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MULTIPLE SOLUTIONS FOR A CLASS OF NON-HOMOGENEOUS $p(x)$-KIRCHHOFF TYPE EQUATIONS

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Abstract. The main aim of this paper is to investigate the existence of nontrivial solutions for a class of variable exponent $p(x)$-Kirchhoff type equations. We prove the existence of three solutions by using the mountain pass theorem and Ekeland's variational principle. Moreover, when $\lambda = 0$, we obtain the existence of infinite many solutions by using the symmetric mountain pass theorem.

Keywords. Multiplicity; $p(x)$-Kirchhoff problem; Variable exponent; Variational methods.

1. Introduction and Main Results

In this work, we consider the following nonlocal problem

$$\begin{cases} -\left(a-b\int_{\Omega}\frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)\Delta_{p(x)}u = g(x,u)+\lambda f(x), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N (N \geq 1)$, with smooth boundary $\partial\Omega$, λ is a positive parameter, $\Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator, and g satisfies the following hypotheses:

(g_1) for any $(x,s) \in \Omega \times \mathbb{R}$, $g(x,s)$ satisfies the following subcritical growth condition:

$$|g(x,s)| \leq C(1+|s|^{q(x)-1}),$$

where C is a positive constant and $p(x) < q(x) < p^*(x)$;

(g_2) $\lim\limits_{s\to 0}\frac{g(x,s)}{|s|^{p(x)-2}s} = 0$;

(g_3) for any $x \in \Omega$, $\lim\limits_{s\to +\infty}\frac{G(x,s)}{|s|^{p^+}} = +\infty$;

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(g_4) there exist a continuous function $\sigma : \overline{\Omega} \to [0, +\infty)$ and $a_1, a_2 > 0$, $\mu \in (p^+, \frac{2(p^-)^2}{p^+})$ such that $\sigma^+ < p^+$ and

$$\frac{1}{\mu} g(x,s)s - G(x,s) \geq -a_1 - a_2 |s|^{\sigma(x)}, \text{ for each } (x,s) \in \Omega \times \mathbb{R};$$

(g_5) $g(x,-t) = -g(x,t)$, for any $(x,t) \in \Omega \times \mathbb{R}$.

In recent years, much attention has been paid to various problems with variable exponential growth conditions, we refer to [2, 8, 12, 21, 24] for related results. The reason why people are interested in this is that the problems with variable exponents have a wide range of real applications, such as elastic mechanics and electrorheological fluids [25], continuum mechanics [5], dielectric breakdown, electrical resistivity and polycrystal plasticity [6] and image restoration [9].

It is known that problem (1.1) is related to the stationary problem of a model introduced by Kirchhoff [22]. To be more precise, Kirchhoff gave the following physical model

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L} \int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 dx \Big) \frac{\partial^2 u}{\partial x^2} = 0.$$

The above model extends the classical D'Alemberts wave equation by considering the effects of the changes in the length of the strings during the vibrations. Lately, Kirchhoff type equations were extended to the following p-Laplacian Kirchhoff type equations by many authors

$$-M \left(\int_\Omega |\nabla u|^p dx \right) \Delta_p u = f(x,u), \text{ in } \Omega, \tag{1.2}$$

where $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian; see, e.g., [1, 7, 15, 26, 29].

Particularly, under the conditions on $M(t) = a - bt$, $f(x,u) = |u|^2 u + \mu h(x)$, $p = 2$, and $\Omega = \mathbb{R}^4$, the existence of at least two positive solutions for (1.2) was obtained in [26] by using variational methods, where a, b are positive constants, μ is a non-negative parameter, and $h(x) \in L^{\frac{4}{3}}(\mathbb{R}^4)$ is a non-negative function. Under the conditions on $M(t) = a - bt$, $f(x,u) = |u|^{q-2} u$, $p = 2$, $\Omega \subset \mathbb{R}^N$, at least a nontrivial non-negative solution and a nontrivial non-positive solution for (1.2) were demonstrated in [29] by variational methods, where $q \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = +\infty$ otherwise.

As a natural generalization of p-Laplacian operator, $p(x)$-Laplacian operator has strong inhomogeneity. In recent years, a large number of results with $p(x)$-Laplacian operator have appeared; see, e.g., [3, 10, 11, 12, 13, 17, 19].

In [17], the following equation

$$\begin{cases} -\left(a - b \displaystyle\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx \right) \Delta_{p(x)} u = g(x,u) + \lambda |u|^{p(x)-2} u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

was studied, in which a mountain pass solution to (1.3) was obtained under the assumption that g satisfies some appropriate conditions, where $a \geq b$ are positive constants, λ is a real parameter, $p \in C(\overline{\Omega})$ with $N > p(x) > 1$, and $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain.

In [10], the following kind of fourth order elliptic variable exponent Kirchhoff type equations

$$\begin{cases} \Delta^2_{p(x)}u - M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\Delta_{p(x)}u = f(x,u)+h(x), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

were considered, in which at least two nontrivial solutions to equation (1.4) were obtained by using variational argument under certain conditions for h, p and Ambrosetti-Rabinowitz type conditions for f, where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a smooth bounded domain, $M(t) = a + bt^k$, $a,k>0$, $b \geq 0$, and $\Delta^2_{p(x)}u = \Delta(|\Delta u|^{p(x)-2}\Delta u)$ is the $p(x)$-biharmonic operator.

Inspired by [10, 17], a natural question is that if we can generalize the results of [26, 29] to the case of $p(x)$-Kirchhoff variable exponent and obtain the existence of multiple nontrivial solutions. This paper will give an affirmative answer to this question. In the present work, we will prove the existence and multiplicity of nontrivial solutions for the perturbation problem (1.1) involving the following negative nonlocal item

$$a - b\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right).$$

The main results extend in several directions previous results recently appeared in the literature (for example [18, 26, 29]). The difficulty in this case is the Palais-Smale condition for the corresponding energy functional could not be checked directly. To overcome this difficulty, we must give a threshold value of J to prove the Palais-Smale condition. As far as we know, the results of the present paper do not appear in the existing literature.

Our main results are as follows.

Theorem 1.1. *Suppose that conditions* (g_1)-(g_4) *hold. If function* $q \in C(\overline{\Omega})$ *satisfies*

$$1 < p^- < p(x) < p^+ < q^- < q(x) < p^*(x) \ \ \text{and} \ \ p^+ < 2p^-, \tag{1.5}$$

then there exist a constant $\lambda^* > 0$ *such that, when* $\lambda \in (0,\lambda^*)$*, problem* (1.1) *has at least three nontrivial weak solutions.*

Theorem 1.2. *Suppose that* $\lambda = 0$ *and* (g_1)-(g_5) *hold. If function* $q \in C(\overline{\Omega})$ *satisfies* (1.5)*, then problem* (1.1) *has a sequence of weak solutions* $\pm u_k$ *such that* $I(\pm u_k) \to +\infty$ *as* $k \to +\infty$*.*

Remark 1.3. In [10, 17], the nonlinearities satisfy the (AR) condition, whereas the nonlinearity g considered in this paper does not satisfy the (AR) condition. Therefore, this paper is a generalization of the results in [10, 17].

Now, we give the organizational structure of this article as follows. In Section 2, we will recall some basic facts about the variable exponent Lebesgue and Sobolev spaces. Proof of Theorem 1.1 is given in Section 3. Section 4 is devoted to proving Theorem 1.2.

2. PRELIMINARIES

Let Ω be a bounded domain of $\mathbb{R}^N$ for any $x \in \overline{\Omega}$. Let $p^- = \inf_\Omega p(x) \leq p(x) \leq p^+ = \sup_\Omega p(x) < N$ and denote $C_+(\overline{\Omega}) = \{p(x) : p(x) \in C(\overline{\Omega}), p(x) > 1\}$.

We introduce the following variable exponential Lebesgue space:

$$L^{p(\cdot)}(\Omega) = \Big\{u : \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\},$$

where $p(x) \in C_+(\overline{\Omega})$, u is a measurable real-valued function, and endowed with the so-called Luxemburg norm

$$||u||_{L^{p(x)}(\Omega)} = |u|_{p(\cdot)} = \inf\Big\{\mu > 0 : \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{p(x)} dx < 1\Big\}.$$

Obviously, $L^{p(\cdot)}(\Omega)$ is a separable and reflexive Banach space. For basic properties of the variable exponent Lebesgue spaces, we refer to [16, 23, 28] and the references therein.

Proposition 2.1. [28] *The space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is uniformly convex , separable, and reflexive, and its the corresponding conjugate space is $(L^{q(x)}(\Omega), |.|_{q(x)})$, where $q(x)$ and $p(x)$ are conjugate functions of each other. That is, for any $x \in \Omega$,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1.$$

For each $v \in L^{q(x)}(\Omega), u \in L^{p(x)}(\Omega)$, the following Hölder type inequality

$$\Big|\int_\Omega uvdx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}$$

holds.

The inclusion between Lebesgue spaces also extends the classical variational framework, that is, if p_1 and p_2 are variable exponents such that $p_1 \leq p_2$ in Ω, and $0 < |\Omega| < \infty$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous. Let the module of generalized Lebesgue-Sobolev space $L^{p(\cdot)}(\Omega)$ be $\rho_{p(\cdot)}$, and

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx,$$

which plays an important role in the generalized Lebesgue-Sobolev space $L^{p(\cdot)}(\Omega)$.

Lemma 2.2. [14] *Assume that $p^+ < +\infty$ and $u_n, u \in L^{p(\cdot)}$. Then we have the following properties:*

1. $|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^+}_{p(\cdot)}$;
2. $|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \leq \rho_{p(\cdot)}(u) \leq |u|^{p^-}_{p(\cdot)}$;
3. $|u|_{p(\cdot)} < 1$ *(individually*$,= 1; > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) \leq 1$ *(individually*$,= 1; > 1)$*;*
4. $|u_n|_{p(\cdot)} \to 0$ *(individually*$, \to +\infty) \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to 0$ *(individually*$, \to +\infty)$*;*
5. $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}|u_n - u|_{p(x)} = 0$.

The Sobolev space $W^{1,p(x)}(\Omega)$ with variable exponent is defined by

$$W^{1,p(x)}(\Omega) := \Big\{u : \Omega \subset \mathbb{R}^N \to \mathbb{R} : u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

and the corresponding norm is

$$||u||_{1,p(x)} = ||\nabla u||_{p(x)} + ||u||_{p(x)}.$$

Then the closure of $C_0^\infty(\Omega)$ under norm $||u||_{1,p(x)}$ is defined by $W_0^{1,p(x)}(\Omega)$. According to the above information, $W^{1,p(x)}(\Omega)$, $W_0^{1,p(x)}(\Omega)$, and $L^{p(x)}(\Omega)$ become reflexive Banach spaces and separeble. For more information, we refer to [14, 21]. In addition, we define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ +\infty, & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.3. *(Sobolev Embedding [21]). For $p, q \in C_+(\overline{\Omega})$ such that $1 \leq q(x) \leq p^*(x)$, there is a continuous embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega) \text{ for each } x \in \overline{\Omega}.$$

Furthermore, if $1 < q(x) < p^*(x)$, then, for any $x \in \overline{\Omega}$, embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.

Proposition 2.4. *(Poincaré Inequality [21]). For any $u \in W_0^{1,p(x)}(\Omega)$, there is a positive constant C such that $||u||_{L^{p(x)}(\Omega)} \leq C||\nabla u||_{L^{p(x)}(\Omega)}$.*

Remark 2.5. According to Proposition 2.4, it is not difficult for us to find that $||\nabla u||_{L^{p(x)}(\Omega)}$ and $||u||_{W^{1,p(x)}(\Omega)}$ are equivalent on $W_0^{1,p(x)}(\Omega)$.

Lemma 2.6. *([20]). For any $u \in W_0^{1,p(x)}(\Omega)$, we first define $A(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx$. Then it is not difficult to verify that $A(u) \in C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$, and, for any $u, v \in W_0^{1,p(x)}(\Omega)$, derivative given by $\langle A'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx$. Moreover, A and A' have the following properties:*

1. for any $u \in W_0^{1,p(x)}(\Omega)$, $A(u) \to +\infty$ as $||u|| \to +\infty$;

2. $A' : W_0^{1,p(x)}(\Omega) \to (W^{-1,p'(x)}(\Omega)) = (W_0^{1,p(x)}(\Omega))^*$ is a strictly monotone and bounded homeomorphism operator, where $p(x)$ and $p'(x)$ satisfies $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$;

3. A' is an S_+-type mapping, that is, if $\limsup\langle A'(u_n), u_n - u\rangle \leq 0$ and $u_n \rightharpoonup u$, then $u_n \to u$ (strongly) in $W_0^{1,p(x)}(\Omega)$.

Definition 2.7. If, for any $\varphi \in W_0^{1,p(x)}(\Omega)$,

$$\Big(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi dx = \int_\Omega g(x,u)\varphi dx - \lambda\int_\Omega f(x)\varphi dx,$$

then the function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to (1.1).

Define the energy functional $J : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ associated with problem (1.1) by

$$J(u) = a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 - \int_\Omega G(x,u)dx$$
$$- \lambda\int_\Omega f(x)udx.$$

It is not difficult to find that $J(u)$ is well defined and of C^1 class on $W_0^{1,p(x)}(\Omega)$. Furthermore, for any $u, \varphi \in W_0^{1,p(x)}(\Omega)$, we have

$$\langle J'(u), \varphi\rangle = \Big(a - b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi dx - \int_\Omega g(x,u)\varphi dx$$
$$- \lambda\int_\Omega f(x)\varphi dx.$$

Therefore, if a function u is the weak solution of problem (1.1) if and only if u is the critical point of functional J. In addition, for the sake of simplicity, we will denote the norm of $W_0^{1,p(x)}(\Omega)$ by $||.||$ instead of $||.||_{W_0^{1,p(x)}(\Omega)}$.

3. Proof of Theorem 1.1

Now, let us briefly review the definition of the Palais-Smale compactness condition.

Definition 3.1. [17] Let $(W_0^{1,p(x)}(\Omega),||.||)$ be a complete normed linear space and $J\in C^1(W_0^{1,p(x)}(\Omega))$. Given $c\in\mathbb{R}$, if any sequence $\{u_n\}\in W_0^{1,p(x)}(\Omega)$ satisfying

$$J(u_n)\to c \text{ and } J'(u_n)\to 0 \text{ in } W_0^{-1,p'(x)}(\Omega) \text{ as } n\to\infty, \tag{3.1}$$

and for sequence $\{u_n\}$ there exists a convergent subsequence, then we say that J satisfies the Palais-Smale condition at the level $c\in\mathbb{R}$ ("$(PS)_c$" condition for short).

Next, we give the proof of the $(PS)_c$ condition of functional J. In the following, we denotes $W_0^{1,p(x)}(\Omega)$ by X for simplicity.

Lemma 3.2. *Suppose that the above conditions $(g_1)-(g_4)$ hold. When $c<\frac{a^2}{2b}$, functional J satisfies the $(PS)_c$ condition.*

Proof. We divide the proof into two steps.

Step 1. Verify that $\{u_n\}$ is bounded in X.

Let $\{u_n\}\subset X$ satisfy

$$J(u_n)\to c \text{ and } J'(u_n)\to 0 \text{ in } W_0^{-1,p'(x)}(\Omega) \text{ as } n\to\infty,$$

and $c<\frac{a^2}{2b}$. For n large enough, by (3.1) and (g_4), we obtain that

$$\begin{aligned}
&o_n(1)+c\\
&\geq J(u_n)-\frac{1}{p^+}\langle J'(u_n),u_n\rangle\\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx-\frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)^2-\int_\Omega G(x,u_n)dx\\
&\quad -\lambda\int_\Omega f(x)u_ndx-\frac{1}{p^+}\Big(\Big[a-b\int_\Omega\frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big]\int_\Omega|\nabla u_n|^{p(x)}dx-\int_\Omega g(x,u_n)u_ndx\Big)\\
&\quad +\frac{1}{p^+}\lambda\int_\Omega f(x)u_ndx\\
&\geq b\Big(\frac{1}{(p^+)^2}-\frac{1}{2(p^-)^2}\Big)\Big(\int_\Omega|\nabla u_n|^{p(x)}dx\Big)^2-\lambda\Big(1-\frac{1}{p^+}\Big)\int_\Omega f(x)u_ndx\\
&\quad -\int_\Omega\Big(G(x,u_n)-\frac{1}{p^+}g(x,u_n)u_n\Big)dx\\
&\geq b\Big(\frac{1}{(p^+)^2}-\frac{1}{2(p^-)^2}\Big)\Big(\int_\Omega|\nabla u_n|^{p(x)}dx\Big)^2-a_1|\Omega|-a_2\int_\Omega|u_n|^{\sigma(x)}dx\\
&\quad -\lambda\Big(1-\frac{1}{p^+}\Big)\int_\Omega f(x)u_ndx.
\end{aligned}$$

So, we can deduce that

$$o_n(1)+c \geq \Big(\frac{1}{(p^+)^2}-\frac{1}{2(p^-)^2}\Big)b||u_n||^{2p^-}-a_1|\Omega|-a_2||u_n||^{p^+}$$
$$-\Big(1-\frac{1}{p^+}\Big)|f|_{\frac{6}{5}}S^{-\frac{1}{2}}||u||.$$

It follows from (1.5) that $\{u_n\}$ is bounded in X.

Step 2. Prove that there exists $u \in X$ such that $u_n \to u$ in X.

According to Proposition 2.3, for $1 \leq s(x) < p^*(x)$, it is not difficult to find that the embedding $X \hookrightarrow L^{s(x)}(\Omega)$ is compact. Due to $\{u_n\}$ is bounded in X, then there exists $u \in X$ such that (up to a subsequence)

$$u_n \rightharpoonup u \text{ in } X,\ u_n \to u \text{ in } L^{s(x)}(\Omega),\ u_n(x) \to u(x) \text{ a.e. in } \Omega. \tag{3.2}$$

By condition (g_1) and (g_2), it is not difficult to verify that, for each $\varepsilon \in (0,1)$, there exists positive constant C_ε such that

$$|g(x,u_n)| \leq \varepsilon|u_n|^{p(x)-1}+C_\varepsilon|u_n|^{q(x)-1}. \tag{3.3}$$

According to (3.3) and Proposition 2.3, we obtain that

$$\lim_{n\to\infty}\int_\Omega g(x,u_n)(u_n-u)dx = 0. \tag{3.4}$$

From (3.1), we have $\langle J'(u_n), u_n-u\rangle = 0$ and then

$$\langle J'(u_n),u_n-u\rangle = \Big(a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx$$
$$-\int_\Omega g(x,u_n)(u_n-u)dx-\lambda\int_\Omega f(x)(u_n-u)dx \to 0.$$

So, using(3.4), we can easily verify that

$$\Big(a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)dx \to 0.$$

Since $\{u_n\}$ is bounded in X, we may assume that

$$\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to t_0 > 0 \text{ as } n\to\infty.$$

Case 1. If $t_0 = 0$, then $u_n \to u = 0$ in X, that is, the conclusion is established immediately.

Case 2. If $t_0 > 0$, we will consider it in two subcases:

Subcase 1. If $t_0 = \frac{a}{b}$, then $a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0$. For each $u \in W_0^{1,p(x)}(\Omega)$, we consider the following functional

$$\varphi(u) = \lambda\int_\Omega f(x)udx+\int_\Omega G(x,u)dx.$$

Obviously, for any $u,v \in W_0^{1,p(x)}(\Omega)$, we obtain that

$$\langle\varphi'(u),v\rangle = \lambda\int_\Omega f(x)vdx+\int_\Omega g(x,u)vdx,$$

from which we deduce

$$\langle\varphi'(u_n)-\varphi'(u),v\rangle = \int_\Omega (g(x,u_n)-g(x,u))vdx.$$

Subcase 2. If $t_0 \neq \frac{a}{b}$, then we can easily verify $a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \nrightarrow 0$ and no subsequence of $\{a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0\}$ converges to zero. Hence, there exists a positive constant δ such that $0<\delta<|a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx|$ when n is large enough. So,

$$\left\{a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0\right\} \text{ is bounded.}$$

To end the proof, we also need the following lemma.

Lemma 3.3. [17] *Let $u_n, u \in X$ satisfies (3.2). Then, for any $v \in X$, the following properties hold:*

(i) $\lim_{n\to\infty}\int_\Omega |g(x,u_n)-g(x,u)||v|dx = 0$;

(ii) $\langle \varphi'(u_n)-\varphi'(u), v\rangle \to 0$ *as* $n \to +\infty$.

Next, we complete the proof of lemma 3.2.

Since $\langle J'(u),v\rangle = (a-b\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx)\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v dx - \langle \varphi'(u),v\rangle$, $\langle J'(u_n),v\rangle \to 0$, and $a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \to 0$, then it follows from lemma 3.3 that $\varphi'(u_n)\to 0$ $(n\to\infty)$, namely, for each $v \in W_0^{1,p(x)}(\Omega)$, we have

$$\langle \varphi'(u),v\rangle = \lambda\int_\Omega f(x)vdx + \int_\Omega g(x,u)vdx.$$

Thus, for a.e. $x\in\Omega$, we obtain that $\lambda f(x)+g(x,u(x))=0$. Then, it follows from the fundamental lemma of the variational method (see [27]) that $u=0$. This implies that

$$\varphi(u_n) = \lambda\int_\Omega f(x)u_n dx + \int_\Omega G(x,u_n)dx \to \lambda\int_\Omega f(x)udx + \int_\Omega G(x,u)dx = 0.$$

Therefore, we can easily see that, for $t_0 = \frac{a}{b}$,

$$\begin{aligned} J(u_n) =& a\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\Big)^2 - \int_\Omega G(x,u_n)dx \\ &-\lambda\int_\Omega f(x)u_n dx \to \frac{a^2}{2b}. \end{aligned}$$

Obviously, this is a contradiction due to $J(u_n)\to c < \frac{a^2}{2b}$. Then $a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx \nrightarrow 0$. Using a method similar to Subcase 1, we obtain

$$\left\{a-b\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx\right\} \text{ is bounded.}$$

Therefore, we arrive at

$$\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n - \nabla u)dx \to 0.$$

On account of S_+ condition (see Lemma 2.6), we can easily deduce that $||u_n|| \to ||u||$ as $n\to\infty$, which implies that, for any $c<\frac{a^2}{2b}$, J satisfies the $(PS)_c$ condition.

Lemma 3.4. *Assume that the above conditions (g_1) and (g_2) hold. Then, for any $u\in X$, there exist $\rho>0$ and $\alpha>0$ such that $0<\alpha\le J(u)$ with $||u||=\rho$.*

Proof. Let $\varepsilon > 0$ be small enough so that $\frac{1}{2p^+} = \frac{\varepsilon}{\lambda_{p(x)}p^-}$. Using conditions (g_1) and (g_2), we obtain that

$$|G(x,u)| \leq \frac{\varepsilon}{p(x)}|u|^{p(x)} + \frac{C_\varepsilon}{q(x)}|u|^{q(x)}. \tag{3.5}$$

For $u \in X$, let $\rho \in (0,1)$ such that $||u|| = \rho$. According to Lemma 2.2, Proposition 2.3, (1.5), and (3.5), we can easily deduce that

$$\begin{aligned}
J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 - \int_\Omega G(x,u)dx - \lambda\int_\Omega f(x)udx \\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 - \varepsilon\int_\Omega \frac{|u|^{p(x)}}{p(x)}dx \\
&\quad - C_\varepsilon\int_\Omega \frac{|u|^{q(x)}}{q(x)}dx - \lambda|f|_{\frac{6}{5}}S^{-\frac{1}{2}}||u|| \\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\Big)^2 - \frac{\varepsilon}{\lambda_{p(x)}}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx \\
&\quad - \frac{CC_\varepsilon}{q^-}\int_\Omega |\nabla u|^{q(x)}dx - \lambda|f|_{\frac{6}{5}}S^{-\frac{1}{2}}||u|| \\
&\geq \Big(\frac{a}{p^+} - \frac{\varepsilon}{\lambda_{p(x)}p^-}\Big)\rho_{p(x)}(\nabla u) - \frac{b}{2(p^-)^2}(\rho_{p(x)}(\nabla u))^2 - \frac{CC_\varepsilon}{q^-}\rho_{q(x)}(\nabla u) - \lambda|f|_{\frac{6}{5}}S^{-\frac{1}{2}}||u|| \\
&\geq \frac{a}{2p^+}||u||^{p^+} - \frac{b}{2(p^-)^2}||u||^{2p^-} - \frac{CC_\varepsilon}{q^-}||u||^{q^-} - \lambda|f|_{\frac{6}{5}}S^{-\frac{1}{2}}||u|| \\
&\geq \Big(\frac{a}{2p^+}||u||^{p^+-1} - \frac{b}{2(p^-)^2}||u||^{2p^--1} - \frac{CC_\varepsilon}{q^-}||u||^{q^--1} - \lambda|f|_{\frac{6}{5}}S^{-\frac{1}{2}}\Big)||u||.
\end{aligned}$$

Consider the function $\gamma_1 : [0,+\infty) \to \mathbb{R}$ given by the formula

$$\gamma_1(s) = \frac{a}{2p^+}s^{p^+-1} - \frac{b}{2(p^-)^2}s^{2p^--1} - \frac{CC_\varepsilon}{q^-}s^{q^--1}.$$

Thanks to $q^- > 2p^- > p^+$, there exists $s = \rho > 0$ such that $\gamma(\rho) = \max_{s\in[0,+\infty)}\gamma_1(s) > 0$. Taking $\Lambda = \frac{\gamma_1(\rho)}{|f|_{\frac{6}{5}}S^{-\frac{1}{2}}}$, for each $u \in X$ and $\lambda \in (0,\Lambda)$, we can choose positive constant α and ρ such that $0 < \alpha \leq J(u)$ with $||u|| = \rho$.

Lemma 3.5. *Suppose that (g_3) holds. Then there exist $e \in X$ such that $J(e) < 0$ with $\rho < ||e||$ (where ρ is defined in Lemma 3.4.).*

Proof. By (g_3), we can infer that, for all $A > 0$, there exists a positive constant c_A such that

$$A|u|^{p^+} - c_A \leq G(x,u), \text{ for any } (x,u) \in \Omega\times\mathbb{R}. \tag{3.6}$$

Let $t>1$, $\psi\in C_0^\infty(\Omega)$ and $\psi>0$. According to (3.6), we obtain that

$$\begin{aligned}
J(t\psi) &= a\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx\Big)^2 - \int_\Omega G(x,t\psi)dx \\
&\quad -\lambda\int_\Omega f(x)t\psi dx \\
&\le a\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|t\nabla\psi|^{p(x)}dx\Big)^2 - At^{p^+}\int_\Omega |\psi|^{p^+}dx + c_A|\Omega| \\
&\le \frac{at^{p^+}}{p^-}\int_\Omega |\nabla\psi|^{p(x)}dx - \frac{bt^{2p^-}}{2(p^+)^2}\Big(\int_\Omega |\nabla\psi|^{p(x)}\Big)^2 - At^{p^+}\int_\Omega |\psi|^{p^+}dx + c_A|\Omega|.
\end{aligned}$$

Due to $p^-<p^+<2p^-$, we can easily see that $J(t\psi)\to-\infty$ as $t\to+\infty$. Then, for $t>1$ large enough, at this very moment, we just take $e=t\psi$ so that $||e||>\rho$ and such that $J(e)<0$.

Proof of the existence of the first solution. According to Lemmas 3.2, 3.4, and 3.5, there exists a positive constant δ such that, for $|f|_{p'(x)}<\delta$, all postulated conditions of the mountain pass theorem [4] hold. Then, there exists a critical point $u_1\in X$ of the functional J, namely u_1 satisfies $J'(u_1)=0$. Hence, problem (1.1) has a nontrivial weak solution $u_1\in X$ and such that $J(u_1)=\overline{c}>0$.

In addition, we need the following lemma to help us verify the condition of Ekeland's variational principle.

Lemma 3.6. *Suppose that the above conditions $(g_1)-(g_3)$ hold. Then, for any $s>0$ small enough, there exists a function $\psi\in X$ and $\psi\not\equiv 0$, such that $J(s\psi)<0$.*

Proof. Similar to the argument in [10], there exist constants C_1, C_2 such that

$$F(x,t)\ge C_1|t|^\mu - C_2|t|^{p^+},\ \forall\, x\in\Omega,\ t\in\mathbb{R}. \tag{3.7}$$

For any $s>0$ small enough, we first prove that there exists a function $\psi\in X$ such that $J(s\psi)<0$. As a matter of fact, let $\psi\in C_0^\infty(\Omega)$ be such that $\int_\Omega f(x)\psi(x)dx>0$. Due to $p^->1$, for any $s>0$ small enough, it follows from (3.7) that

$$\begin{aligned}
J(s\psi) &= a\int_\Omega \frac{1}{p(x)}|\nabla(t\psi)|^{p(x)}dx - \frac{b}{2}\Big(\int_\Omega \frac{1}{p(x)}|\nabla t\psi|^{p(x)}dx\Big)^2 - \int_\Omega G(x,s\psi)dx \\
&\quad -\lambda\int_\Omega f(x)s\psi dx \\
&\le \frac{as^{p^-}}{p^-}\int_\Omega |\nabla\psi|^{p(x)}dx - \frac{bt^{2p^+}}{2(p^-)^2}\Big(\int_\Omega |\nabla\psi|^{p(x)}dx\Big)^2 - C_1s^\mu\int_\Omega |\psi|^\mu dx \\
&\quad + C_2s^{p^+}\int_\Omega |\psi|^{p^+}dx - \lambda s\int_\Omega f(x)\psi dx \\
&< 0.
\end{aligned}$$

Proof of the existence of the second solution. By using Ekeland's variational principle, we will show that the existence of the second non-trivial weak solution $u_2\in X$ and $u_1\neq u_2$.

In fact, it follows from Lemma 3.4 that on the boundary of the ball centered at the origin and of radius ρ in X, denoted by $B_\rho(0)$

$$\overline{c}=\inf_{u\in\partial B_\rho(0)} J(u)>0.$$

It follows from Lemma 3.4 again that functional J is bounded from below on $B_\rho(0)$. In addition, according to Lemma 3.6, for any $\tau > 0$ small enough, there exists a function $\phi \in X$ such that $J(\tau\phi) < 0$, which implies that

$$-\infty < \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J(u) < 0.$$

Choose $\varepsilon > 0$ such that $0 < \varepsilon < \inf_{u\in\partial B_\rho(0)} J(u) - \inf_{u\in\overline{B}_\rho(0)} J(u)$. We apply the Ekeland's variational principle [27] to $J : \overline{B}_\rho(0) \to \mathbb{R}$. It is not difficult to infer that there exists $u_\varepsilon \in \overline{B}_\rho(0)$ such that

$$J(u_\varepsilon) < \inf_{u\in\overline{B}_\rho(0)} J(u) + \varepsilon,$$

and

$$J(u_\varepsilon) < J(u) + \varepsilon||u - u_\varepsilon||,\ u \neq u_\varepsilon.$$

Then, we have $J(u_\varepsilon) < \inf_{u\in B_\rho(0)} J(u)$ and then $u_\varepsilon \in B_\rho(0)$.

Next, we define the functional $I(u) = J(u) + \varepsilon||u - u_\varepsilon||$. It is not hard to find $I : \overline{B}_\rho(0) \to \mathbb{R}$ and easily to verify that u_ε is a minimum point of I. Thus

$$\frac{I(u_\varepsilon + \tau v) - I(u_\varepsilon)}{\tau} \geq 0,$$

for any $v \in B_\rho(0)$ and each $\tau > 0$ small enough. Hence, we have

$$\frac{J(u_\varepsilon + \tau v) - J(u_\varepsilon)}{\tau} + \varepsilon||v|| \geq 0.$$

Letting $\tau \to 0^+$ in the above inequality, we obtain that

$$0 \leq \langle J'(u_\varepsilon), v\rangle + \varepsilon||v||,$$

which implies that $||J'(u_\varepsilon)||_{X^*} \leq \varepsilon$. Hence, there exists a sequence $\{u_n\} \subset B_\rho(0)$ such that

$$J(u_n) \to \underline{c} = \inf_{u\in\overline{B}_\rho(0)} J(u) < 0 \text{ and } J'(u_n) \to 0 \text{ in } X^* \text{ as } n \to \infty. \tag{3.8}$$

According to Lemma 3.2, we can easily find that $u_n \to u_2$ as $n \to \infty$. Furthermore, due to $J \in C^1(X,\mathbb{R})$, it follows from (3.8) that $J'(u_2) = 0$. Therefore, u_2 is a nontrivial weak solution to problem (1.1) with $J(u_2) = \underline{c} < 0$. In conclusion, thanks to $J(u_1) = \overline{c} > 0 > \underline{c} = J(u_2)$, we find the fact that $u_1 \neq u_2$. $J(u_2) = \underline{c} < 0$. This completes the proof.

Proof of the existence of the third solution. Letting $M = \sup_{u\in X} J(u)$, we have $M < +\infty$. Hence $-M = \inf_{u\in X} -J(u)$. Using Ekeland's variational principle on space X for $-J(u)$, there exists a $(PS)_{-M}$ sequence of $-J(u)$, so it is a $(PS)_M$ sequence of $J(u)$. Since $-M$ is a global minimum of $-J(u)$ on X, we let $\{u_k\}_{k\in\mathbb{N}}$ be a minimizing sequence. By standard argument, we can prove that $\{u_k\}_{k\in\mathbb{N}}$ converge strongly to u_3. So, $-M$ is a critical value of $-J(u)$ and the corresponding critical point is u_3. Moreover, u_3 is a critical point of $J(u)$ and then we obtain our third solution.

4. Proof of Theorem 1.2

In this section, we prove Theorem 1.2 by using the symmetric mountain pass lemma. The method of proving $(PS)_c$ condition is similar to Lemma 2.6.

Next, let us recall the detail of symmetric mountain pass lemma.

Lemma A. (Symmetric mountain pass lemma, See[27]) For finite dimensional space Y, let infinite dimensional Banach space $X = Y \bigoplus Z$, and functional $J \in C^1(X,\mathbb{R})$ satisfy the $(PS)_c$ condition as well as the following properties:
(i) $J(u) = 0$ and there exist two positive constants r and α, such that $J|_{\partial B_r} \geq \alpha$;
(ii) for any $u \in X$, $J(-u) = J(u)$, that is, J is even;
(iii) for any finite dimensional subspaces $\widetilde{X} \subset X$, there exists $R = R(\widetilde{X}) > 0$ such that $J(u) \leq 0$ for each $u \in X \setminus B_R(\widetilde{X})$, where $B_R(\widetilde{X}) = \{u \in \widetilde{X} : ||u|| < R\}$.
Then J there exist an unbounded sequence of critical points.

Proof of Theorem 1.2 Under all the hypothetical conditions on Theorem 1.2, J satisfies the $(PS)_c$ condition. We now prove that J satisfies conditions (i)-(iii) of Lemma A.

(i) Obviously, $J(0) = 0$. Since $p^+ < (p^+)^2 < q^- < q(x) < p_2^*(x), X \hookrightarrow L^{2p^+}(\Omega)$, $X \hookrightarrow L^{q(x)}(\Omega)$, then

$$|u|_{2p^+} \leq C_3||u||,\ |u|_{q(x)} \leq C_4||u||,$$

for some $C_3, C_4 > 0$. It follows from (g_1) and (g_2) that

$$|G(x,u)| \leq \frac{\varepsilon}{p(x)}|u|^{p(x)} + \frac{C_\varepsilon}{q(x)}|u|^{q(x)}, \text{for all } (x,u) \in \Omega \times \mathbb{R}. \tag{4.1}$$

Let $r \in (0,1)$ and $u \in X$ be such that $||u|| = r$. Thus, by considering (4.1), Propositions 2.3, 2.4, and (1.5), we have

$$\begin{aligned}
J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega G(x,u)dx \\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \\
&\quad - \frac{\varepsilon}{p(x)}\int_\Omega |u|^{p(x)}dx - \frac{C_\varepsilon}{q(x)}\int_\Omega |u|^{q(x)}dx \\
&\geq a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 \\
&\quad - \varepsilon C_1||u||^{p(x)} - C_\varepsilon C_2||u||^{q(x)} \\
&\geq \frac{a}{p^+}||u||^{p^+} - \frac{b}{2(p^-)^2}||u||^{2p^-} - \varepsilon C_3||u||^{p^-} - C_\varepsilon C_4||u||^{q^-} \\
&= ||u||^{p^+}\left(\frac{a}{p^+} - \frac{b}{2(p^-)^2}||u||^{2p^- - p^+} - \varepsilon C_3||u||^{p^- - p^+} - C_\varepsilon C_4||u||^{q^- - p^+}\right) \\
&= r^{p^+}\left(\frac{a}{p^+} - \frac{b}{2(p^-)^2}r^{2p^- - p^+} - \varepsilon C_3 r^{p^- - p^+} - C_\varepsilon C_4 r^{q^- - p^+}\right),
\end{aligned}$$

so, we can choose $\varepsilon, r > 0$ small enough such that $u \in X$ and $J(u) \geq \alpha > 0$, where $||u|| = r$.

(ii) It is clear that J is even.

(iii) Using (g_3), we obtain that

$$C|t|^{p^+} - C \leq G(x,t). \tag{4.2}$$

Let $R = R(\widetilde{X}) > 1$ for any $u \in \widetilde{X}$ with $||u|| > R$. By (4.2), one has

$$\begin{aligned}
J(u) &= a\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{b}{2}\left(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx\right)^2 - \int_\Omega G(x,u)dx \\
&\leq \frac{a}{p^-}\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{2(p^+)^2}\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 - C\int_\Omega |u|^{p^+}dx + C\int_\Omega dx \\
&= \frac{a}{p^-}\int_\Omega |\nabla u|^{p(x)}dx - \frac{b}{2(p^+)^2}\left(\int_\Omega |\nabla u|^{p(x)}dx\right)^2 - C\int_\Omega |u|^{p^+}dx + \lambda C|\Omega|.
\end{aligned}$$

Moreover, the equivalence of all norms on the finite dimensional space $\widetilde{X}$ implies that there exists a positive constant C_W such that

$$\int_\Omega |u|^{p^+}dx \geq C_W||u||^{p^+}.$$

Therefore, we obtain

$$J(u) \leq \frac{a}{p^-}||u||^{p^+} - \frac{b}{2(p^+)^2}||u||^{2p^-} - CC_W||u||^{p^+} + C|\Omega|.$$

Due to $p^+ < 2p^-$, we can deduce $J(u) \leq 0$ some $||u|| > R$ large enough. Subsequently, the conclusion of Theorem 1.2 can be obtained by using the symmetric mountain pass lemma.

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