Then there exist $\frac{b}{xy} - a \leqslant \theta < a+b$ such that $|a(x-y) + b(\frac{1}{x} - \frac{1}{y})| \leqslant \theta \cdot |x-y|$ for all $x, y \in [\frac{1}{2}, 2]$. Above all, for all $x, y \in [\frac{1}{2}, 2]$ and any $a > \frac{3}{2}b$, T is a (a,b,θ)-enriched contraction. □

This suggests us that a mapping T is a (a,b,θ)-enriched contraction but it may be not a Banach contraction. Hence, studying Theorem 2.5 is necessary and significative. The following is to use Theorem 2.5 and obtain a Maia type fixed point theorem for (a,b,θ)-enriched contractions in Banach spaces.

Theorem 2.8. *Let X be a linear vector space endowed with d and a norm $\|\cdot\|$ satisfying the condition $d(x,y) \leqslant \|x-y\|$ for all $x, y \in X$. Suppose*

(i) (X,d) is a complete metric space;

(ii) $T : X \to X$ is continuous with respect to d;

(iii) T is a (a,b,θ)-enriched contraction with respect to $\|\cdot\|$, that is, there exist $a, b \in (0, +\infty)$ and $\theta \in [0, a+b)$ such that $\|a(x-y) + b(Tx - Ty)\| \leqslant \theta \|x-y\|$ for all $x, y \in X$. Then

(i) Fix $(T) = x^*$, for some $x^* \in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ give by

$$x_{n+1} = \frac{a}{a+b} x_n + \frac{b}{a+b} T x_n,\ n \geqslant 0, \tag{2.8}$$

converges to x^*, for any $x_0 \in X$;

(iii) the following estimates hold

$$\|x_n - x^*\| \leqslant \frac{\delta}{1-\delta} \cdot \|x_n - x_{n-1}\|, \; n \geqslant 1, \tag{2.9}$$

and

$$\|x_n - x^*\| \leqslant \frac{\delta^n}{1-\delta} \cdot \|x_1 - x_0\|, \; n \geqslant 1, \tag{2.10}$$

where $\delta = \frac{\theta}{a+b}$.

Proof. We consider the mapping T_{ab} defined by (2.5). By (iii), similar to the proof of Theorem 2.5, we deduce that $\{x_n\}_{n=0}^{\infty}$, defined by (2.8), which is in fact the Picard iteration associated to T_{ab} and a Cauchy sequence in $(X, \|\cdot\|)$. Since $d(x,y) \leqslant \|x-y\|$ for all $x,y \in X$, one has that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in (X,d). By (i), it converges. Let $x^* = \lim_{n\to\infty} x_n$. By (ii), we obtain that $x^* \in$Fix(T_{ab}) and by (iii) that Fix$(T_{ab}) = \{x^*\}$. Again, Fix(T_{ab}) =Fix(T). Then, these conclusions follow. □

Remark 2.9. If $d(x,y) = \|x-y\|$ for all $x,y \in X$, then by Theorem 2.8 we obtain Theorem 2.5. In this case, the two estimates (2.9) and (2.10) in Theorem 2.8 can be merged to yield the unified estimate in Theorem 2.5.

3. (a,b,k)-Enriched Kannan Contractions

A mapping $T : X \to X$ on a metric space (X,d) is called a Kannan mapping ([17, 18]) if there exist $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx,Ty) \leqslant \alpha(d(x,Tx) + d(y,Ty)), \; \forall x,y \in X, \tag{3.1}$$

Berinde [4] introduced and studied the concept of the enriched Kannan contraction, which is a generalization of Kannan mappings.

Definition 3.1. [4] Let $(X, \|\cdot\|)$ be a linear normal space. A mapping $T : X \to X$ is said to be a (k,a)-enriched Kannan mappping if there exist $a \in [0, \frac{1}{2})$ and $k \in [0, \infty)$ such that

$$\|k(x-y) + Tx - Ty\| \leqslant a(\|x - Tx\| + \|y - Ty\|), \; \forall x,y \in X. \tag{3.2}$$

Theorem 3.2. [4] *Let* $(X, \|\cdot\|)$ *be a Banach space and let* $T : X \to X$ *be a* (k,a)*-enriched Kannan mappping. Then*

(i) Fix $(T) = p$ *for some* $p \in X$;

(ii) there exists $\lambda \in (0,1]$ *such that the iterative method* $\{x_n\}_{n=0}^{\infty}$*, give by* $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$*,* $n \geqslant 0$*, converges to* p*, for any* $x_0 \in X$;

(iii) the following estimate holds

$$\|x_{n+i-1} - p\| \leqslant \frac{c^i}{1-c} \cdot \|x_n - x_{n-1}\|, \; n = 1,2,\ldots; i = 1,2,\ldots,$$

where $c = \frac{a}{1-a}$.

In this section, we give a new notion of (a,b,k)-enriched Kannan contractions, which extends the (k,a)-enriched Kannan mappping. Some results concerning this contractions are presented as follows.

Definition 3.3. Let $(X,\|\cdot\|)$ be a linear normal space and let $T:X\to X$ be a given mapping. We say that T is a (a,b,k)-enriched Kannan contraction if there exist $a,b\in(0,+\infty)$ and $k\in[0,\frac{b}{2})$ such that

$$\|a(x-y)+b(Tx-Ty)\|\leqslant k(\|x-Tx\|+\|y-Ty\|),\ \forall x,y\in X. \tag{3.3}$$

Theorem 3.4. *Let $(X,\|\cdot\|)$ be a Banach space and $T:X\to X$ be a (a,b,k)-enriched Kannan contraction. Then*

(i) Fix $(T)=x^*$ for some $x^*\in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1}=\frac{a}{a+b}x_n+\frac{b}{a+b}Tx_n,\ n\geqslant 0, \tag{3.4}$$

converges to x^* for any $x_0\in X$;

(iii) the following estimate holds

$$\|x_{n+i-1}-x^*\|\leqslant\frac{\gamma^i}{1-\gamma}\cdot\|x_n-x_{n-1}\|,\ n=1,2,\dots;i=1,2,\dots,$$

where $\gamma=\frac{k}{b-k}$.

Proof. Consider the mapping T_{ab} defined by (2.5). Thus contractive condition (3.3) becomes

$$\|\frac{a}{a+b}(x-y)+\frac{b}{a+b}(Tx-Ty)\|\leqslant\frac{k}{a+b}(\|x-Tx\|+\|y-Ty\|),\ \forall x,y\in X,$$

which can be written equivalently as

$$\|T_{ab}x-T_{ab}y\|\leqslant\frac{k}{b}(\|x-T_{ab}x\|+\|y-T_{ab}y\|),\ \forall x,y\in X. \tag{3.5}$$

The above inequality demonstrates that T_{ab} is a Kannan mapping.

Now we analyze the case that $k>0$, while the case $k=0$ is immediate due to the proof of Theorem 2.5. According to (2.5), the iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (3.4) is the Picard iteration associated to T_{ab}, that is, $x_{n+1}=T_{ab}x_n$ for $n\geqslant 0$. Letting $x=x_n$ and $y=x_{n-1}$ in (3.5), one has

$$\|x_{n+1}-x_n\|\leqslant\frac{k}{b}(\|x_n-x_{n+1}\|+\|x_{n-1}-x_n\|),\ n\geqslant 1.$$

So, $\|x_{n+1}-x_n\|\leqslant\frac{k}{b-k}\cdot\|x_n-x_{n-1}\|$, $n\geqslant 1$. Since $k\in(0,\frac{b}{2})$, we denote $\gamma=\frac{k}{b-k}$ and have $\gamma\in(0,1)$. Therefore $\|x_{n+1}-x_n\|\leqslant\gamma\|x_n-x_{n-1}\|$ for $n\geqslant 1$. By iteration, we have $\|x_{n+1}-x_n\|\leqslant\gamma^n\|x_1-x_0\|$ for $n\geqslant 0$. Hence, we arrive at

$$\|x_{n+m}-x_n\|\leqslant\gamma\cdot\frac{1-\gamma^m}{1-\gamma}\cdot\|x_n-x_{n-1}\|,\ n\geqslant 1,m\geqslant 1, \tag{3.6}$$

and

$$\|x_{n+m}-x_n\|\leqslant\gamma^n\cdot\frac{1-\gamma^m}{1-\gamma}\cdot\|x_1-x_0\|,\ n\geqslant 0,m\geqslant 1. \tag{3.7}$$

Indeed, (3.7) implies that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and thus it converges to some x^* in the Banach space $(X,\|\cdot\|)$, that is, $\lim_{n\to\infty}x_n=x^*$. We first prove that x^* is a fixed point of T_{ab}. Observe that

$$\|x^*-T_{ab}x^*\|\leqslant\|x^*-x_{n+1}\|+\|x_{n+1}-T_{ab}x^*\|=\|x_{n+1}-x^*\|+\|T_{ab}x_n-T_{ab}x^*\|. \tag{3.8}$$

It follows from (3.5) that

$$\|T_{ab}x_n - T_{ab}x^*\| \leqslant \frac{k}{b}(\|x_n - T_{ab}x_n\| + \|x^* - T_{ab}x^*\|).$$

From (3.8), we have

$$\|x^* - T_{ab}x^*\| \leqslant \frac{b+k}{b-k}\|x_{n+1} - x^*\| + \gamma\|x_n - x^*\|. \tag{3.9}$$

By letting $n \to \infty$ in (3.9), we see that $\|x^* - T_{ab}x^*\| = 0$, that is, $x^* = T_{ab}x^*$. So, $x^* \in$Fix(T_{ab}).

Now, we prove that x^* is the unique fixed point of T_{ab}. Assume that $y^* \neq x^*$ is another fixed point of T_{ab}. Then, by (3.5) we obtain $0 < \|x^* - y^*\| \leqslant \frac{k}{b} \cdot 0$, which is a contradiction. Hence Fix(T)=Fix(T_{ab})=$\{x^*\}$, and (i) is proven. In view of $\lim_{n\to\infty} x_n = x^*$, one obtains Conclusion (ii). To prove (iii), letting $m \to \infty$ in (3.6) and (3.7), one has $\|x_n - x^*\| \leqslant \frac{\gamma}{1-\gamma} \cdot \|x_n - x_{n-1}\|$ for $n \geqslant 1$, and $\|x_n - x^*\| \leqslant \frac{\gamma^n}{1-\gamma} \cdot \|x_1 - x_0\|$, $n \geqslant 1$, respectively, where $\gamma = \frac{k}{b-k}$. Hence, we obtain unifying error estimate (iii). □

Remark 3.5. (1) If $a \geqslant 0$ and $b = 1$, then the (a,b,k)-enriched Kannan contractions are (a,k)-enriched Kannan contractions, i.e., they satisfies (3.2). Fron Theorem 3.4, we can obtain Theorem 3.2. (2) If $a = 0, b \neq 0$, then the (a,b,k)-enriched Kannan contractive mappings are Kannan mappings. Using Theorem 3.4, we can obtain Kannan fixed point theorem; see, e.g., [18, 19].

In the following theorem, we use Theorem 3.4 to obtain a Maia type fixed point theorem for (a,b,k)-enriched Kannan contractions in Banach spaces.

Theorem 3.6. *Let X be a linear vector space endowed with d and a norm $\|\cdot\|$ satisfying the condition $d(x,y) \leqslant \|x-y\|$ for all $x,y \in X$. Suppose*

(i) (X,d) is a complete metric space;

(ii) $T : X \to X$ is continuous with respect to d;

(iii) T is a (a,b,k)-enriched Kannan contraction with respect to $\|\cdot\|$, that is, there exist $a,b \in (0,+\infty)$ and $k \in [0, \frac{b}{2})$ such that (3.3) holds.

Then

(i) Fix $(T) = x^*$ for some $x^* \in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = \frac{a}{a+b}x_n + \frac{b}{a+b}Tx_n,\ n \geqslant 0, \tag{3.10}$$

converges to x^* for any $x_0 \in X$;

(iii) the following estimates hold

$$\|x_n - x^*\| \leqslant \frac{\gamma}{1-\gamma} \cdot \|x_n - x_{n-1}\|,\ n \geqslant 1, \tag{3.11}$$

and

$$\|x_n - x^*\| \leqslant \frac{\gamma^n}{1-\gamma} \cdot \|x_1 - x_0\|,\ n \geqslant 1, \tag{3.12}$$

where $\gamma = \frac{k}{b-k}$.

Proof. Consider the mapping T_{ab} defined by (2.5). Similar to the proof of Theorem 3.4, we can follow that $\{x_n\}_{n=0}^{\infty}$ defined by (3.10) is a Cauchy sequence in $(X, \|\cdot\|)$. By $d(x,y) \leqslant \|x-y\|$ for all $x,y \in X$, one has that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the complete metric space (X,d).

Hence, it is convergent. Let us denote $x^* = \lim_{n\to\infty} x_n$. By (ii), we can obtain $x^* \in$Fix(T_{ab}). By (iii), we have that Fix(T)=Fix$(T_{ab}) = \{x^*\}$. Hence, all conclusions have been proved. □

Remark 3.7. If $d(x,y) = \|x-y\|$ for all $x,y \in X$, then by Theorem 3.6 we obtain Theorem 3.4. In this case, the two estimates (3.11) and (3.12) in Theorem 3.6 can be merged to yield the unified estimate in Theorem 3.4.

4. (a,b,p,q)-ENRICHED ĆIRIĆ-REICH-RUS CONTRACTIONS

In 1971, Rus [28] established the following fixed point theorem.

Theorem 4.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a mapping which there exist numbers $\alpha, \beta \in [0,+\infty)$, $\alpha + 2\beta < 1$, such that*

$$d(Tx,Ty) \leqslant \alpha d(x,y) + \beta(d(x,Tx) + d(y,Ty)),\ \forall x,y \in X, \tag{4.1}$$

Then T has a unique fixed point.

Note that this result was proved independently also by Ćirić [12] and Reich [27]. One also say that the mapping T satisfies (4.1) is a Ćirić-Reich-Rus type contraction mapping; see, e.g., [1, 20, 24].

On the other hand, Theorem 4.1 combines and improves both the Banach contraction principle and the Kannan fixed point theorem. Inspired by Berinde and Păcurar [5], the aim of this section is to unify and use Theorems 2.5, 2.8, 3.4, and 3.6 to obtain a Maia type fixed point theorem for (a,b,p,q)-enriched Ćirić-Reich-Rus contractions in Banach spaces.

For what follows, we recall the following concept and theorem.

Definition 4.2. [5] Let $(X,\|\cdot\|)$ be a linear normal space. A mapping $T : X \to X$ is said to be a (k,a,b)-enriched Ćirić-Reich-Rus contraction if there exist $a,b \geqslant 0$ satisfying $a+2b < 1$ and $k \in [0,\infty)$ such that

$$\|k(x-y) + Tx - Ty\| \leqslant a\|x-y\| + b(\|x-Tx\| + \|y-Ty\|),\ \forall x,y \in X. \tag{4.2}$$

Theorem 4.3. [5] *Let $(X,\|\cdot\|)$ be a Banach space and $T : X \to X$ be a (k,a,b)-enriched Ćirić−Reich-Rus contraction. Then*

(i) Fix $(T) = p$ for some $p \in X$;

(ii) There exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, give by $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$ for $n \geqslant 0$, converges to p, for any $x_0 \in X$;

(iii) The following estimate holds $\|x_{n+i-1} - p\| \leqslant \frac{c^i}{1-c} \cdot \|x_n - x_{n-1}\|$ for $n = 1,2,\ldots; i = 1,2,\ldots,$ where $c = \frac{a+b}{1-b}$.

A new definition which unifies and generalizes Definitions 2.3 and 3.3 and some results concerning the new definition are given below.

Definition 4.4. Let $(X,\|\cdot\|)$ be a linear normal space and $T : X \to X$ be a given mapping. We say that T is a (a,b,p,q)-enriched Ćirić−Reich-Rus contraction, if there exist $a,b \in (0,+\infty)$ and $p,q \geqslant 0$ satisfying $p+2q < b$ such that

$$\|a(x-y) + b(Tx-Ty)\| \leqslant p\|x-y\| + q(\|x-Tx\| + \|y-Ty\|),\ \forall x,y \in X. \tag{4.3}$$

Theorem 4.5. *Let $(X,\|\cdot\|)$ be a Banach space and $T:X\to X$ be a (a,b,p,q)-enriched Ćirić$-$Reich-Rus contraction. Then*

(i) Fix $(T)=x^*$, for some $x^*\in X$;

(ii) The iterative sequence $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1}=\frac{a}{a+b}x_n+\frac{b}{a+b}Tx_n,\ n\geqslant 0, \tag{4.4}$$

converges to x^*, for any $x_0\in X$;

(iii) The following estimate holds $\|x_{n+i-1}-x^*\|\leqslant\frac{\sigma^i}{1-\sigma}\cdot\|x_n-x_{n-1}\|, n=1,2,\dots;i=1,2,\dots,$ where $\sigma=\frac{p+q}{b-q}$.

Proof. First, we consider the mapping T_{ab} defined by (2.5). By hypothesis, one has

$$\|\frac{a}{a+b}(x-y)+\frac{b}{a+b}(Tx-Ty)\|\leqslant\frac{p}{a+b}\|x-y\|+\frac{q}{a+b}(\|x-Tx\|+\|y-Ty\|),\ \forall x,y\in X,$$

which yields that

$$\|T_{ab}x-T_{ab}y\|\leqslant\frac{p}{a+b}\|x-y\|+\frac{q}{b}(\|x-T_{ab}x\|+\|y-T_{ab}y\|),\ \forall x,y\in X.$$

Since $a,b>0$, we have

$$\|T_{ab}x-T_{ab}y\|\leqslant\frac{p}{b}\|x-y\|+\frac{q}{b}(\|x-T_{ab}x\|+\|y-T_{ab}y\|),\ \forall x,y\in X. \tag{4.5}$$

Using the triangle inequality in (4.5) yields that

$$\|T_{ab}x-T_{ab}y\|\leqslant\sigma\|x-y\|+2\sigma\|y-T_{ab}x\|,\ \forall x,y\in X, \tag{4.6}$$

where $\sigma=\frac{p+q}{b-q}<1$. According to (2.5), the iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (4.4) is the Picard iteration associated to T_{ab}, that is, $x_{n+1}=T_{ab}x_n$ for $n\geqslant 0$. Letting $x=x_{n-1}$ and $y=x_n$ in (4.6), one has $\|x_{n+1}-x_n\|\leqslant\sigma\|x_n-x_{n-1}\|$ for $n\geqslant 1$, and further $\|x_{n+1}-x_n\|\leqslant\sigma^n\|x_1-x_0\|$ for $n\geqslant 0$. Hence, one has

$$\|x_{n+m}-x_n\|\leqslant\sigma\cdot\frac{1-\sigma^m}{1-\sigma}\cdot\|x_n-x_{n-1}\|,\ n\geqslant 1,m\geqslant 1,$$

and

$$\|x_{n+m}-x_n\|\leqslant\sigma^n\cdot\frac{1-\sigma^m}{1-\sigma}\cdot\|x_1-x_0\|,\ n\geqslant 0,m\geqslant 1.$$

Because $\sigma\in(0,1)$, we claim that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since X is complete, there is $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$, that is, $\lim_{n\to\infty}x_n=x^*$. Now we prove that x^* is a fixed point of T_{ab}. Observe that

$$\|x^*-T_{ab}x^*\|\leqslant\|x^*-x_{n+1}\|+\|x_{n+1}-T_{ab}x^*\|=\|x_{n+1}-x^*\|+\|T_{ab}x_n-T_{ab}x^*\|. \tag{4.7}$$

It follows from (4.6) that $\|T_{ab}x_n-T_{ab}x^*\|\leqslant\sigma\|x_n-x^*\|+2\sigma\|x^*-T_{ab}x_n\|$. Therefore, (4.7) yields that $\|x^*-T_{ab}x^*\|\leqslant(2\sigma+1)\|x_{n+1}-x^*\|+\sigma\|x_n-x^*\|$. By letting $n\to\infty$ in the inequality above, we see that $\|x^*-T_{ab}x^*\|=0$, that is, $x^*=T_{ab}x^*$. So, $x^*\in$Fix(T_{ab}).

Next, we prove that x^* is the unique fixed point of T_{ab}. Suppose that $y^*\neq x^*$ is another fixed point of T_{ab}. By (4.5), we have $0<\|x^*-y^*\|\leqslant\frac{p}{b}\cdot\|x^*-y^*\|<\|x^*-y^*\|$, which is a contradiction. Hence Fix(T)=Fix(T_{ab})=$\{x^*\}$. (i) is proven. Conclusion (ii) is obvious. The rest of the proof is similar to that of Theorem 2.5. □

Theorem 4.6. *Let X be a linear vector space endowed with d and a norm $\|\cdot\|$, which satisfies the condition $d(x,y) \leqslant \|x-y\|$ for all $x,y \in X$. Suppose*

(i) (X,d) is a complete metric space;

(ii) $T: X \to X$ is continuous with respect to d;

(iii) T is a (a,b,p,q)-enriched Ćirić−Reich-Rus contraction, with respect to $\|\cdot\|$, that is, there exist $a,b \in (0,+\infty)$ and $p+2q<b$ such that (4.3) holds.

Then

(i) Fix $(T) = x^*$ for some $x^* \in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = \frac{a}{a+b}x_n + \frac{b}{a+b}Tx_n$ for $n \geqslant 0$, converges to x^*, for any $x_0 \in X$;

(iii) the following estimates hold

$$\|x_n - x^*\| \leqslant \frac{\sigma}{1-\sigma} \cdot \|x_n - x_{n-1}\|,\ n \geqslant 1 \tag{4.8}$$

and

$$\|x_n - x^*\| \leqslant \frac{\sigma^n}{1-\sigma} \cdot \|x_1 - x_0\|,\ n \geqslant 1, \tag{4.9}$$

where $\sigma = \frac{p+q}{b-q}$.

Proof. From Theorems 2.8 and 4.5, the proof can be immediately proved. □

Remark 4.7. (1) It is easy to see that any (a,b,p,q)-enriched Ćirić-Reich-Rus contraction with $a=0, b\neq 0$ is a Ćirić-Reich-Rus type contraction mapping. From Theorem 4.5, we can deduce Theorem 4.1. While, if $a \geqslant 0, b=1$, then (a,b,p,q)-enriched Ćirić-Reich-Rus contractions are (a,p,q)-enriched Ćirić-Reich-Rus contractions, i.e., it satisfies (4.2). By Theorem 4.5, we can obtain Theorem 4.3.

(2) Obviously, any (a,b,θ)-enriched contraction satisfies (4.3) with $q=0$, and any (a,b,k)-enriched Kannan contractive mapping also satisfies (4.3) with $p=0$. Then by Theorem 4.5, we can have Theorem 2.5 and Theorem 3.4.

(3) If $d(x,y) = \|x-y\|$ for all $x,y \in X$, then by Theorem 4.6 we obtain Theorem 4.5. In this case, estimates (4.8) and (4.9) in Theorem 4.6 can be merged to yield the unified estimate in Theorem 4.5.

5. (a,b,l)-Enriched Chatterjea Mappings

In 1972, Chatterjea [11] established a fixed point theorem for a similar type of Kannan contractive condition.

Theorem 5.1. *Let (X,d) be a complete metric space and $T: X \to X$ be satisfied*

$$d(Tx,Ty) \leqslant \beta(d(x,Ty)+d(y,Tx)),\ \forall x,y \in X, \tag{5.1}$$

where $\beta \in [0,\frac{1}{2})$ (known as Chatterjea contractions), then T has a unique fixed point.

It is easy to verify that all Banach contractions with constant $r<\frac{1}{3}$ and all Kannan mappings with Kannan constant $\alpha < \frac{1}{4}$ are Chatterjea contractions. That is, if $r \in [0,\frac{1}{3}), \alpha \in [0,\frac{1}{4})$, we can deduce Theorem 5.1 by Theorem 1.1 and Kannan fixed point theorem satisfies (3.1), respectively. Other fixed point theorems, related to the Chatterjea contractions, were subsequently established by various authors recently; see, e.g, [10, 15, 25]. The concept of enriched

Chatterjea mappings was introduced and studied in [6] as a generalization of that of Chatterjea contractions [11].

Definition 5.2. [11] Let $(X,\|\cdot\|)$ be a linear normal space. A mapping $T: X \to X$ is said to be a (k,a)-enriched Chatterjea mapping if there exist $b \in [0,\frac{1}{2})$ and $k \in [0,\infty)$ such that

$$\|k(x-y)+Tx-Ty\| \leqslant b[\|(k+1)(x-y)+y-Ty\|+\|(k+1)(y-x)+x-Tx\|], \ \forall x,y \in X. \tag{5.2}$$

Theorem 5.3. [11] *Let $(X,\|\cdot\|)$ be a Banach space and $T: X \to X$ be a (k,b)-enriched Chatterjea mapping. Then*

(i) Fix $(T) = p$ for some $p \in X$;

(ii) there exists $\lambda \in (0,1]$ such that the iterative method $\{x_n\}_{n=0}^{\infty}$, give by $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$, $n \geqslant 0$, converges to p, for any $x_0 \in X$;

(iii) the following estimate holds

$$\|x_{n+i-1}-p\| \leqslant \frac{c^i}{1-c} \cdot \|x_n - x_{n-1}\|, \ n = 1,2,\ldots; i = 1,2,\ldots,$$

where $c = \frac{b}{1-b}$.

The aim of this section is to apply the method for (a,b)-enriching contractive type mappings to the class of Chatterjea mappings, that is, (a,b,l)-enriched Chatterjea mappings, that generalize (k,a)-enriched Chatterjea mappings. We prove a fixed point theorem and a Maia type fixed point theorem for this new class of mappings in Banach spaces.

Definition 5.4. Let $(X,\|\cdot\|)$ be a linear normal space. A mapping $T: X \to X$ is said to be a (a,b,l)-enriched Chatterjea mappings if there exist $a,b \in (0,+\infty)$ and $l \in [0,\frac{1}{2}]$ such that

$$\|a(x-y)+b(Tx-Ty)\| \leqslant l[\|(a+b)(x-y)+b(y-Ty)\|+\|(a+b)(y-x)+b(x-Tx)\|], \ \forall x,y \in X. \tag{5.3}$$

Theorem 5.5. *Let $(X,\|\cdot\|)$ be a Banach space and $T: X \to X$ be a (a,b,l)-enriched Chatterjea mappings. Then*

(i) Fix $(T) = x^*$, for some $x^* \in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = \frac{a}{a+b}x_n + \frac{b}{a+b}Tx_n, \ n \geqslant 0, \tag{5.4}$$

converges to x^*, for any $x_0 \in X$;

(iii) the following estimate holds $\|x_{n+i-1}-x^*\| \leqslant \frac{\tau^i}{1-\tau} \cdot \|x_n - x_{n-1}\|$ for $n = 1,2,\ldots; i = 1,2,\ldots$, where $\tau = \frac{l}{1-l}$.

Proof. Here we still consider the mapping T_{ab} defined by (2.5). By (5.3), we have

$$\|\frac{a}{a+b}(x-y)+\frac{b}{a+b}(Tx-Ty)\| \leqslant l[\|x-y+\frac{b}{a+b}(y-Ty)\|+\|y-x+\frac{b}{a+b}(x-Tx)\|], \ \forall x,y \in X,$$

and then,

$$\|T_{ab}x - T_{ab}y\| \leqslant l(\|x - T_{ab}y\| + \|y - T_{ab}x\|), \ \forall x,y \in X, \tag{5.5}$$

which indicates that T_{ab} is a Chatterjea contraction in the sense of (5.1).

Now we work in the case that $l > 0$ (the case $l = 0$ is immediate). According to (2.5), the iteration sequence $\{x_n\}_{n=0}^{\infty}$ defined by (5.4) is the Picard iteration associated to T_{ab}, that is, $x_{n+1} = T_{ab}x_n$, $n \geqslant 0$. Setting $x = x_n$ and $y = x_{n-1}$ in (5.5), one has

$$\|x_{n+1} - x_n\| \leqslant l(\|x_{n+1} - x_n\| + \|x_n - x_{n-1}\|),$$

which yields $\|x_{n+1} - x_n\| \leqslant \frac{l}{1-l}\|x_n - x_{n-1}\|$ for $n \geqslant 1$. Since $l \in (0, \frac{1}{2})$, we can denote $\tau = \frac{l}{1-l}$ and have $\tau \in (0,1)$. Hence, we have $\|x_{n+1} - x_n\| \leqslant \tau\|x_n - x_{n-1}\|$ for $n \geqslant 1$, and then $\|x_{n+1} - x_n\| \leqslant \tau^n\|x_1 - x_0\|$ for $n \geqslant 0$. These imply that

$$\|x_{n+m} - x_n\| \leqslant \tau \cdot \frac{1-\tau^m}{1-\tau} \cdot \|x_n - x_{n-1}\|, \ n \geqslant 1, m \geqslant 1,$$

and

$$\|x_{n+m} - x_n\| \leqslant \tau^n \cdot \frac{1-\tau^m}{1-\tau} \cdot \|x_1 - x_0\|, \ n \geqslant 0, m \geqslant 1.$$

Hence, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(X, \|\cdot\|)$ and it is convergent. Hence, there is $x^* \in X$ such that $\lim\limits_{n\to\infty} x_n = x^*$. Now we prove that x^* is a fixed point of T_{ab}. It is easy to see that

$$\|x^* - T_{ab}x^*\| \leqslant \|x^* - x_{n+1}\| + \|x_{n+1} - T_{ab}x^*\| = \|x_{n+1} - x^*\| + \|T_{ab}x_n - T_{ab}x^*\|. \tag{5.6}$$

It follows from (5.5) that $\|T_{ab}x_n - T_{ab}x^*\| \leqslant l(\|x_n - T_{ab}x^*\| + \|x^* - T_{ab}x_n\|)$, In view of (5.6), we have $\|x^* - T_{ab}x^*\| \leqslant (l+1)\|x_{n+1} - x^*\| + l(\|x_n - x^*\| + \|x^* - T_{ab}x^*\|)$, which yields

$$\|x^* - T_{ab}x^*\| \leqslant \frac{l+1}{1-l} \cdot \|x_{n+1} - x^*\| + \tau\|x_n - x^*\|, \ n \geqslant 0.$$

letting $n \to \infty$ in the inequality above, we have $\|x^* - T_{ab}x^*\| = 0$, that is, $x^* = T_{ab}x^*$. So, $x^* \in$Fix(T_{ab}).

Next we prove that x^* is the unique fixed point of T_{ab}. Suppose that $y^* \neq x^*$ is another fixed point of T_{ab}. By (5.5) with $x = x^*$ and $y = y^*$, it follows $0 < \|x^* - y^*\| \leqslant 2l\|x^* - y^*\| < \|x^* - y^*\|$, which a contradiction. Hence Fix(T)=Fix(T_{ab})=$\{x^*\}$. Claim (i) is proven. In view of $\lim_{n\to\infty} x_n = x^*$, one has Conclusion (ii). The proof of Conclusion (iii) is similar to that of Theorem 2.5. This completes the proof. □

Theorem 5.6. *Let X be a linear vector space endowed with d and a norm $\|\cdot\|$ satisfying the condition $d(x,y) \leqslant \|x - y\|$ for all $x, y \in X$. Suppose*

(i) (X,d) is a complete metric space;

(ii) $T : X \to X$ is continuous with respect to d;

(iii) T is a (a,b,l)-enriched Chatterjea mappings, with respect to $\|\cdot\|$, that is, there exist $a, b \in (0,+\infty)$ and $l \in [0, \frac{1}{2})$ such that (5.3) holds.

Then

(i) Fix $(T) = x^*$, for some $x^* \in X$;

(ii) the iterative sequence $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = \frac{a}{a+b}x_n + \frac{b}{a+b}Tx_n$ for $n \geqslant 0$, converges to x^* for any $x_0 \in X$;

(iii) the following estimates hold

$$\|x_n - x^*\| \leqslant \frac{\tau}{1-\tau} \cdot \|x_n - x_{n-1}\|, \ n \geqslant 1, \tag{5.7}$$

and

$$\|x_n - x^*\| \leqslant \frac{\tau^n}{1-\tau} \cdot \|x_1 - x_0\|,\ n \geqslant 1, \tag{5.8}$$

where $\tau = \frac{l}{1-l}$.

Proof. From Theorems 2.8 and 5.5, we obtain the desired conclusion immediately. □

Remark 5.7. (1) If $a = 0, b \neq 0$, then the (a,b,l)-enriched Chatterjea mappings are Chatterjea contractions. By using Theorem 5.5, we can obtain Theorem 5.1. If $a \geqslant 0, b = 1$, then the (a,b,l)-enriched Chatterjea mappings are (a,l)-enriched Chatterjea mappings, that is, they satisfy (5.2). From Theorem 5.5, we can obtain Theorem 5.3.

(2) If $d(x,y) = \|x-y\|$ for all $x,y \in X$, then Theorem 5.6 yields Theorem 5.5. In this case, estimates (5.7) and (5.8) in Theorem 5.6 can be merged to yield the unified estimate in Theorem 5.3.

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