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A RELAXED EXTENDED CQ ALGORITHM FOR THE SPLIT FEASIBILITY PROBLEM IN HILBERT SPACES

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Abstract. In this paper, we study a split feasibility problem in Hilbert spaces. To solve the problem, Byrne introduced the extended CQ algorithm that involves the projections onto convex and closed subsets. However, the projections onto convex and closed subsets might be hard to be implemented in general. To overcome this difficulty, we propose a relaxed extended CQ algorithm in which the projections onto convex and closed subsets are replaced by the projections onto half-spaces. Under mild conditions, we establish the weak convergence of the proposed algorithm to a solution of the split feasibility problem.

Keywords. CQ algorithm; Split feasibility problem; Numerical experiment; Weak convergence.

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let C and Q be nonempty, convex, and closed subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a nonzero bounded linear operator and let A^* be its adjoint. The split feasibility problem (SFP) is formulated to find a point $x \in H_1$ satisfying

$$x \in C \quad \text{and} \quad Ax \in Q. \tag{1.1}$$

The SFP was first introduced by Censor and Elfving [5] in Euclidean spaces. Their aim is to model an inverse problem, which arises from intensity-modulated radiation therapy (IMRT) [6, 7, 8]. Since the celebrated the result was established, the problem received great attention and was applied to numerous situations; see, e.g., [13, 11, 15, 16, 19, 20, 24].

The SFP can be reformulated as the following constrained minimization:

$$\min_{x\in C} \frac{1}{2}\|(I-P_Q)Ax\|^2,$$

where P_Q is the orthogonal projection of H_2 onto set Q. In view of this reformulation, Byrne [2, 3] introduced and studied the CQ algorithm, which is based on the projected-gradient method. The iterative step of the CQ algorithm is formulated as follows:

$$x_{n+1} = P_C(x_n - \tau A^*(I-P_Q)Ax_n),$$

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where P_C is the orthogonal projection of H_1 onto set C, I is the identity operator on H_1, and $\tau \in (0, \frac{2}{\|A\|^2})$. Since the CQ algorithm requires calculating the orthogonal projections on both C and Q in each step, this can be applied only whenever an explicit formula available for these projections exist and this is mainly the case when the sets are quite "simple". In real scenarios, the involved sets C and Q are often given as level sets of some convex functions. In a way of handling these cases, Yang [22] proposed the so-called relaxed CQ algorithm, in which C and Q are given by

$$C = \{x \in H_1 \mid c(x) \leq 0\} \tag{1.2}$$

and

$$Q = \{y \in H_2 \mid q(y) \leq 0\}, \tag{1.3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are two lower semicontinuous convex functions on H_1 and H_2, respectively. The relaxed CQ algorithm is given as follows:

$$x_{n+1} = P_{C_n}(x_n - \tau A^*(I - P_{Q_n})Ax_n),$$

where $\tau \in (0, \frac{2}{\|A\|^2})$ and

$$C_n = \{x \in H_1 \mid c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\}, \quad \xi_n \in \partial c(x_n), \tag{1.4}$$

and

$$Q_n = \{y \in H_2 \mid q(Ax_n) + \langle \eta_n, y - Ax_n \rangle \leq 0\}, \quad \eta_n \in \partial q(Ax_n), \tag{1.5}$$

where $\partial c(\cdot)$ and $\partial q(\cdot)$ are the subdifferential of c and q, respectively (see Definition 2.3). In the relaxed CQ algorithm, since C_n and Q_n are both half-spaces, the projections P_{C_n} and P_{Q_n} are easily calculated.

In a recent paper [4], to avoid the hard constraint that x lies in C, Byrne proposed to consider the following problem

$$\min\Big(\frac{\alpha}{2}\|(I - P_C)x\|^2 + \frac{1-\alpha}{2}\|(I - P_Q)Ax\|^2\Big),$$

for some α in the interval $(0,1)$. Byrne [4] introduced the extended CQ algorithm, which is based on the forward-backward splitting method (see also [14]). The extended CQ algorithm is defined as follows:

$$x_{n+1} = \frac{1}{1+\gamma\alpha}(I + \gamma\alpha P_C)\big(I - \gamma(1-\alpha)A^*(I - P_Q)A\big)x_n, \tag{1.6}$$

where $\gamma \in (0, \frac{2}{(1-\alpha)\|A\|^2})$. Let $f_1(x) = \frac{\alpha}{2}\|(I - P_C)x\|^2$ and $f_2(x) = \frac{1-\alpha}{2}\|(I - P_Q)Ax\|^2$. It is easy to verify that

$$\text{prox}_{\gamma f_1}(x) = (I + \gamma\nabla f_1)^{-1}(x) = \frac{1}{1+\gamma\alpha}(I + \gamma\alpha P_C)(x),$$

where $\text{prox}_{\gamma f_1}$ is the proximal operator of f_1 of order γ (see Definition 2.8). So the extended CQ algorithm can also be written as

$$x_{n+1} = \text{prox}_{\gamma f_1}(I - \gamma\nabla f_2)x_n,$$

which is just the famous proximal gradient algorithm (see [10, 21]).

Similar to the CQ algorithm, the extended CQ algorithm also involves two projections P_C and P_Q, which are still hard to be implemented in the case where one of them fails to have a closed-form expression. We here propose a relaxed extended CQ algorithm in which the projections onto C and Q are replaced by projections onto half-spaces C_n and Q_n, respectively. Under mild assumptions, we prove the weak convergence of the proposed algorithm in this paper.

2. Preliminaries

In this section, we review some definitions and basic results that are used in this paper. From now on, we denote by H a Hilbert space and by I the identity operator on H. If $f : H \to \mathbb{R}$ is a differentiable functional, then we denote by ∇f the gradient of f. Given a sequence $\{x_n\}$ in H, $\omega_w(x_n)$ (resp., $\omega(x_n)$) stands for the set of cluster points in the weak (resp., strong) topology. '$x_n \rightharpoonup x$' (resp.,'$x_n \to x$') means the weak (resp., strong) convergence of $\{x_n\}$ to x.

Definition 2.1. [1, 3] Let D be a nonempty subset of H and let $T : D \to H$. Then T is

(1) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in D;$$

(2) firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in D;$$

(3) ν-inverse strongly monotone (ν-ism) if there is $\nu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \nu \|Tx - Ty\|^2, \quad \forall x, y \in D.$$

For any $x \in H$, the orthogonal projection onto a nonempty, convex, and closed subset C is defined as

$$P_C x = \arg\min\{\|y - x\| \mid y \in C\}.$$

The projection P_C has the following useful properties [1, 3].

Lemma 2.2. *Let $C \subseteq H$ be a nonempty, convex and closed subset. Then, for all $x, y \in H$ and $z \in C$,*

(1) $\langle x - P_C x, z - P_C x\rangle \leq 0$;
(2) *P_C and $I - P_C$ are both nonexpansive;*
(3) *P_C and $I - P_C$ are both firmly nonexpansive;*
(4) *P_C and $I - P_C$ are both 1-ism.*

Definition 2.3. Let $\lambda \in (0, 1)$ and $f : H \to (-\infty, +\infty]$ be a proper function.

(1) f is convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y), \quad \forall x, y \in H.$$

(2) A vector $u \in H$ is a subgradient of f at a point x if

$$f(y) \geq f(x) + \langle u, y - x\rangle, \quad \forall y \in H.$$

(3) The set of all subgradients of f at x, denoted by $\partial f(x)$, is called the subdifferential of f.

Definition 2.4. Let $f : H \to (-\infty, +\infty]$ be a proper function.

(1) f is lower semi-continuous at x if $x_n \to x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(2) f is weakly lower semi-continuous at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(3) f is lower semi-continuous on H if it is lower semi-continuous at every point $x \in H$. f is weakly lower semi-continuous on H if it is weakly lower semi-continuous at every point $x \in H$.

Lemma 2.5. [1, 17] *Suppose that H is finite-dimensional and let $f : H \to \mathbb{R}$ be a convex function. Then*

(1) *The function f is continuous;*
(2) *The function f is subdifferentiable everywhere;*
(3) *The subdifferentials of f are uniformly bounded on any bounded subset.*

Lemma 2.6. [1] *Let $f : H \to (-\infty, +\infty]$ be a proper convex function. Then f is semi-continuous if and only if it is weakly semi-continuous.*

Lemma 2.7. [3, 12] *Let $f(x) = \frac{1}{2}\|(I-P_Q)Ax\|^2$. Then*

(1) *f is convex and differential.*
(2) $\nabla f(x) = A^*(I-P_Q)Ax, \quad x \in H$.
(3) *f is weakly lower semi-continuous on H.*
(4) *∇f is $\|A\|^2$-Lipschitz: $\|\nabla f(x) - \nabla f(y)\| \le \|A\|^2\|x-y\|$ for all $x, y \in H$.*

Let $\Gamma_0(H)$ be the space of convex functions in H that are proper and lower semi-continuous.

Definition 2.8. The proximal operator of $\varphi \in \Gamma_0(H)$ is defined by

$$\operatorname{prox}_\varphi(x) := \arg\min_{v\in H}\{\varphi(v) + \frac{1}{2}\|v-x\|^2\}, \quad x \in H.$$

The proximal operator of φ of order $\lambda > 0$ is defined as the proximal operator of $\lambda\varphi$, that is,

$$\operatorname{prox}_{\lambda\varphi}(x) := \arg\min_{v\in H}\{\varphi(v) + \frac{1}{2\lambda}\|v-x\|^2\}, \quad x \in H.$$

The proximal operators can be used to minimize the sum of two convex functions

$$\min_{x\in H} f(x) + g(x), \tag{2.1}$$

where $f, g \in \Gamma_0(H)$. It is often the case where one of them is differentiable. The following is an equivalent fixed point formulation of (2.1).

Lemma 2.9. [21] *Let $f, g \in \Gamma_0(H)$. Let $x^* \in H$ and $\lambda > 0$. Assume that f is finite-valued and differentiable on H. Then x^* is a solution to (2.1) if and only if x^* solves the equation*

$$x^* = prox_{\lambda g} \circ (I - \lambda\nabla f)x^*.$$

Lemma 2.10. *Let H be a Hilbert space. Then*

$$\|tx+sy\|^2 = t(t+s)\|x\|^2 + s(t+s)\|y\|^2 - st\|x-y\|^2, \quad \forall x,y \in H, \forall s,t \in \mathbb{R}.$$

The convergence analysis of the proposed algorithm is based on the Fejér monotonicity.

Definition 2.11. Let C be a nonempty, convex, and closed subset in H. A vector sequence $\{x_n\}$ in H is said to be Fejér monotone with respect to C if

$$\|x_{n+1} - z\| \le \|x_n - z\|, \quad \forall n \ge 1, \quad \forall z \in C.$$

Lemma 2.12. [9] *Let C be a nonempty, convex, and closed subset in H. If the vector sequence $\{x_n\}$ is Fejér monotone with respect to C, then the following hold:*

(1) *$x_n \rightharpoonup x^* \in C$ if and only if $\omega_w(x_n) \subseteq C$;*

(2) *the sequence* $\{P_C x_n\}$ *converges strongly;*
(3) *if* $x_n \rightharpoonup x^* \in C$*, then* $x^* = \lim_{n\to\infty} P_C x_n$.

3. The Relaxed Extended CQ Algorithm

In what follows, we will treat the SFP (1.1) under the following assumptions.

(A1) The solution set $S = \{x \in C \mid Ax \in Q\}$ is nonempty.
(A2) The sets C and Q are given by (1.2) and (1.3), respectively.
(A3) For any $x \in H_1$ and $y \in H_2$, at least one subgradient $\xi \in \partial c(x)$ and $\eta \in \partial q(y)$ can be calculated, respectively. We also assume that the subdifferential operators ∂c and ∂q are bounded on bounded sets.

Remark 3.1. (1) It is worth noting that every convex function defined on a finite dimensional Hilbert space satisfies assumption (A3) by Lemma 2.5.

(2) Since $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex, one sees from Lemma 2.6 that both c and q are weakly lower semicontinuous by condition (A2).

Let us now give the relaxed extended CQ algorithm for solving the SFP (1.1).

Algorithm 3.2. Let x_0 be arbitrary. Given x_n, construct x_{n+1} via the formula

$$x_{n+1} = \frac{1}{1+\gamma\alpha}(I+\gamma\alpha P_{C_n})\big(I-\gamma(1-\alpha)A^*(I-P_{Q_n})A\big)x_n, \tag{3.1}$$

where $\gamma \in (0, \frac{2}{(1-\alpha)\|A\|^2})$, C_n and Q_n are given by (1.4) and (1.5), respectively. If $x_{n+1} = x_n$, then stop; otherwise, set $n := n+1$ and go to (3.1) to compute the next iterate x_{n+2}.

Remark 3.3. It is easily seen that $C \subseteq C_n$ and $Q \subseteq Q_n$ for all $n \in \mathbb{N}$. Note that C_n and Q_n are half-spaces and thus the corresponding projections have closed-form expressions.

Subsequently, we give the convergence analysis of the sequence $\{x_n\}$ generated by Algorithm 3.2.

Theorem 3.4. *In Algorithm 3.2, if* $x_{n+1} = x_n$ *for some* $n \geq 0$*, then* x_n *is a solution to the SFP* (1.1).

Proof. Let

$$g_n(x) = \frac{\alpha}{2}\|(I-P_{C_n})x\|^2$$

and

$$f_n(x) = \frac{1-\alpha}{2}\|(I-P_{Q_n})Ax\|^2.$$

From (3.1), if $x_{n+1} = x_n$, then

$$\begin{aligned} x_n &= \frac{1}{1+\gamma\alpha}(I+\gamma\alpha P_{C_n})\big(I-\gamma(1-\alpha)A^*(I-P_{Q_n})A\big)x_n \\ &= \operatorname{prox}_{\gamma g_n} \circ (I-\gamma\nabla f_n)x_n. \end{aligned}$$

According to Lemma 2.9, x_n is a solution to the following optimization problem:

$$\min_{x\in H} f_n(x) + g_n(x).$$

Since $C \subseteq C_n$, $Q \subseteq Q_n$, and $S \neq \emptyset$, we obtain that $f_n(x_n) = 0$ and $g_n(x_n) = 0$, which implies that $x_n \in C_n$ and $Ax_n \in Q_n$. From (1.4) and (1.5), we have that x_n is in C and Ax_n is in Q. The proof is complete. □

By Theorem 3.4, we see that if Algorithm 3.2 terminates in a finite (say n) step of iterations, then x_n is a solution to the SFP. Thus in the rest of this section, we assume that Algorithm 3.2 does not terminate in a finite number of iterations, and hence generates an infinite sequence $\{x_n\}$. The convergence of Algorithm 3.2 is proved below.

Theorem 3.5. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.2. Then $\{x_n\}$ converges weakly to a solution of the SFP (1.1).*

Proof. Let $x^* \in S$ be arbitrarily chosen. Put $y_n = x_n - \gamma(1-\alpha)A^*(I-P_{Q_n})Ax_n$. Then we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|(x_n - x^*) - \gamma(1-\alpha)A^*(I-P_{Q_n})Ax_n\|^2 \\ &= \|x_n - x^*\|^2 - 2\gamma(1-\alpha)\langle A^*(I-P_{Q_n})Ax_n, x_n - x^*\rangle \\ &\quad + \gamma^2(1-\alpha)^2\|A^*(I-P_{Q_n})Ax_n\|^2.\end{aligned}$$

Note that $I - P_{Q_n}$ is 1-ism, which implies that

$$\begin{aligned}\langle A^*(I-P_{Q_n})Ax_n, x_n - x^*\rangle &= \langle (I-P_{Q_n})Ax_n, Ax_n - Ax^*\rangle \\ &= \langle (I-P_{Q_n})Ax_n - (I-P_{Q_n})Ax^*, Ax_n - Ax^*\rangle \\ &\geq \|(I-P_{Q_n})Ax_n\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\gamma(1-\alpha)\|(I-P_{Q_n})Ax_n\|^2 \\ &\quad + \gamma^2(1-\alpha)^2\|A^*(I-P_{Q_n})Ax_n\|^2 \\ &\leq \|x_n - x^*\|^2 - 2\gamma(1-\alpha)\|(I-P_{Q_n})Ax_n\|^2 \\ &\quad + \gamma^2(1-\alpha)^2\|A\|^2\|(I-P_{Q_n})Ax_n\|^2 \\ &= \|x_n - x^*\|^2 - \gamma(1-\alpha)(2-\gamma(1-\alpha)\|A\|^2)\|(I-P_{Q_n})Ax_n\|^2.\end{aligned}$$

From (3.1), we have

$$x_{n+1} = \frac{1}{1+\gamma\alpha}y_n + \frac{\gamma\alpha}{1+\gamma\alpha}P_{C_n}y_n.$$

Since $C \subseteq C_n$, then $x^* = P_C(x^*) = P_{C_n}(x^*)$. It follows from Lemma 2.10 that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\frac{1}{1+\gamma\alpha}(y_n - x^*) + \frac{\gamma\alpha}{1+\gamma\alpha}(P_{C_n}y_n - x^*)\|^2 \\ &= \frac{1}{1+\gamma\alpha}\|y_n - x^*\|^2 + \frac{\gamma\alpha}{1+\gamma\alpha}\|P_{C_n}y_n - x^*\|^2 \\ &\quad - \frac{\gamma\alpha}{(1+\gamma\alpha)^2}\|(I-P_{C_n})y_n\|^2 \\ &\leq \|y_n - x^*\|^2 - \frac{\gamma\alpha}{(1+\gamma\alpha)^2}\|(I-P_{C_n})y_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \gamma(1-\alpha)(2-\gamma(1-\alpha)\|A\|^2)\|(I-P_{Q_n})Ax_n\|^2 \\ &\quad - \frac{\gamma\alpha}{(1+\gamma\alpha)^2}\|(I-P_{C_n})y_n\|^2. \end{aligned} \tag{3.2}$$

Hence, the sequence $\{x_n\}$ is Fejér monotone with respect to S. This implies that the sequence $\{\|x_n - x^*\|\}$ is convergent and $\{x_n\}$ is a bounded sequence. Furthermore, from (3.2) and the assumption on γ, we can immediately reach that

$$\sum_{n=0}^{\infty} \|(I-P_{Q_n})Ax_n\|^2 < \infty,$$

and

$$\sum_{n=0}^{\infty} \|(I-P_{C_n})y_n\|^2 < \infty.$$

In particular, we have

$$\lim_{n\to\infty} \|(I-P_{Q_n})Ax_n\| = 0,$$

and

$$\lim_{n\to\infty} \|(I-P_{C_n})y_n\| = 0.$$

By Lemma 2.12, to prove the weak convergence of $\{x_n\}$ to a solution of the SFP (1.1), it suffices to prove that $\omega_w(x_n) \subseteq S$, since $\{x_n\}$ is Fejér monotone with respect to S. Now let $\bar{x} \in \omega_w(x_n)$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$. In the following, we prove that $\bar{x} \in S$. First we prove $A\bar{x} \in Q$. Since ∂q is bounded on bounded sets, there is a constant $\delta_1 > 0$ such that $\|\eta_n\| \le \delta_1$ for all $n \ge 0$. From (1.5) and the fact that $P_{Q_n}(Ax_n) \in Q_n$, it follows that

$$q(Ax_n) \le \langle \eta_n, Ax_n - P_{Q_n}(Ax_n)\rangle \le \delta_1 \|(I-P_{Q_n})Ax_n\|. \tag{3.3}$$

The weakly lower semicontinuity of q and (3.3) imply that

$$q(A\bar{x}) \le \liminf_{k\to\infty} q(Ax_{n_k}) \le \lim_{k\to\infty} \delta_1 \|(I-P_{Q_{n_k}})Ax_{n_k}\| = 0.$$

It turns out that $A\bar{x} \in Q$.

We next turn to $\bar{x} \in C$. By the definition of y_n, we have

$$\|y_n - x_n\| = \gamma(1-\alpha)\|A^*(I-P_{Q_n})Ax_n\| \le \gamma(1-\alpha)\|A\|\|(I-P_{Q_n})Ax_n\| \to 0.$$

Since ∂c is bounded on bounded sets, there is a constant $\delta_2 > 0$ such that $\|\xi_n\| \le \delta_2$ for all $n \ge 0$. By the definition of C_n and the fact that $P_{C_n}(y_n) \in C_n$, we obtain

$$\begin{aligned} c(x_n) &\le \langle \xi_n, x_n - P_{C_n} y_n\rangle \\ &= \langle \xi_n, (x_n - y_n) + (I-P_{C_n})y_n\rangle \\ &\le \delta_2(\|x_n - y_n\| + \|(I-P_{C_n})y_n\|). \end{aligned} \tag{3.4}$$

Again, the weakly lower semicontinuity of c and (3.4) imply that

$$c(\bar{x}) \le \liminf_{k\to\infty} c(x_{n_k}) \le \lim_{k\to\infty} \delta_2(\|x_{n_k} - y_{n_k}\| + \|(I-P_{C_{n_k}})y_{n_k}\|) = 0.$$

Consequently, $\bar{x} \in C$. In conclusion, we conclude that $\bar{x} \in S$. This completes the proof. □

From Theorem 3.5 and Remark 3.1, the following corollary can be obtained immediately.

Corollary 3.6. *Let H_1 and H_2 be the finite dimensional Hilbert space $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Assume that conditions (A1) and (A2) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges to a solution to the SFP* (1.1).

Remark 3.7. (1) The relaxed CQ algorithm (1.6) is the projection gradient algorithm, while the relaxed extended CQ algorithm (3.1) is the proximal gradient algorithm.

(2) Let $y_n = x_n - \gamma(1-\alpha)A^*(I-P_{Q_n})Ax_n$. In algorithm (3.1), the half-space C_n is constructed via x_n. In [23], the authors constructed the half-space C_n' via y_n instead x_n, where C_n' is defined as follows:

$$C_n' = \{x \in H_1 \mid c(y_n) + \langle \xi_n', x - y_n\rangle \le 0\}, \quad \xi_n' \in \partial c(y_n).$$

Inspired by [23], we introduce the following algorithm:

$$x_{n+1} = \frac{1}{1+\gamma\alpha}(I+\gamma\alpha P_{C_n'})\big(I-\gamma(1-\alpha)A^*(I-P_{Q_n})A\big)x_n.$$

Similar to the proof of Theorem 3.5, the sequence $\{x_n\}$ generated by this algorithm converges weakly to a solution of the SFP (1.1).

4. Numerical Experiments

To give some insight into the behavior of the algorithm presented in this paper, we implemented it in MATLAB to solve the following example. For simplicity, we denote the relaxed CQ algorithm and the relaxed extended CQ algorithm by RCQA and RECQA, respectively. The experiments were carried out in the environment of MATLAB2014a and the CPU is Intel(R) Core(TM) i5-8265U with @3.20GHz. In the results reported below, all CPU times reported are in seconds.

Example 4.1. In this example, we apply RCQA and RECQA to solve the celebrated LASSO problem. Let us first recall the LASSO problem [18] which is given as follows:

$$\begin{array}{l} \min_{x\in\mathbb{R}^n} \frac{1}{2}\|Ax-b\|^2, \\ s.t. \quad \|x\|_1 \le t, \end{array} \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$, and $t > 0$ is a given constant. Let $C = \{x \mid c(x) \le 0\}$, where $c(x) = \|x\|_1 - t$ and $Q = \{b\}$. Then problem (4.1) can be seen as an SFP (1.1). In this example, the vector x in $\mathbb{R}^n$ is a K-sparse signal that is generated from uniform distribution in the interval $[-2,2]$ with K non-zero elements. The matrix $A \in \mathbb{R}^{m\times n}$ is generated from a normal distribution with mean zero and one variance. The vector b is taken as equal to Ax, so no noise is assumed. The goal is then to recover the K-sparse signal x by solving the LASSO problem (4.1).

Throughout the experiment, the parameters used in these algorithms are set with $m = 50$, $n = 100$, $t = K$, $\tau = \frac{0.1}{\|A\|^2}$, $\alpha = 0.1$, and $\gamma = \frac{0.1}{(1-\alpha)\|A\|^2}$. It is worth noting that the initial point in this experiment is generated randomly. As a stopping criterion, 1000 iterations are the maximum allowed. The numerical behavior of each algorithm is demonstrated in Table 1 and Figure 1.

TABLE 1. Numerical results for Example 4.1

K-sparse signal	RCQA (CPU (s))	RECQA (CPU (s))
$K = 5$	0.0413	0.0371
$K = 10$	0.0438	0.0373
$K = 20$	0.0412	0.0370
$K = 30$	0.0405	0.0365

As demonstrated in Table 1 and Figure 1, our proposed algorithm converges more quickly than the RCQA. The algorithm proposed in this research is therefore efficient.

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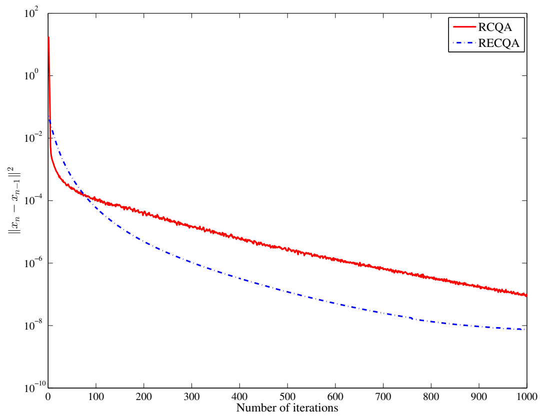
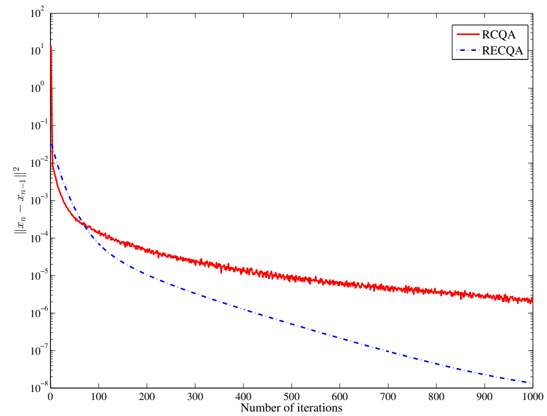
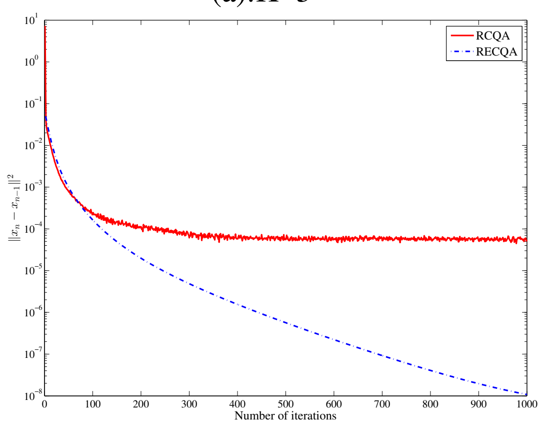


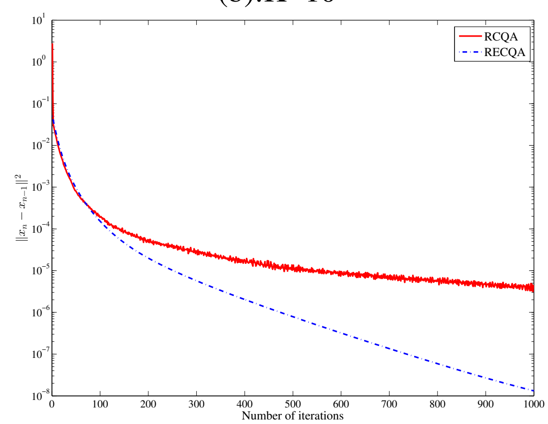
(a):K=5



(b):K=10



(c):K=20



(d):K=30

FIGURE 1. Numerical results for Example 4.1

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