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A NOVEL ACCELERATED ALGORITHM FOR SOLVING SPLIT VARIATIONAL INCLUSION PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. Motivated by the Tseng's extragradient method and the Moudafi's viscosity method, a new hybrid inertial accelerated algorithm with the line search technique is proposed for solving fixed point problems of demimetric mappings and split variational inclusion problems. A strong convergence theorem is established under some mild conditions. Our proof is different with from those presented in the literatures. In addition, numerical results are reported to support the main results.

Keywords. Demimetric mappings; Hybrid inertial accelerated algorithm; Line search; Numerical experiment; Split variational inclusion.

1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces. Let $A : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. Let $B_1 : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ and $B_2 : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ be multi-valued maximally monotone mappings. The split variational inclusion problem (SVIP) aims to find a point $x^* \in \mathscr{H}_1$ such that

$$0 \in B_1(x^*) \text{ and } 0 \in B_2(Ax^*), \tag{1.1}$$

whose solutions set is denoted by $\mathrm{SVIP}(B_1, B_2)$ from now on. A wide variety of important problems, such as convex minimization, monotone variational inequalities over convex sets, equilibrium problems, and so on, can be reformulated and investigated in the form of (1.1). Because of its wide applications in signal processing and image reconstruction, with particular progress in intensity-modulated radiation therapy [5], split variational inclusion problems attracted many researchers worldwide. For efficient iterative methods to split variational inclusion problems, we refer to [4, 8, 9, 12, 14, 18, 19]. Byrne et al. [3] introduced the following iterative method in Hilbert spaces. For a given $x_1 \in \mathscr{H}_1$, let $\{x_n\}$ be generated as follows.

$$x_{n+1} = J_r^{B_1}(x_n - \gamma A^*(I - J_{r_n}^{B_2})Ax_n), \forall n \in \mathbb{N}, \tag{1.2}$$

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where $J_r^B = (I + rB)^{-1}$ for all $x \in \mathscr{H}$ and r, γ are real positive constants. They proved that the sequence $\{x_n\}$ defined by (1.2) converges to a solution of the SVIP under some suitable conditions.

In 2001, Alvarez and Attouch [1] used the heavy ball method that was studied in [13] for maximally monotone operators on the proximal point algorithm. Their algorithm is said to be the inertial proximal point algorithm and it is as follows

$$\begin{cases} x_{n+1} = (I + r_n T)^{-1} y_n, \\ y_n = x_n + \theta_n (x_n - x_{n-1}), n \geq 1. \end{cases} \tag{1.3}$$

They also proved that the sequence $\{x_n\}$ constructed by (1.3) converges weakly to a zero point of T under some suitable conditions.

Recently, using the idea of Alvarez and Attouch [1], Chuang [6] proposed the following hybrid inertial proximal algorithm for the SVIP in Hilbert spaces

Algorithm 1: Hybrid inertial proximal algorithm for the SVIP (HSVIP).

Initialization. Let $x_1 \in \mathscr{H}_1$ be arbitrary.

Step 1. Compute $y_n = J_{r_n}^{B_1}(x_n - r_n A^*(I - J_{r_n}^{B_2})Ax_n)$,
where $r_n \subseteq [\gamma, \frac{\delta}{\|A\|^2}] \subseteq (0, \infty)$ satisfies

$$r_n \|A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ay_n\| \leq \delta \|x_n - y_n\|.$$

Step 2. Compute

$$x_{n+1} = J_{r_n}^{B_1}(x_n - d_n e_n),$$

where

$$d_n = x_n - y_n - r_n(A^*(I - J_{r_n}^{B_2})Ax_n - A^*(I - J_{r_n}^{B_2})Ay_n),$$

and

$$e_n = \frac{\langle x_n - y_n, d_n \rangle}{\|d_n\|^2}.$$

Set $n = n + 1$ and go to **Step 1.**

Assume that $S : \mathscr{H}_1 \to \mathscr{H}_1$ is a k-demicontractive mapping, $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with adjoint $A^* : \mathscr{H}_2 \to \mathscr{H}_1$, $B_1 : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ and $B_2 : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ are multi-valued maximally monotone mappings. They proved, under certain appropriate assumptions, Algorithm 1 converges strongly to the unique element.

Recently, Jolaoso and Karahan [7] proposed the following splitting algorithm for solving the SVIP in Hilbert spaces and proved its weak convergence under suitable conditions

Assume that $S : \mathscr{H}_1 \to \mathscr{H}_1$ is a k-demicontractive mapping, $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator with adjoint $A^* : \mathscr{H}_2 \to \mathscr{H}_1$, $B_1 : \mathscr{H}_1 \to 2^{\mathscr{H}_1}$ and $B_2 : \mathscr{H}_2 \to 2^{\mathscr{H}_2}$ are multi-valued maximally monotone mappings. They proved, under certain appropriate assumptions, Algorithm 2 converges weakly to the unique element.

Motivated and inspired by Alvarez and Attouch [1], Chuang [6], Jolaoso and Karahan [7], and Song [15], we propose and analyze a hybrid inertial accelerated method for finding common solutions of split variational inclusion problems and fixed point problems of a demimetric mapping in a real Hilbert space. Strong convergence of presented method is proved under some mild conditions. Numerical experiments are provided to demonstrate the efficiency of the proposed method over some existing ones.

Algorithm 2: Splitting algorithm for solving the SVIP (SSVIP).

Initialization. Set $\zeta > 0, l \in (0,1), \delta \in (0,1), \eta \in (0,2)$, and let $x_1 \in \mathscr{H}_1$ be arbitrary.

Step 1. Compute

$$w_n = x_n + \mu_n(x_n - x_{n-1}), y_n = J_{r_n}^{B_1}(u_n - r_n A^*(I - J_{r_n}^{B_2})Au_n),$$

where $r_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$r_n \|A^*(I - J_{r_n}^{B_2})Au_n - A^*(I - J_{r_n}^{B_2})Ay_n\| \leq \delta\|u_n - y_n\|.$$

Step 2. Compute

$$x_{n+1} = u_n - \phi d_n e_n,$$

where $\phi \in (0,2)$,

$$d_n = u_n - y_n - r_n(A^*(I - J_{r_n}^{B_2})Au_n - A^*(I - J_{r_n}^{B_2})Ay_n),$$

and

$$e_n = \frac{\langle u_n - y_n, d_n\rangle + \gamma_n\|(I - J_{r_n}^{B_2})Ay_n\|^2}{\|d_n\|^2}.$$

Set $n = n+1$ and go to **Step 1.**

2. Preliminaries

Throughout this paper, The set of fixed points of a mapping T is denoted by $\mathrm{Fix}(T)$.

Let C be a nonempty, convex and closed subset of a real Hilbert space $\mathscr{H}$. For $u \in \mathscr{H}$ and $v \in C$, $v = P_C u$ if and only if $\langle u - v, w - v\rangle \leq 0$ for all $w \in C$, where P_C is the metric projection from $\mathscr{H}$ onto C.

One has the following results in real Hilbert space $\mathscr{H}$

(1) $\|ku + (1-k)v\|^2 = k\|u\|^2 + (1-k)\|v\|^2 - k(1-k)\|u-v\|^2$, $\forall u, v \in \mathscr{H}$ and $k \in [0,1]$.

(2)$\|u \pm v\|^2 = \|u\|^2 \pm 2\langle u, v\rangle + \|v\|^2, u, v \in \mathscr{H}$.

(3) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle, u, v \in \mathscr{H}$.

Recall that a mapping $S : C \to \mathscr{H}$ is said to be:

(1) nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \forall u, v \in C;$$

(2) γ-contractive if there exists $\gamma \in [0,1)$ such that

$$\|Su - Sv\| \leq \gamma\|u - v\|, \forall u, v \in C;$$

(3) quasi-nonexpansive if $\mathrm{Fix}(S) \neq \emptyset$ and

$$\|Su - x^*\| \leq \|u - x^*\|, \forall u \in C, x^* \in \mathrm{Fix}(S);$$

(4) α-strongly pseudo-contractive if there exists a constant $\alpha \in [0,1)$, such that

$$\langle Su - Sv, u - v\rangle \leq \alpha\|u - v\|^2,\ \forall u, v \in C;$$

(5) pseudo-monotone if

$$\langle Sv, u - v\rangle \geq 0 \Rightarrow \langle Su, u - v\rangle \geq 0,\ \forall u, v \in C;$$

(6) k-demicontractive if $\mathrm{Fix}(S) \neq \emptyset$ and there exists $k \in [0,1)$, such that

$$\|Su - x^*\|^2 \leq \|u - x^*\|^2 + k\|u - Su\|^2, \forall u \in C, x^* \in \mathrm{Fix}(S);$$

(7) k-demimetric if $\mathrm{Fix}(S) \neq \emptyset$ and there exists $k \in (-\infty, 1)$, such that

$$\|Su - x^*\|^2 \leq \|u - x^*\|^2 + k\|u - Su\|^2, \forall u \in C, x^* \in \mathrm{Fix}(S).$$

Obviously, k-demimetric mappings includes k-demicontractive mappings. we remark that Takahashi in [21] presented a specific example that is a demimetric mapping but not a demicontractive mapping.

Finally, we need the following lemmas to obtain our main results.

Lemma 2.1. *([2]) Let C be a nonempty, convex and closed subset of a real Hilbert space $\mathscr{H}$, and let $T : C \to C$ be a nonexpansive mapping. Then, $I - T$ is demiclosed at zero, i.e., if $\{x_n\}$ converges weakly to a point $x \in C$ and $\{I - T)x_n\}$ converges to zero, then $x = Tx$.*

Lemma 2.2. *([16, 20]) Let C be a nonempty, convex and closed subset of a real Hilbert space $\mathscr{H}$. Assume that $S : C \to \mathscr{H}$ is k-demimetric such that $\mathrm{Fix}(S)$ is nonempty. Let κ be a real number with $\kappa \in (0, \infty)$ and define $T = (1 - \kappa)I + \kappa S$. Then*

(1) $\mathrm{Fix}(T) = \mathrm{Fix}(S)$ *if $\kappa \neq 0$;*
(2) T is a quasi-nonexpansive mapping for $\kappa \in (0, 1 - k]$;
(3) $\mathrm{Fix}(S)$ *is a closed convex subset of $\mathscr{H}$.*

Lemma 2.3. *([11]) Assume that $\{a_n\}$ is a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ with $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_j\} \subseteq \mathbb{N}$ such that $m_j \to \infty$ and the following properties are satisfied for all (sufficiently large) numbers $j \in \mathbb{N}$: $a_j \leq a_{m_j+1}$ and $a_{m_j} \leq a_{m_j+1}$. Indeed, $m_j = \max\{k \leq j : a_k < a_{k+1}\}$.*

Lemma 2.4. *([10]) Assume that $\{a_n\}$ is a sequence of nonnegative numbers satisfying the following inequality: $a_{n+1} \leq (1 - \beta_n)a_n + \gamma_n + \beta_n\delta_n$ for all $n \in \mathbb{N}$, where $\{\beta_n\}, \{\gamma_n\}$, and $\{\delta_n\}$ satisfy the restrictions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$,
(ii) $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$,
(iii) $\limsup_{n\to\infty} \delta_n \leq 0$.

Then $\lim_{n\to\infty} a_n = 0$.

In order to study the SVIP, we recall some other lemmas which are needed in our proof. We denote by $B^{-1}(0) = \{x \in \mathscr{H} : 0 \in Bx\}$, $D(T)$ the domain of T and $Fix(T)$ the fixed point set of T, that is, $Fix(T) = \{x \in \mathscr{H} : x = Tx\}$.

Lemma 2.5. *Let $\mathscr{H}$ be a real Hilbert space, and let $B : \mathscr{H} \to 2^{\mathscr{H}}$ be a set-valued maximal monotone mapping. Then,*

(1) J_r^B is a single-valued and firmly nonexpansive mapping for each $r > 0$;
(2) $D(J_r^B) = \mathscr{H}$ and $Fix(J_r^B) = \{x \in D(B) : 0 \in B(x)\}$;
(3) $\|x - J_{r_1}^B x\| \leq \|x - J_{r_2}^B x\|$ for all $0 < r_1 \leq r_2$ and for all $x \in \mathscr{H}$;
(4) Suppose that $B^{-1}(0) \neq \emptyset$. Then $\|x - J_{r_1}^B x\|^2 + \|x^* - J_{r_1}^B x^*\|^2 \leq \|x - x^*\|^2$ for each $x \in \mathscr{H}$, each $x^* \in B^{-1}(0)$, and each $r > 0$;
(5)) Suppose that $B^{-1}(0) \neq \emptyset$. Then $\langle x - J_r^B x, J_r^B x - w \rangle \geq 0$ for each $x \in \mathscr{H}$, each $w^* \in B^{-1}(0)$, and each $r > 0$.

Lemma 2.6. *Assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces, and $A : \mathscr{H}_1 \to \mathscr{H}_2$ is a linear and bounded operator with its adjoint A^*. Let $B : \mathscr{H}_2 \to \mathscr{H}_2$ be a set-valued maximal monotone* *mapping, and let J_r^B be a resolvent mapping of B in order r. Define a mapping $\tilde{T}:\mathscr{H}_1\to\mathscr{H}_1$ as $\tilde{T}x := A^*(I-J_r^B)Ax$ for each $x\in\mathscr{H}_1$. Then, the following statements hold:*

(1) $\|\tilde{T}x-\tilde{T}y\|^2 \le \|A\|^2\langle \tilde{T}x-\tilde{T}y, x-y\rangle$;

(2) $\|(I-J_r^B)Ax-(I-J_r^B)Ay\|^2 \le \big\langle \tilde{T}x-\tilde{T}y, x-y\big\rangle$.

Lemma 2.7. *Assume that $\mathscr{H}_1$ and $\mathscr{H}_2$ are real Hilbert spaces, and $A:\mathscr{H}_1\to\mathscr{H}_2$ is a linear and bounded operator with its adjoint A^*. Let $B_1:\mathscr{H}_1\to\mathscr{H}_1$ and $B_2:\mathscr{H}_2\to\mathscr{H}_2$ be a set-valued maximal monotone mappings, and let $r, r>0$. Then, the following statements hold:*

(1) If $\hat{x}$ is a solution of the SVIP, *then* $J_r^{B_1}\left(\hat{x}-rA^*(I-J_r^{B_2})A\hat{x}\right)=\hat{x}$;

(2) If $J_r^{B_1}\left(\hat{x}-rA^*(I-J_r^{B_2})A\hat{x}\right)=\hat{x}$ *and the solution set of (SFVIP) is nonempty, then $\hat{x}$ is a solution to* SVIP.

3. Main Results

Let C and Q be nonempty, convex and closed subsets of real Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, respectively, and let $A:\mathscr{H}_1\to\mathscr{H}_2$ be a bounded linear operator with adjoint $A^*:\mathscr{H}_2\to\mathscr{H}_1$. Let $B_1:\mathscr{H}_1\to\mathscr{H}_1$ and $B_2:\mathscr{H}_2\to\mathscr{H}_2$ be a set-valued maximally monotone mappings. Assume that $S:\mathscr{H}_1\to\mathscr{H}_1$ is k-demimetric and $I-S$ is demiclosed at zero. Let $g:\mathscr{H}_1\to\mathscr{H}_1$ be contractive with constant $\alpha\in(0,1)$. Assume that $\mathrm{Sol} := \mathrm{SVIP}(B_1,B_2)\bigcap \mathrm{Fix}(S)\neq\emptyset$ and the following conditions are satisfied:

(C1) $\{\alpha_n\}\subset(0,1)$, $\lim_{n\to\infty}\alpha_n=0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$;
(C2) $\{\beta_n\}\subset(0,1)$, $0<\liminf_{n\to\infty}\beta_n\le\limsup_{n\to\infty}\beta_n<1-k$;
(C3) $\{\mu_n\}\subset[0,1]$ and $\lim_{n\to\infty}\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|=0$, where $\{x_n\}$ is generated by Algorithm 3;
(C4) $\{r_n\}\subset(0,\infty)$ and $\liminf_{n\to\infty} r_n>0$.

We now introduce the following algorithm.

Algorithm 3: Hybrid inertial accelerated method for finding a common solution of the split variational inclusion problem and the fixed point problem (HSVIPP).

Initialization: Set $\{r_n\}\subset(0,+\infty)$, $\{\alpha_n\}\subset(0,+\infty)$, $r_1>0$, $\upsilon>0$. Choose a nonnegative real sequence $\{\sigma_n\}$ such that $\sum_{n=0}^{\infty}\sigma_n<\infty$. Let $x_0,x_1\in\mathscr{H}_1$ be arbitrary.

Step 1. Given x_{n-1} and $x_n (n\ge1)$, compute

$$u_n = x_n+\mu_n(x_n-x_{n-1}).$$

Step 2. Compute $y_n = J_{r_n}^{B_1}\left(u_n-r_nA^*(I-J_{r_n}^{B_2})Au_n\right)$, where $\{r_n\}$ is updated by

$$r_{n+1}=\begin{cases}\min\{\frac{\upsilon\|u_n-y_n\|}{\|A^*(I-J_{r_n}^{B_2})Au_n-A^*(I-J_{r_n}^{B_2})Ay_n\|}, r_n+\sigma_n\},\\ \text{ if } A^*(I-J_{r_n}^{B_2})Au_n-A^*(I-J_{r_n}^{B_2})Ay_n\neq0;\\ r_n+\sigma_n, \text{ otherwise.}\end{cases} \tag{3.1}$$

Step 3. Compute

$$z_n = y_n-r_n\left(A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\right).$$

Step 4. Compute $x_{n+1}=T_n\left(\alpha_n g x_n+(1-\alpha_n)z_n\right)$,
where $T_n=(1-\beta_n)I+\beta_n S$. Set $n=n+1$ and go to **Step 1.**

Lemma 3.1. *Suppose $VI(C,A)\neq\emptyset$. Then the sequence $\{r_n\}$ generated by (3.1) is well defined and $\lim_{n\to\infty} r_n = r$ and $r\in[\min\{\frac{\upsilon}{\|A\|}, r_1\}, r_1+\zeta]$, where $\zeta=\sum_{i=1}^{\infty}\sigma_n$.*

Proof. Indeed, using Lemma 2.6 (1), one sees that

$$\|A^*(I-J_{r_n}^{B_2})Ay_n - A^*(I-J_{r_n}^{B_2})Au_n\| \leq \|A^2\|\|y_n-u_n\|.$$

By using Tan and Qin [17, Lemma 4], we can obtain the desired result immediately. □

Lemma 3.2. *Suppose that Conditions (C1)-(C4) hold. Let $\{u_n\},\{y_n\}$ and $\{z_n\}$ be three sequences created by Algorithm 3. Then, for $\forall q\in \mathrm{SVIP}(B_1,B_2)$,*

$$\|z_n-q\|^2 \leq \|u_n-q\|^2 - \left(1-\frac{\upsilon^2 r_n^2}{r_{n+1}^2}\right)\|y_n-u_n\|^2.$$

Proof. Let $q\in\mathrm{SVIP}(B_1,B_2)$, that is, $J_{r_n}^{B_1}q=q$, $J_{r_n}^{B_2}Aq=Aq$. From the definitions of z_n, we find

$$\begin{aligned}
&\|z_n-q\|^2\\
=\ &\|y_n-r_n\left(A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\right)-q\|^2\\
=\ &\|y_n-q\|^2+r_n^2\|A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\|^2\\
&-2r_n\langle y_n-q, A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\rangle\\
=\ &\|y_n-u_n\|^2+\|u_n-q\|^2+2\langle y_n-u_n,u_n-q\rangle\\
&+r_n^2\|A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\|^2-2r_n\langle y_n-q,A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\rangle\\
=\ &\|y_n-u_n\|^2+\|u_n-q\|^2-2\langle y_n-u_n,y_n-u_n\rangle\\
&+2\langle y_n-u_n,y_n-q\rangle-2r_n\langle y_n-q,A^*(I-J_{r_n}^{B_2})Ay_n\\
&-A^*(I-J_{r_n}^{B_2})Au_n\rangle+r_n^2\|A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\|^2\\
=\ &\|u_n-q\|^2-\|y_n-u_n\|^2+2\langle y_n-u_n+r_nA^*(I-J_{r_n}^{B_2})Au_n,y_n-q\rangle\\
&-2r_n\langle A^*(I-J_{r_n}^{B_2})Ay_n,y_n-q\rangle+r_n^2\|A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\|^2.
\end{aligned}\tag{3.2}$$

From Lemma 2.5, we obtain that

$$\langle y_n-u_n+r_nA^*(I-J_{r_n}^{B_2})Au_n,y_n-q\rangle\leq 0.\tag{3.3}$$

Noticing $Aq=J_{r_n}^{B_2}Aq$, we obtain from Lemma 2.5(2) that

$$\langle A^*(I-J_{r_n}^{B_2})Ay_n,y_n-q\rangle\geq 0.\tag{3.4}$$

It follows from (3.2), (3.3), and (3.4) that

$$\begin{aligned}
&\|z_n-q\|^2\\
\leq\ &\|u_n-q\|^2-\|y_n-u_n\|^2+r_n^2\|A^*(I-J_{r_n}^{B_2})Ay_n-A^*(I-J_{r_n}^{B_2})Au_n\|^2\\
\leq\ &\|u_n-q\|^2-\|y_n-u_n\|^2+\frac{\upsilon^2r_n^2}{r_{n+1}^2}\|y_n-u_n\|^2\\
=\ &\|u_n-q\|^2-\left(1-\frac{\upsilon^2r_n^2}{r_{n+1}^2}\right)\|y_n-u_n\|^2.
\end{aligned}$$

The proof is completed. □

Lemma 3.3. *Let $\{u_n\}$ and $\{y_n\}$ be created by Algorithm 3. If $u_{n_k} \rightharpoonup z^*$ and $\lim_{n\to\infty}\|u_n - y_n\| = 0$, then $z^* \in \mathrm{SVIP}(B_1, B_2)$.*

Proof. Taking any $q \in \mathrm{SVIP}(B_1, B_2)$, we know that $J_{r_n}^{B_2}Aq = Aq$. It implies that $A^*(I - J_{r_n}^{B_2})Aq = 0$. By Lemma 2.5, we obtain that

$$\langle A^*(I - J_{r_n}^{B_2})Ay_n - A^*(I - J_{r_n}^{B_2})Aq, y_n - q\rangle \geq \|(I - J_{r_n}^{B_2})Ay_n\|^2.$$

This together with (3.3) and Lemma 2.6(1) yields that

$$\begin{aligned}
& r_n\|Ay_n - J_{r_n}^{B_2}Ay_n\|^2 \\
\leq\ & r_n\langle A^*(I - J_{r_n}^{B_2})Ay_n - A^*(I - J_{r_n}^{B_2})Aq, y_n - q\rangle \\
\leq\ & r_n\langle A^*(I - J_{r_n}^{B_2})Ay_n - A^*(I - J_{r_n}^{B_2})Aq, y_n - q\rangle - \langle y_n - (u_n - r_nA^*(I - J_{r_n}^{B_2})Au_n), y_n - q\rangle \\
=\ & \langle u_n - y_n - r_nA^*(I - J_{r_n}^{B_2})Au_n + r_nA^*(I - J_{r_n}^{B_2})Ay_n, y_n - q\rangle \\
\leq\ & \|u_n - y_n - r_nA^*(I - J_{r_n}^{B_2})Au_n + r_nA^*(I - J_{r_n}^{B_2})Ay_n\|\|y_n - q\| \\
\leq\ & \left(\|u_n - y_n\| + r_n\|A^*(I - J_{r_n}^{B_2})Au_n - A^*(I - J_{r_n}^{B_2})Ay_n\|\right)\|y_n - q\| \\
\leq\ & (1 + r_n\|A\|^2)\|u_n - y_n\|\|y_n - q\|.
\end{aligned}$$

By Lemma 3.1 and $\lim_{n\to\infty}\|u_n - y_n\| = 0$, we find that

$$\lim_{n\to\infty}\|Ay_n - J_{r_n}^{B_2}Ay_n\| = 0.$$

Moreover, by Lemma 2.5, one can deduce that

$$\begin{aligned}
\|Au_n - J_{r_n}^{B_2}Au_n\| &\leq \|Au_n - Ay_n + (J_{r_n}^{B_2}Ay_n - J_{r_n}^{B_2}Au_n)\| + \|Ay_n - J_{r_n}^{B_2}Ay_n\| \\
&\leq 2\|A\|\|u_n - y_n\| + \|Ay_n - J_{r_n}^{B_2}Ay_n\|.
\end{aligned}$$

This indicates that

$$\lim_{n\to\infty}\|Au_n - J_{r_n}^{B_2}Au_n\| = 0. \tag{3.5}$$

Again using Lemma 2.5 and the definition of y_n, we derive

$$\begin{aligned}
\|y_n - J_{r_n}^{B_1}u_n\| &= \|J_{r_n}^{B_1}(u_n - r_nA^*(I - J_{r_n}^{B_2})Au_n) - J_{r_n}^{B_1}u_n\| \\
&\leq \|r_nA^*(I - J_{r_n}^{B_2})Au_n\| \\
&\leq r_n\|A\|\|Au_n - J_{r_n}^{B_2}Au_n\|,
\end{aligned}$$

which together with (3.5) gives that $\lim_{n\to\infty}\|y_n - J_{r_n}^{B_1}u_n\| = 0$. From $\lim_{n\to\infty}\|y_n - u_n\| = 0$, we can obtain that $\lim_{n\to\infty}\|u_n - J_{r_n}^{B_1}u_n\| = 0$. According to (C4), there exist a positive number r and some positive integer N_0 such that $0 < r < r_n$ $(\forall n \geq N_0)$. It follows from Lemma 2.5(3) that

$$\lim_{n\to\infty}\|u_n - J_r^{B_1}u_n\| \leq \lim_{n\to\infty}\|u_n - J_{r_n}^{B_1}u_n\| = 0.$$

This combining with Lemma 2.1, Lemma 2.5(1)(ii) and $u_{n_k} \rightharpoonup z^*$ yield $z^* \in \mathrm{Fix}(J_r^{B_1}) = B_1^{-1}(0)$. Due to the fact that A is a linear bounded operator and $u_{n_k} \rightharpoonup z^*$, we get that

$$Au_{n_k} \rightharpoonup Az^*. \tag{3.6}$$

Using (3.5) and Lemma 2.5, we obtain that

$$\lim_{n\to\infty}\|Au_n - J_r^{B_2}Au_n\| \leq \lim_{n\to\infty}\|Au_n - J_{r_n}^{B_2}Au_n\| = 0. \tag{3.7}$$

By (3.6), (3.7), Lemma 2.1, and Lemma 2.5, we reach $Az^* \in \mathrm{Fix}(J_r^{B_2}) = B_2^{-1}(0)$. Thus, we deduce that $z^* \in \mathrm{SVIP}(B_1, B_2)$. The proof is completed. □

Theorem 3.4. *Assume that conditions (C1)-(C4) are satisfied. Then the iterative sequence $\{x_n\}$ constructed by Algorithm 3 converges to q in norm, where $q = P_{\mathrm{Sol}} gq$.*

Remark 3.5. We note that condition (C3) can be easily implemented due to the fact that the value of $\|x_n - x_{n-1}\|$ is known before choosing μ_n. Indeed, the parameter μ_n can be chosen such that

$$\mu_n = \begin{cases} \omega, \ x_n = x_{n-1}, \\ \frac{\xi_n}{\|x_n - x_{n-1}\|}, \ x_n \neq x_{n-1}, \end{cases}$$

where $\omega \geq 0$ and $\{\xi_n\}$ is a positive sequence such that $\xi_n = o(\alpha_n)$.

We now prove the Theorem 3.4.

Proof. Firstly, we prove that the sequence $\{x_n\}$ is bounded. Taking any $q \in \mathrm{Sol}$, and noting $\upsilon \in (0,1)$ and Lemma 3.2, there exists $N_1 \geq 1$ such that

$$\|z_n - q\| \leq \|u_n - q\|, \ \forall n \geq N_1. \tag{3.8}$$

In view of the definition of u_n, one deduces that

$$\|u_n - q\| \leq \|x_n - q\| + \mu_n \|x_n - x_{n-1}\|. \tag{3.9}$$

Invoking (C3), there exists a positive constant $M_1 < \infty$ such that $\frac{\mu_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1$. From (3.8), (3.9), and Lemma 2.2, one obtains that

$$\begin{aligned}
\|x_{n+1} - q\| &= \|T_n(\alpha_n g x_n + (1-\alpha_n) z_n) - q\| \\
&\leq \|\alpha_n (g x_n - q) + (1-\alpha_n)(z_n - q)\| \\
&\leq \alpha_n \|g x_n - f(q)\| + \alpha_n \|f(q) - q\| + (1-\alpha_n)\|z_n - q\| \\
&\leq \alpha_n \alpha \|x_n - q\| + \alpha_n \|f(q) - q\| + (1-\alpha_n)(\|x_n - q\| + \mu_n \|x_n - x_{n-1}\|) \\
&\leq (1 - (1-\alpha)\alpha_n)\|x_n - q\| + \alpha_n \|f(q) - q\| + \alpha_n M_1 \\
&\leq (1 - (1-\alpha)\alpha_n)\|x_n - q\| + \alpha_n (1-\alpha) \frac{\|f(q) - q\| + M_1}{1-\alpha} \\
&\leq \max\{\|x_n - q\|, \frac{\|f(q) - q\| + M_1}{1-\alpha}\} \\
&\leq \cdots \leq \max\{\|x_1 - q\|, \frac{\|f(q) - q\| + M_1}{1-\alpha}\}.
\end{aligned}$$

This implies that sequence $\{x_n\}$ is bounded. Using (3.8), (3.9) and the definition of $\{y_n\}$, one concludes that $\{z_n\}$, $\{y_n\}$, and $\{u_n\}$ are bounded. It follows that

$$\begin{aligned}
\|u_n - q\|^2 &= \|x_n - q + \mu_n (x_n - x_{n-1})\|^2 \\
&\leq \|x_n - q\|^2 + 2\mu_n \langle x_n - x_{n-1}, u_n - q \rangle \\
&\leq \|x_n - q\|^2 + 2\mu_n \|x_{n-1} - x_n\| \|u_n - q\| \\
&\leq \|x_n - q\|^2 + \mu_n \|x_{n-1} - x_n\| M_2,
\end{aligned} \tag{3.10}$$

where $M_2 = \sup_{n\geq 0}\{2\|u_n - q\|\} < \infty$. It follows from Lemma 2.2, Lemma 3.2 and (3.10) that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \|T_n(\alpha_n g x_n + (1-\alpha_n)z_n) - q\|^2 \\
&\leq \|\alpha_n g x_n + (1-\alpha_n)z_n - q\|^2 \\
&\leq \alpha_n \|gx_n - q\|^2 + (1-\alpha_n)\|z_n - q\|^2 \\
&\leq \alpha_n \|gx_n - q\|^2 + \|u_n - q\|^2 - \left(1 - \frac{\upsilon^2 r_n^2}{r_{n+1}^2}\right)\|y_n - u_n\|^2 \\
&\leq \alpha_n \|gx_n - q\|^2 + \|x_n - q\|^2 + \mu_n \|x_{n-1} - x_n\| M_2 - \left(1 - \frac{\upsilon^2 r_n^2}{r_{n+1}^2}\right)\|y_n - u_n\|^2 \\
&\leq \alpha_n M_3 + \|x_n - q\|^2 - \left(1 - \frac{\upsilon^2 r_n^2}{r_{n+1}^2}\right)\|y_n - u_n\|^2, \qquad (3.11)
\end{aligned}$$

where $M_3 = \sup_{n\geq 1}\{\|gx_n - q\|^2 + \frac{\mu_n}{\alpha_n}\|x_{n-1} - x_n\| M_2\} < \infty$. Let us rewrite (3.11) as

$$\left(1 - \frac{\upsilon^2 r_n^2}{r_{n+1}^2}\right)\|y_n - u_n\|^2 \leq \alpha_n M_3 + \|x_n - q\|^2 - \|x_{n+1} - q\|^2. \qquad (3.12)$$

Setting $g_n = \alpha_n g x_n + (1-\alpha_n)z_n$ and using (3.8) and (3.10), we infer

$$\begin{aligned}
&\|x_{n+1}-q\|^2 \\
= \;& \|((1-\beta_n)I + \beta_n S)g_n - q\|^2 \\
= \;& (1-\beta_n)\|g_n - q\|^2 + \beta_n \|Sg_n - q\|^2 - \beta_n(1-\beta_n)\|g_n - Sg_n\|^2 \\
\leq \;& ((1-\beta_n)\|g_n - q\|^2 + \beta_n(\|g_n - q\|^2 + k\|g_n - Sg_n\|^2) - \beta_n(1-\beta_n)\|g_n - Sg_n\|^2 \\
= \;& \|g_n - q\|^2 - \beta_n(1-\beta_n-k)\|g_n - Sg_n\|^2 \\
\leq \;& \alpha_n \|gx_n - q\|^2 + (1-\alpha_n)\|z_n - q\|^2 - \beta_n(1-\beta_n-k)\|g_n - Sg_n\|^2 \\
\leq \;& \alpha_n \|gx_n - q\|^2 + \|x_n - q\|^2 + \mu_n\|x_{n-1} - x_n\| M_2 - \beta_n(1-\beta_n-k)\|g_n - Sg_n\|^2.
\end{aligned}$$

Thus

$$\begin{aligned}
&\beta_n(1-\beta_n-k)\|g_n - Sg_n\|^2 \\
\leq \;& \|x_n - q\|^2 - \|x_{n+1} - q\|^2 + \alpha_n\|gx_n - q\|^2 + \mu_n\|x_{n-1} - x_n\| M_2. \qquad (3.13)
\end{aligned}$$

We next demonstrate that the convergence of $\{\|x_n - q\|\}$ to zero by the following two cases:

Case 1. Assume that there exists $N_0 \in \mathbb{N}$ such that the sequence $\{\|x_n - q\|\}_{n\geq N_0}$ is monotone decreasing; then, $\lim_{n\to\infty}\|x_n - q\|$ exists. Noting $\upsilon \in (0,1)$, from (C1) and putting n tend to infinity in (3.12), we derive that

$$\lim_{n\to\infty}\|y_n - u_n\| = 0. \qquad (3.14)$$

It follows from (C1), (C3) and the definitions of u_n that

$$\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\mu_n\|x_n - x_{n-1}\| = 0. \qquad (3.15)$$

By (C1), (C2), (C3), and (3.13), we obtain that

$$\lim_{n\to\infty}\|g_n - Sg_n\| = 0. \qquad (3.16)$$

Due to the fact that $g_n = \alpha_n g x_n + (1-\alpha_n) z_n$, we infer that

$$\lim_{n\to\infty} \|g_n - z_n\| = \lim_{n\to\infty} \alpha_n \|g x_n - z_n\| = 0. \tag{3.17}$$

In view of Lemma 3.1, the definition of z_n, Lemma 2.5, and (3.14), one deduces that

$$\begin{aligned} \lim_{n\to\infty} \|z_n - y_n\| &= r_n \left\| A^*(I - J_{r_n}^{B_2}) A y_n - A^*(I - J_{r_n}^{B_2}) A u_n \right\| \\ &\leq \lim_{n\to\infty} r_n \|A\|^2 \|y_n - u_n\| = 0. \end{aligned} \tag{3.18}$$

Thanks to (3.14), (3.15), and (3.18), one infers that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.19}$$

Taking into consideration that

$$\begin{aligned} \|x_{n+1} - z_n\| &= \|((1-\beta_n)I + \beta_n S) g_n - z_n\| \\ &\leq (1-\beta_n)\|g_n - z_n\| + \beta_n(\|S g_n - g_n\| + \|g_n - z_n\|), \end{aligned}$$

we can deduce from (3.16) and (3.17) that $\lim_{n\to\infty} \|x_{n+1} - z_n\| = 0$. Notice $\|x_{n+1} - x_n\| \leq \|x_{n+1} - z_n\| + \|z_n - x_n\|$, which together with (3.19) implies

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.20}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to some $z \in \mathscr{H}_1$ and

$$\limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle = \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle = \langle f(q) - q, z - q\rangle.$$

According to (3.14), (3.15), and Lemma 3.3, we derive that $z \in \mathrm{SVIP}(B_1, B_2)$. By the assumption that $I - S$ is demiclosed and noticing (3.16), (3.17), and (3.19), we deduce $z \in \mathrm{Fix}(S)$. Therefor, $z \in \mathrm{Sol}$. It is easy to see that $P_{\mathrm{Sol}} g$ is a contractive mapping. Banach's Contraction Mapping Principle implies that $P_{\mathrm{Sol}} g$ has a unique fixed point, say $q \in \mathscr{H}_1$. Namely, $q = P_{\mathrm{Sol}} g q$. It follows that $\langle f(q) - q, y - q\rangle \leq 0$ for ally $\in$ Sol. Therefore, we have that

$$\begin{aligned} \limsup_{n\to\infty} \langle f(q) - q, x_n - q\rangle &= \lim_{k\to\infty} \langle f(q) - q, x_{n_k} - q\rangle \\ &= \langle f(q) - q, z - q\rangle \leq 0. \end{aligned}$$

This together with (3.20) implies that

$$\limsup_{n\to\infty} \langle f(q) - q, x_{n+1} - q\rangle \leq 0. \tag{3.21}$$

It follows from (3.8), (3.10), and Lemma 2.2 that

$$\begin{aligned}
&\|x_{n+1}-q\|^2\\
\le\ & \|\alpha_n g x_n+(1-\alpha_n)z_n-q\|^2\\
=\ & \|\alpha_n(gx_n-f(q))+(1-\alpha_n)(z_n-q)+\alpha_n(f(q)-q)\|^2\\
\le\ & \|\alpha_n(gx_n-f(q))+(1-\alpha_n)(z_n-q)\|^2+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\
\le\ & \alpha_n\|gx_n-f(q)\|^2+(1-\alpha_n)\|z_n-q\|^2+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\
\le\ & \alpha_n\alpha\|x_n-q\|^2+(1-\alpha_n)(\|x_n-q\|^2+\mu_n\|x_n-x_{n-1}\|M_2)+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle\\
\le\ & (1-\alpha_n(1-\alpha))\|x_n-q\|^2+\alpha_n\left(\frac{\mu_n}{\alpha_n}\|x_n-x_{n-1}\|M_2\right)+2\alpha_n\langle f(q)-q,x_{n+1}-q\rangle.
\end{aligned}$$

Thus, from (3.21), (C1), (C3), and Lemma 2.4, we conclude that $x_n\to q$.

Case 2. Assume that $\{\|x_n-q\|\}$ is not monotone decreasing. Then there exists a subsequence $\{\|x_{n_i}-q\|\}$ of $\{\|x_n-q\|\}$ such that

$$\|x_{n_i}-q\|<\|x_{n_i+1}-q\|,\ \forall i\in\mathbb{N}. \tag{3.22}$$

According to Lemma 2.3, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that

$$\max\{\|x_{m_k}-q\|,\|x_k-q\|\}\le\|x_{m_k+1}-q\|. \tag{3.23}$$

Following similar argument as in Case I, it is easy to obtain

$$\lim_{k\to\infty}\|x_{m_k+1}-x_{m_k}\|=0 \tag{3.24}$$

We want to show that

$$\limsup_{k\to\infty}\langle f(q)-q,x_{m_k+1}-q\rangle\le 0, \tag{3.25}$$

where $q=P_{\mathrm{Sol}}gq$. Without loss of generality, there exists a subsequence $\{x_{m_{k_j}}\}$ of $\{x_{m_k}\}$ such that $x_{m_{k_j}}\rightharpoonup w$ for some $w\in\mathscr{H}_1$ and $\limsup_{k\to\infty}\langle f(q)-q,x_{m_k}-q\rangle=\lim_{j\to\infty}\langle f(q)-q,x_{m_{k_j}}-q\rangle$. Like Case 1, we can also obtain $\omega\in\mathrm{Sol}$. Thus, we have that

$$\begin{aligned}
\limsup_{k\to\infty}\langle f(q)-q,x_{m_k}-q\rangle &= \lim_{j\to\infty}\langle f(q)-q,x_{m_{k_j}}-q\rangle\\
&=\langle f(q)-P_{\mathrm{Sol}}f(q),w-P_{\mathrm{Sol}}f(q)\rangle\le 0.
\end{aligned}$$

This together with (3.24) implies that $\limsup_{k\to\infty}\langle f(q)-q,x_{m_k+1}-q\rangle\le 0$. Resorting to (3.8), (3.10), and Lemma 2.2, one deduces that

$$\begin{aligned}
\|x_{m_k+1}-q\|^2 \le\ & \|\alpha_{m_k}(fx_{m_k}-f(q))+(1-\alpha_{m_k})(z_{m_k}-q)+\alpha_{m_k}(f(q)-q)\|^2\\
\le\ & \|\alpha_{m_k}(fx_{m_k}-f(q))+(1-\alpha_{m_k})(z_{m_k}-q)\|^2+2\alpha_{m_k}\langle f(q)-q,x_{m_k+1}-q\rangle\\
\le\ & \alpha_{m_k}\|fx_{m_k}-f(q)\|^2+(1-\alpha_{m_k})\|z_{m_k}-q\|^2+2\alpha_{m_k}\langle f(q)-q,x_{m_k+1}-q\rangle\\
\le\ & \alpha_{m_k}\alpha\|x_{m_k}-q\|^2+(1-\alpha_{m_k})\|u_{m_k}-q\|^2+2\alpha_{m_k}\langle f(q)-q,x_{m_k+1}-q\rangle\\
\le\ & \alpha_{m_k}\alpha\|x_{m_k}-q\|^2+(1-\alpha_{m_k})(\|x_{m_k}-q\|^2\\
& +\mu_{m_k}\|x_{m_k}-x_{m_k-1}\|M_2)+2\alpha_{m_k}\langle f(q)-q,x_{m_k+1}-q\rangle\\
\le\ & (1-\alpha_{m_k}(1-\alpha))\|x_{m_k}-q\|^2+\alpha_{m_k}\left(\frac{\mu_n}{\alpha_{m_k}}\|x_{m_k}-x_{m_k-1}\|M_2\right)\\
& +2\alpha_{m_k}\langle f(q)-q,x_{m_k+1}-q\rangle,
\end{aligned}$$

which yields that

$$\begin{aligned} &(1-\alpha)\alpha_{m_k}\left\|x_{m_k+1}-q\right\|^2 \\ \leq\ &(1-\alpha_{m_k}(1-\alpha))\left(\|x_{m_k}-q\|^2-\left\|x_{m_k+1}-q\right\|^2\right)+\alpha_{m_k}\left(\frac{\mu_n}{\alpha_{m_k}}\|x_{m_k}-x_{n-1}\|M_2\right) \\ &+2\alpha_{m_k}\langle f(q)-q, x_{m_k+1}-q\rangle. \end{aligned}$$

Noticing (3.22), we infer

$$\left\|x_{m_k+1}-q\right\|^2 \leq \frac{1}{1-\alpha}\left(\frac{\mu_n}{\alpha_{m_k}}\|x_{m_k}-x_{n-1}\|M_2\right)+\frac{2}{1-\alpha}\langle f(q)-q, x_{m_k+1}-q\rangle.$$

By using (C1), (C3) and (3.25), we have that $\lim_{k\to\infty}\left\|x_{m_k+1}-q\right\|=0$. It then follows from (3.23) that $\lim_{k\to\infty}\|x_k-q\|=0$. Hence we can obtain that the sequences constructed by Algorithm 3 strongly converge to the unique fixed point $q\in \mathrm{Sol}$ of the contractive mapping $P_{\mathrm{Sol}}g$. Then the proof is completed. □

Remark 3.6. The main results in this paper have the following aspects compared with the known results in the literature.

(1) The approach for proving the main results are simpler and different from those in the early and recent literature manly duo to Lemma 2.5(3). In fact, Lemma 2.5(3) together with Lemma 3.3 presents an interesting and simple method to prove $u_n \to p \in \mathrm{SVIP}(B_1,B_2)$ under conditions $u_n \rightharpoonup p$ and $\left\|u_n - T_{r_n}^F u_n\right\| \to 0$.

(2) Theorem 3.4 strengthens the corresponding results in [1, 6, 7] including finding a solution for the VIP, a solution to the SVIP, or common solutions of the SVIP and fixed point problems for demicontractive mappings. Moreover, our proof is also different from those.

Corollary 3.7. *Assume that Conditions (C1)-(C4) are satisfied. Then the sequence* $\{x_n\}$ *constructed by Algorithm 3 converges strongly to a point q, where* $q = P_{\mathrm{SVIP}(C,Q)}f(q)$.

4. Numerical Results

In this section, we report the preliminary numerical results of our proposed method in comparison with related methods in the literature. We compare Algorithm 3 (HSVIPP) with Algorithm 1 (HSVIP, Algorithm 2 of Chuang [6]) and Algorithm 2 (SSVIP, Algorithm 3.1 of Jolaoso and Karahan [7]) using the following example. The performance of these three algorithms is demonstrated in Figure 1.

Example 4.1. Let $\mathscr{H}_1=\mathscr{H}_2=\mathbb{R}^2$. Let $g:\mathbb{R}^2\to\mathbb{R}^2$ be given by $gx=\frac{1}{2}x$, $A:\mathbb{R}^2\to\mathbb{R}^2$ be given by $Ax=x$, $B_1,B_2:\mathbb{R}^2\to\mathbb{R}^2$ be given by $B_1x=B_2x=2x$, and $S:\mathbb{R}^2\to\mathbb{R}^2$ be given by $Sx=\frac{1}{3}x$ for all $x\in\mathbb{R}^2$. One immediately deduces that g is $\frac{1}{2}$-contraction, A is a bounded linear operator, and S is 0-demimetric mapping. Let us choose $\alpha_n=\frac{1}{2n+1}$, $\beta_n=\frac{1}{2}$, $\sigma_n=\frac{1}{(n+1)^2}$, and $r_1=\frac{1}{3}$.

We test Algorithm 1 (HSVIP), Algorithm 2 (SSVIP) and Algorithm 3 (HSVIPP) from different initial points x_0 as follows:
Case I: $x_0=(20,5)^T$;
Case II: $x_0=(10,1)^T$;

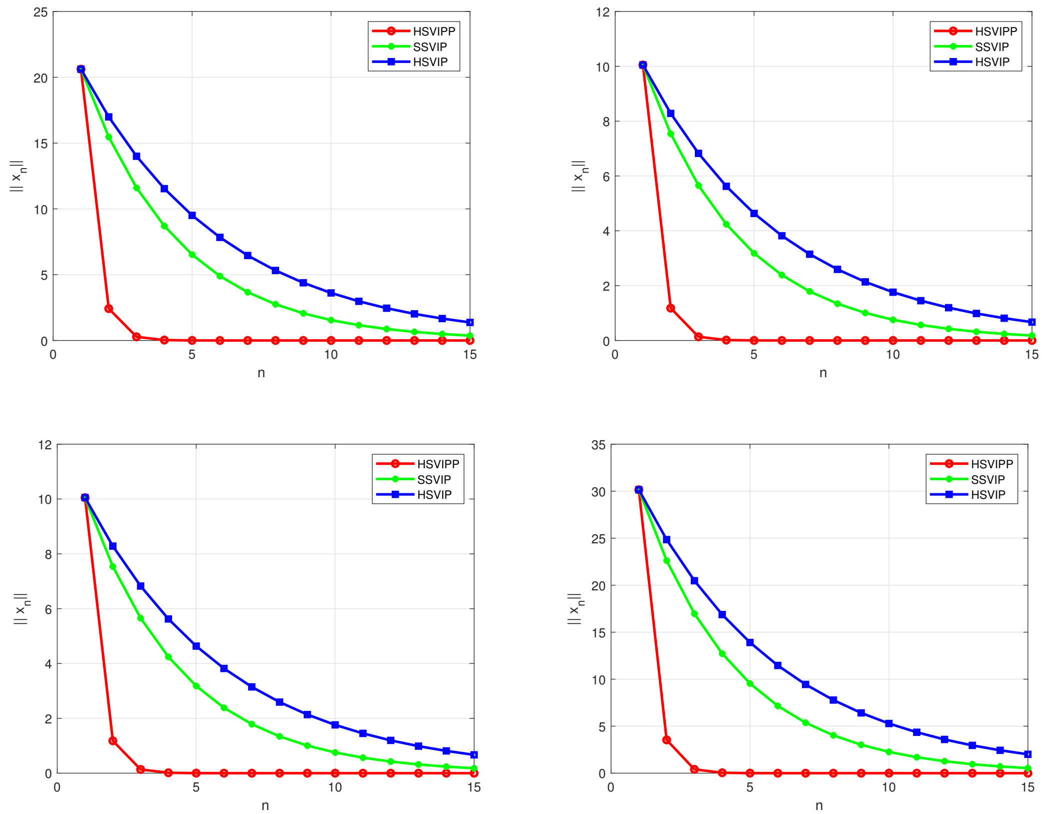

FIGURE 1. Top left: Case I; top right: Case II, bottom left: Case III, bottom right: Case IV.

Case III: $x_0 = (1,10)^T$;
Case IV: $x_0 = (3,30)^T$.

According to the performance figures, we can see that Algorithm 3 has a better convergence behavior than Algorithm 1 and Algorithm 2 for Example 4.1.

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