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EXISTENCE AND STABILITY OF GENERALIZED WEAKLY-MIXED VECTOR EQUILIBRIUM PROBLEMS

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Abstract. The purpose of this paper is to investigate a new model, called the generalized weakly-mixed vector equilibrium problem and denoted by GWMVEP, which is an extension of vector equilibrium problems, vector variational inequalities and vector optimization problems. We first verify the existence of the GWMVEP on a noncompact domain by using the KKMF lemma. In addition, we identify a class of the GWMVEP with weaker assumptions such that most of the GWMVEPs are structurally stable and robust in the sense of the Baire classification.

Keywords. Bounded rationality; KKMF lemma; Robust; Structurally stable; Vector equilibrium problem.

1. Introduction

Let X be a nonempty subset of a Hausdorff topological vector space G and C be defined by the closed convex pointed cone of a Hausdorff topological vector space Z. Assume that $g : X \times X \to Z,\ T : X \to L(G,Z),\ f : X \to Z$ are three mappings, where $L(G,Z)$ is denoted as all bounded linear operators from G to Z. We consider the GWMVEP and it is our main object of interest in this paper. The GWMVEP is formulated as finding

$$x^* \in X \text{ such that } g(x^*,y) + \langle T(x^*), y - x^* \rangle + f(y) - f(x^*) \notin -intC,\ \forall y \in X,$$

where $\langle T(x^*), y - x^* \rangle$ denotes the evaluation of the linear mapping $T(x^*)$ at $y - x^*$.

Some special cases of the GWMVEPs are as follows.

(i) If $g = 0$, then the GWMVEP is equivalent to a weakly-mixed vector variational inequality problem (briefly, WMVVIP), which is formulated as finding

$$x^* \in X \text{ such that } \langle T(x^*), y - x^* \rangle + f(y) - f(x^*) \notin -intC,\ \forall y \in X.$$

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(ii) If $f = 0$, then the GWMVEP is equivalent to a weakly-mixed vector equilibrium problem (briefly, WMVEP), which is used to find

$$x^* \in X \text{ such that } g(x^*, y) + \langle T(x^*), y - x^* \rangle \notin -intC, \ \forall y \in X.$$

(iii) If $g = 0$ and $f = 0$, then the GWMVEP reduces to a weak vector variational inequality problem (briefly, WVVIP), which is formulated as finding

$$x^* \in X \text{ such that } \langle T(x^*), y - x^* \rangle \notin -intC, \ \forall y \in X.$$

(iv) If $T = 0$ and $f = 0$, then the GWMVEP is equivalent to a weak vector equilibrium problem (briefly, WVEP), which is used to find

$$x^* \in X \text{ such that } g(x^*, y) \notin -intC, \ \forall y \in X.$$

The equilibrium problem (briefly, EP) was introduced by Blum and Oettli [3]. Recently, the GWMVEP, which has attracted much attention, played an important role in many areas, such as optimization problems, variational inequalities, equilibrium problems, and fixed point problems; see, e.g., [9, 10, 12, 15]. Many scholars proposed existence theorems of solutions, the method of duality, and the iterative method for mixed equilibrium problems by combining equilibrium problems with variational inequalities under different conditions; see, e.g., [4, 5, 7, 18, 19, 20, 22]. Farajzadeh [8] proposed two new classes of mixed vector equilibrium problems and proved the existence of solutions under noncompact domains. In [16], Lee et al. constructed a new class of generalized mixed vector equilibrium problems and obtained the existence theorems by using KKMF lemma and Nadler lemma. In [11], Izuchukwu studied the existence and uniqueness of the solution of the mixed equilibrium problem.

As it is known, many economic models are based on the assumption of fully rational players. However, the assumption is often not satisfied in real life. Thus, many scholars applied these models to the bounded rationality, which has important applications. In 2001, Anderlini and Canning [1] established a bounded rationality model $M = \{\Lambda, X, \Gamma, R\}$, but the assumption conditions were too strong to satisfy many current models. In [24, 25, 26], the authors weakened some conditions: (1) the compact space was reduced to a complete metric space; (2) the continuous behavioral mapping were reduced to upper semicontinuous; (3) the rational function was reduced to lower pseudo-continuous. Furthermore, they expanded the applications of the game model and obtained some interesting results. In [13], Kilicman et al. proved the existence of strong mixed vector equilibrium problems with multivalued mappings by applying the Fan-Browder fixed point lemma. In 2008, Peng and Yao [18] introduced a new class of generalized mixed equilibrium problems consisting of equilibrium problems, variational inequality problems, and minimization problems and obtained some strong convergence theorems via hybrid methods and extragradient methods. Subsequently, some weakly convergence theorems were obtained by using the extragradient methods in [19]. In [21], Shan and Huang extended [18] and [19] to generalized mixed vector equilibrium problems and proved the existence and uniqueness of solutions with the aid of iterative schemes. Recently, Chen et al. [4] constructed bilevel generalized mixed equilibrium problems and investigated its existence theorems and well-posedness under some mild assumptions. However, the assumption of full rationality is not easy to satisfy. The introduction of the bounded rationality is more realistic. Therefore, it is necessary to study the structural stability and robustness of the GWMVEP solutions under bounded rationality.

Inspired by the results mentioned above, we fill the gaps in this work by discussing the existence, structural stability, and robustness with the bounded rationality of the GWMVEP. The outline of this paper is organized as follows. In Section 2, we provide some necessary mathematical preliminaries. In Section 3, we construct a function for GWMVEP, and prove the existence of the GWMVEP solution by using the KKMF lemma. In Section 4, we define rational functions for GWMVEP and prove that most of the GWMVEPs are structurally stable and robust to ε-equilibrium in the Baire classification sense. Finally, we come to the conclusion in Section 5.

2. Preliminaries

In this section, we describe some necessary mathematical preliminaries. Throughout this paper, the upper semicontinuous, low semicontinuous, C-upper semicontinuous, and C-low semicontinuous are represented as *u.s.c* , *l.s.c*, *C-u.s.c*, and *C-l.s.c*, respectively.

Definition 2.1. (see [2, 14]) Assume that X and Y are two topological spaces and $K : X \to P_0(Y)$ represents a set-valued correspondence, where all nonempty subsets of Y are represented by $P_0(Y)$. K is said to be

(1) *u.s.c* (resp. *l.s.c*) at $x \in X$ if, for any open set M of Y with $K(x) \subset M$ (resp. $K(x) \bigcap M \neq \emptyset$), then there exists an open neighborhood $O(x)$ of x such that $K(x') \subset M$ (resp. $K(x') \bigcap M \neq \emptyset$) for any $x' \in O(x)$;

(2) *u.s.c* (resp. *l.s.c*) on X if it is *u.s.c* (resp. *l.s.c*) at each point $x \in X$; K is continuous at $x \in X$ if K is both *u.s.c* and *l.s.c* at $x \in X$;

(3) *usco* correspondence at $x \in X$ if K is *u.s.c* at x and $K(x)$ is a compact set.

Definition 2.2. (see [17]) Assume that X is a nonempty subset of a Hausdorff topological space, Z is a Hausdorff linear topological space, C is a cone in Z, and $K : X \to Z, x \in X$. If for any open neighborhood V of zero element in Z, then there exists an open neighborhood $O(x)$ of $x \in X$ satisfies $K(x') \in K(x) + V - C$ (resp. $K(x') \in K(x) + V + C$) for each $x' \in O(x)$. Then, K is *C-u.s.c* (resp. *C-l.s.c*)) at $x \in X$. If K is *C-u.s.c* (resp. *C-l.s.c*) for each $x \in X$, then K is a *C-u.s.c* (resp. *C-l.s.c*) on X.

Definition 2.3. (see [17]) Assume that X is a nonempty convex subset of a linear space, Z is a Hausdorff linear topological space, C is a cone of Z, and $K : X \to Z$. If, for all $x_1, x_2 \in X$, $\lambda \in (0,1)$, $y \in Z$ with $K(x_1) \in y - C$, and $K(x_2) \in y - C$ (resp. $K(x_1) \in y + C$, $K(x_2) \in y + C$) satisfy $K(\lambda x_1 + (1-\lambda)x_2) \in y - C$ (resp. $K(\lambda x_1 + (1-\lambda)x_2) \in y + C$), then K is called *C-quasiconvex* (resp. *C-quasiconcave*)) on X.

Lemma 2.4. *(see [27]) If $K : X \to Y$ is C-quasiconvex mapping on X, then the set $\{x \in X : K(x) \in -intC\}$ is convex.*

Lemma 2.5. *(see [23]) Assume that X is a nonempty subset of a Hausdorff topological space. For the function $\Psi = (\Psi_1, \cdots, \Psi_k) : X \to R^k$, where $\Psi_i : X \to R$ $(i = 1, \cdots, k)$. The vector-valued function Ψ is R^k_+-u.s.c on X iff Ψ_i is u.s.c on X for $i = 1, \cdots, k$.*

Lemma 2.6. *(KKMF Lemma, see [6]) Assume that X is a nonempty subset in a Hausdorff linear topological space Y. If $K : X \to P_0(Y)$ is a KKM correspondence, $K(x)$ is a nonempty closed subset in Y for each $x \in X$, and there exists $x_0 \in X$ such that $K(x_0)$ is a compact set, and for any finite subset $\{x_i\}_{i \in \Delta}$ (Δ is the index set) of X, $co\{x_i : i \in \Delta\} \subset \cup_{i \in \Delta} K(x_i)$, then $\cap_{x \in X} K(x) \neq \emptyset$.*

Definition 2.7. (see [24]) If the equilibrium mapping $E : \Lambda \to P_0(X)$ is continuous at $\varphi \in \Lambda$, then model M is said to be structurally stable at φ. For any $\varphi \in \Lambda$, $\delta > 0$, and $\varepsilon > 0$, if $\varepsilon^* > \varepsilon$ and $\rho(\varphi, \varphi') < \varepsilon^*$, there exists $h(E(\varphi', \varepsilon), E(\varphi')) < \delta$, then the game model M is said to be robust to the ε-equilibrium point set at φ.

Lemma 2.8. *(see [26]) Assume that (Λ, ρ) is a complete metric space, X is a compact metric space, $\Gamma : \Lambda \to P_0(X)$ is a u.s.c set-valued mapping, $\Gamma(\varphi)$ is a nonempty compact set for any $\varphi \in \Lambda$, $R : Graph(\Gamma) \to R^k_+$ is l.s.c, and $E(\varphi) \neq \emptyset$ for any $\varphi \in \Lambda$. Then,*
(i) The equilibrium mapping $E : \Lambda \to P_0(X)$ is a usco correspondence.
(ii) There exists a dense residue set Q in Λ such that M is structurally stable at φ for any $\varphi \in Q$.
(iii) If M is structurally stable at $\varphi \in \Lambda$, then M is robust to the ε-equilibrium at φ.
(iv) There exists a dense residue set Q in Λ such that, for any $\varphi \in Q$, $\varphi_n \to \varphi$ and $\varepsilon_n \to 0$, $h(E(\varphi_n, \varepsilon_n), E(\varphi)) \to 0$.
(v) If $\varphi \in \Lambda$ and $E(\varphi) = \{x\}$ (a single point set), then M is structurally stable at $\varphi \in \Lambda$ and robust to the ε-equilibrium at φ.

3. EXISTENCE OF THE GWMVEP

In this section, we prove the existence result for the GWMVEP.

Theorem 3.1. *Assume that X is a nonempty closed convex subset of a Hausdorff topological vector space G, C is a closed convex pointed cone of the Hausdorff topological vector space Z, and $intC \neq \emptyset$. Suppose that $g : X \times X \to Z$, $T : X \to L(G, Z)$, and $f : X \to Z$ satisfy the following conditions:*

(i) $g(x,x) = 0$, $\forall x \in X$;

(ii) $g(\cdot, y)$, $\langle T(\cdot),\ y - \cdot\rangle$ and $f(y) - f(\cdot)$ are C-u.s.c for any $y \in X$;

(iii) $g(x, \cdot)$, $\langle T(x),\ \cdot - x\rangle$ and $f(\cdot) - f(x)$ are C-quasiconvex for any $x \in X$;

(iv) there exists a compact set X_0 of X satisfying $y_0 \in X_0$ with $g(x, y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x) \in -intC$ for any $x \in X/X_0$.

Then, there exists $x_0 \in X$ satisfying $g(x_0, y) + \langle T(x_0), y - x_0\rangle + f(y) - f(x_0) \notin -intC$ for any $y \in X$.

Proof. Define a set-valued mapping $\mathscr{F} : X \to P_0(X)$ by

$$\mathscr{F}(y) = \{x \in X : g(x,y) + \langle T(x), y - x\rangle + f(y) - f(x) \notin -intC\},\ \forall y \in X.$$

One finds from condition (i) that $\mathscr{F}(y) \neq \emptyset$ for any $y \in \mathscr{F}(y)$. Next, one proves that $\mathscr{F}(y)$ is a closed subset of X for any $x \in \mathscr{F}(y)$. Assume that $x_n \in \mathscr{F}(y)$ and $x_n \to x \in X$. From condition (ii), one sees that there exists an open neighborhood V of zero element in Z satisfying

$$\begin{aligned} g(x,y) + \langle T(x), y - x\rangle + f(y) - f(x) &\in g(x_n, y) + \langle T(x_n), y - x_n\rangle + f(y) - f(x_n) + V - C \\ &\notin -intC. \end{aligned}$$

That is, $x \in \mathscr{F}(y)$. Hence, $\mathscr{F}(y)$ is a closed set.

Next, one proves that $\mathscr{F}$ is a KKM correspondence. Assume that $\{y_i : i \in \Delta\}$, where Δ is the index set, is a finite subset of X and $\omega \in co\{y_i : i \in \Delta\}$. Observe that $\omega \in co\{y_i : i \in \Delta\} \subseteq \bigcup_{i \in \Delta} \mathscr{F}(y_i)$. By contradiction, one supposes that $\mathscr{F}$ is not a KKM correspondence, that is, $\omega = \sum_{i \in \Delta} \lambda_i y_i \in co\{y_i : i \in \Delta\}$ $(y_i \geq 0, \sum_{i \in \Delta} \lambda_i = 1)$, but $\omega \notin \bigcup_{i \in \Delta} \mathscr{F}(y_i)$. Then

$$g(\omega, y_i) + \langle T(\omega), y_i - \omega\rangle + f(y_i) - f(\omega) \in -intC.$$

Furthermore, $y_i \in \{y \in X : g(\omega, y) + \langle T(\omega), y - \omega\rangle + f(y) - f(\omega) \in -intC\}$. From condition (iii) and Lemma 2.4, one has that $\{y \in X : g(x,y) + \langle T(x), y-x\rangle + f(y) - f(x) \in -intC\}$ is convex for any $x \in X$. Since $co\{y_i : i \in \Delta\}$ is the smallest convex set of $\{y_i : i \in \Delta\}$, we have

$$co\{y_i : i \in \Delta\} \subseteq \{y \in X : g(\omega, y) + \langle T(\omega), y - \omega\rangle + f(y) - f(\omega) \in -intC\},$$

that is,

$$\omega = \sum_{i\in\Delta} \lambda_i y_i \in \{y \in X : g(\omega, y) + \langle T(\omega), y - \omega\rangle + f(y) - f(\omega) \in -intC\}.$$

This implies that $g(\omega,\omega) + \langle T(\omega), \omega - \omega\rangle + f(\omega) - f(\omega) = 0$, which contradicts the fact that $(0 \notin -intC)$. Thus, $\mathscr{F}$ is a KKM correspondence.

Finally, we prove that there exists $y_0 \in X$ such that $\mathscr{F}(y_0)$ is a compact subset. From condition (iv), we see that there exists a compact set X_0 in X such that $y_0 \in X_0$ satisfies

$$g(x, y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x) \in \ -intC, \forall x \in X/X_0.$$

Then, $\mathscr{F}(y_0) = \{x \in X : g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x) \notin -intC\} \subset X_0$. $\mathscr{F}(y_0)$ is a closed subset from the proof above and X_0 is a compact set. Hence, $\mathscr{F}(y_0)$ is a compact subset. We see that $\mathscr{F}$ satisfies all the assumptions of Lemma 2.6. Hence, $\bigcap\limits_{y\in X} \mathscr{F}(y) \neq \emptyset$ and there exists $x_0 \in X$ satisfying

$$g(x_0, y) + \langle T(x_0), y - x_0\rangle + f(y) - f(x_0) \notin -intC, \ \forall y \in X.$$

This completes the proof. □

Remark 3.2. Comparing with [13, Theorem 14], we have the following assertions. (1) Our model adds the weak vector optimization problems to the mixed vector equilibrium problem and extends it to the generalized weakly-mixed vector equilibrium problem. (2) We replace the set-valued mapping with the vector-valued mapping and generalize the space to non-compact domain. (3) We replace the monotonicity, the convexity, and the continuity of [13, Theorem 14] with some weaker convexity and continuity. Hence, Theorem 3.1 extends and improves [13, Theorem 14].

4. Stability of GWMVEP

In this section, we study the structural stability and robustness of a class of GWMVEP. Let $C = R^k_+$ and X be a nonempty compact subset of the Banach space. The space of GWMVEP is defined as follows:

$$\Lambda = \left\{ \varphi = (g,T,f) : \begin{array}{l} g(x,x) = 0, \forall x \in X; \\ g(x,\cdot),\ \langle T(x), \cdot - x\rangle \text{ and } f(\cdot) - f(x) \text{ are } C\textit{-quasiconvex} \text{ for any } x \in X; \\ g(\cdot,y),\ \langle T(\cdot),\ y - \cdot\rangle \text{ and } f(y) - f(\cdot) \text{ are } R^k_+\textit{-u.s.c} \text{ for any } y \in X; \\ \text{There exists a compact set } X_0 \text{ in } X \text{ satisfying } y_0 \in X_0 \text{ with} \\ \ g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x) \in -intR^k_+ \text{ for any } x \in X/X_0. \end{array} \right\}$$

For any $\varphi_1 = (g_1, T_1, f_1)$ and $\varphi_2 = (g_2, T_2, f_2) \in \Lambda$, the distance ρ of Λ is defined as

$$\rho(\varphi_1, \varphi_2) = \sup_{(x,y)\in X\times X} ||g_1(x,y) - g_2(x,y)|| + \sup_{x\in X} ||T_1(x) - T_2(x)|| + \sup_{x\in X} ||f_1(x) - f_2(x))||.$$

We can easily prove that (Λ, ρ) is a complete metric space.

The following gives the bounded rationality model $M=\{\Lambda,X,\Gamma,R\}$ of the GWMVEP, $\forall\varphi=(g,T,f)\in\Lambda$, $\forall x\in X$,

(1) Λ is a complete metric space and X is a compact metric space;

(2) the solution set of the GWMVEP is expressed by

$$E(\varphi)=\{x\in X: g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x)\notin -intR^k_+,\forall y\in X\};$$

(From Theorem 3.1, we have $E(\varphi)\neq\emptyset$)

(3) the feasible mapping and behavior mapping for the GWMVEP are defined by

$$G(\varphi,x)=X,\ \Gamma(\varphi)=\{x\in X: x\in G(\varphi,x)\}=X;$$

(4) the rational function of the GWMVEP is constructed by

$$\begin{aligned}R(\varphi,x)&=\sup_{y\in X}\min_{z\in Z}\langle z,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle\\&=-\inf_{y\in X}\min_{z\in Z}\langle z,g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x)\rangle,\\R(\varphi,x)&\geq\min_{z\in Z}\langle z,-(g(x,x)+\langle T(x),x-x\rangle+f(x)-f(x))\rangle=0,\end{aligned}$$

where $Z=\{z\in R^k_+: ||z||=1\}$ is a compact subset of R^k_+.

Theorem 4.1. *For each $\varphi=(g,T,f)\in\Lambda$, $R(\varphi,x)=0$ if and only if $x\in E(\varphi)$.*

Proof. If $R(\varphi,x)=0$, then $\min_{z\in Z}\langle z,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle\leq 0$, $\forall y\in X$. If $x\notin E(\varphi)$, then there exists $y\in X$ such that $g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x)\in -intR^k_+$. Furthermore, for any $z\in Z$, one has $\langle z,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle>0$. Since Z is a compact set, we have $\min_{z\in Z}\langle z,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle>0$, which contradicts $\min_{z\in Z}\langle z,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle\leq 0$. Thus, $x\in E(\varphi)$.

Next, one assumes that $x\in E(\varphi)$, that is, $g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x)\notin -intR^k_+$, $\forall y\in X$, where $g(x,y)=(g_1(x,y),\cdots,g_k(x,y))$, $T(x)=(T_1(x),\cdots,T_k(x))$ and $f(x)=(f_1(x),\cdots,f_k(x))$. Let $\Phi(y)=\{i: g_i(x,y)+\langle T_i(x),y-x\rangle+f_i(y)-f_i(x)\geq 0\}$. Then $\Phi(y)\neq\emptyset$. Taking $i_0\in\Phi(y)$, $z^*=(z_1,\cdots,z_k)$, where $z_{i0}=1, z_i=0\ (i\neq i_0)$, one has $z^*\in Z$ and

$$\begin{aligned}R(\varphi,x)&=\min_{z\in Z}\langle z,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle\\&\leq\langle z^*,-(g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x))\rangle\\&=-(g_{i_0}(x,y)+\langle T_{i_0}(x),y-x\rangle+f_{i_0}(y)-f_{i_0}(x))\\&\leq 0.\end{aligned}$$

Thus, $R(\varphi,x)=0$. □

Theorem 4.2. *For each $(\varphi,x)\in\Lambda\times\Gamma(\varphi)$, $R(\varphi,x)$ is l.s.c at (φ,x).*

Proof. Assume that

$$\hbar(\varphi,x)=-R(\varphi,x)=\inf_{y\in X}\min_{z\in Z}\langle z,g(x,y)+\langle T(x),y-x\rangle+f(y)-f(x)\rangle.$$

Obviously, the lower semicontinuity of $R(\varphi,x)$ is equivalent to the upper semicontinuity of $\hbar(\varphi,x)$. By the definition of $u.s.c$, we only need to prove that, $\forall\varepsilon>0$, $\forall\varphi_n=\{g_n,T_n,f_n\}\in\Lambda$ with $\varphi_n\to\varphi\in\Lambda$, and $\forall x_n\in X, x_n\to x$, there exists a sufficiently large number $N^+(\varepsilon)$ and

$n > N^+(\varepsilon)$ such that $\hbar(\varphi_n, x_n) < \hbar(\varphi, x) + \varepsilon$. Based on the definition of the infimum, there exists $y_0 \in X$ such that

$$\min_{z\in Z} \langle z, g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x)\rangle < \hbar(\varphi, x) + \frac{\varepsilon}{3}.$$

In view of $\varphi_n \to \varphi$, we have that there exists a sufficiently large number $N_1^+(\varepsilon)$ and $n \geq N_1^+(\varepsilon)$, $\varphi_n = (g_n, T_n, f_n) \to \varphi = (g, T, f)$. For all $n \geq N_1^+(\varepsilon)$ and $z \in Z$, we have

$$\begin{aligned}
&|\langle z, g_n(x,y_0) + \langle T_n(x), y_0 - x\rangle + f_n(y_0) - f_n(x) - (g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x))\rangle| \\
&\leq ||z|| ||g_n(x,y_0) + \langle T_n(x), y_0 - x\rangle + f_n(y_0) - f_n(x) - (g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x))|| \\
&\leq ||z||(||g_n(x,y_0) - g(x,y_0)|| + ||\langle T_n(x), y_0 - x\rangle - \langle T(x), y_0 - x\rangle|| \\
&\quad + ||f_n(y_0) - f_n(x) - (f(y_0) - f(x))||) \\
&\leq ||z||(||g_n(x,y_0) - g(x,y_0)|| + ||T_n(x) - T(x)|| ||y_0 - x|| + ||f_n(y_0) - f(y_0)|| + ||f(x) - f_n(x)||) \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

$y_0 \in X$ is fixed, and $g(\cdot, y_0)$, $\langle T(\cdot), y_0 - \cdot\rangle$ and $f(y_0) - f(\cdot)$ are R_+^k-$u.s.c$. Hence, $x \to g_i(x,y_0) + \langle T_i(x), y_0 - x\rangle + f_i(y_0) - f_i(x)$ is $u.s.c$ on X for any $i = 1, 2, 3, \cdots, k$, where

$$g(x,y_0) = (g_1(x,y_0), \cdots, g_k(x,y_0)),$$

$$\langle T(x), y_0 - x\rangle = (\langle T_1(x), y_0 - x\rangle, \cdots, \langle T_k(x), y_0 - x\rangle),$$

and

$$f(y_0) - f(x) = (f_1(y_0) - f_1(x), \cdots, f_k(y_0) - f_k(x)).$$

Since $x_n \to x$, there exists $N^+(\varepsilon) \geq N_1^+(\varepsilon)$ and $\forall n \geq N^+(\varepsilon)$ such that

$$\begin{aligned}
&g_i(x_n,y_0) + \langle T_i(x_n), y_0 - x_n\rangle + f_i(y_0) - f_i(x_n) \\
&< g_i(x,y_0) + \langle T_i(x), y_0 - x\rangle + f_i(y_0) - f_i(x) + \frac{\varepsilon}{3k}, \ \forall i = 1, \cdots, k.
\end{aligned}$$

For each $z \in Z$ ($\sum_{i=1}^{n} z_i < k$), we obtain

$$\begin{aligned}
&\langle z, g(x_n,y_0) + \langle T(x_n), y - x_n\rangle + f(y_0) - f(x_n) - (g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x))\rangle \\
&= \sum_{i=1}^{n} z_i \left[g_i(x_n,y_0) + \langle T_i(x_n), y_0 - x_n\rangle + f_i(y_0) - f_i(x_n) - (g_i(x,y_0) + \langle T_i(x), y_0 - x\rangle + f_i(y_0) - f_i(x))\right] \\
&< \frac{\varepsilon}{3}.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\langle z, g_n(x_n,y_0) + \langle T_n(x_n), y - x_n\rangle + f_n(y_0) - f_n(x_n) - (g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x))\rangle \\
&= \langle z, g_n(x_n,y_0) + \langle T_n(x_n), y - x_n\rangle + f_n(y_0) - f_n(x_n) - (g(x_n,y_0) + \langle T(x_n), y - x_n\rangle + f(y_0) - f(x_n))\rangle \\
&\quad + \langle z, g(x_n,y_0) + \langle T(x_n), y - x_n\rangle + f(y_0) - f(x_n) - (g(x,y_0) + \langle T(x), y_0 - x\rangle + f(y_0) - f(x))\rangle \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.
\end{aligned}$$

That is,

$$\begin{aligned}\hbar(\varphi_n,x_n) &\le \min_{z\in Z}\langle z, g_n(x_n,y_0)+\langle T_n(x_n),y-x_n\rangle + f_n(y_0)-f_n(x_n)\rangle\\ &< \min_{z\in Z}\langle z, g(x,y_0)+\langle T(x),y_0-x\rangle + f(y_0)-f(x)\rangle + \frac{2\varepsilon}{3}\\ &< \hbar(\varphi,x)+\varepsilon.\end{aligned}$$

Thus, $\hbar(\varphi,x)$ is *u.s.c* at (φ,x). That is, $R(\varphi,x)$ is *l.s.c* at (φ,x). □

From Theorems 4.1 and 4.2, the conditions of Lemma 2.8 are all satisfied. Then we have the following results.
(1) The equilibrium mapping $E:\Lambda\to P_0(X)$ is a *usco* correspondence.
(2) There exists a dense residue set Q in Λ such that M is structurally stable at φ for any $\varphi\in Q$.
(3) If M is structurally stable at $\varphi\in\Lambda$, then M is robust to the ε-equilibrium at φ.
(4) There exists a dense residue set Q in Λ such that, for any $\varphi\in Q$, $\varphi_n\to\varphi$ and $\varepsilon_n\to 0$, $h(E(\varphi_n,\varepsilon_n),E(\varphi))\to 0$.
(5) If $\varphi\in\Lambda$ and $E(\varphi)=\{x\}$ (a single point set), then M is structurally stable at $\varphi\in\Lambda$ and robust to the ε-equilibrium at φ.

Remark 4.3. Since Λ is a complete metric space and Q is a dense residual set in Λ, then (2) and (3) demonstrate that most of *GWMVEPs* are structurally stable and robust to ε-equilibrium in the Baire classification sense. It means that the equilibrium mapping $E:\Lambda\to P_0(X)$ is continuous at φ (i.e., game φ is essential).

Remark 4.4. Form conclusion (4), for any $\varphi\in Q$, $\varphi_n\to\varphi$ and $\varepsilon_n\to 0$, we have $h(E(\varphi_n,\varepsilon_n), E(\varphi))\to 0$, which demonstrates that the game φ_n is approximate ($\varphi_n\to\varphi$) and the solution method is also approximate ($\varepsilon_n\to 0$) when $\varphi\in Q$, but the set of ε_n-equilibrium points $E(\varphi_n,\varepsilon_n)$ obtained by bounded rationality can replace the set of equilibrium points $E(\varphi)$ obtained by full rationality ($n\to\infty$). Q is the dense residual set of the second category, which demonstrates that the introduction of bounded rationality does not have a large impact or influence on the results of model analysis with the assumption of full rationality in the sense of Baire classification or in the sense of nonlinear analysis.

If $Z=R$, one has the following result.

Theorem 4.5. *Let Z, G, C, and X satisfy the conditions of model Λ. Let Z^* be the topological dual of Z, and let the dual cone of C be represented by C^*, and $intC\neq\emptyset$, where*

$$C^*=\{z^*\in Z^*:\langle z^*,z\rangle\ge 0,\forall z\notin -intC\}.$$

Assume that $g':X\times X\to R$, $T':X\to G$, and $f':X\to R$ satisfy the following conditions:
(i) for all $x\in X$, $g'(x,x)=0$;
(ii) $g'(\cdot,y)$, $\langle T'(\cdot),\ y-\cdot\rangle$ and $f'(y)-f'(\cdot)$ are u.s.c for any $y\in X$;
(iii) $g'(x,\cdot)$, $\langle T'(x),\cdot-x\rangle$ and $f'(\cdot)-f'(x)$ are quasiconvex for any $x\in X$;
(iv) there exists a compact set X_0 of X satisfying $y_0\in X_0$ with $g'(x,y_0)+\langle T'(x),y_0-x\rangle+f'(y_0)-f'(x)\le 0$ for any $x\in X/X_0$.
Then, there exists $x_0\in X$ satisfying $g'(x_0,y)+\langle T'(x_0),y-x_0\rangle+f'(y)-f'(x_0)\ge 0$ for any $y\in X$.

Note that Λ' satisfying the conditions of Theorem 3.1 is the set of generalized weakly mixed scalar equilibrium problem (briefly, the *GWMSEP*). For each $\varphi_1' = (g_1', T_1', f_1'), \varphi_2' = (g_2', T_2', f_2') \in \Lambda'$, define the distance ρ on Λ' as

$$\rho(\varphi_1', \varphi_2') = \sup_{(x,y)\in X\times X} |g_1'(x,y) - g_2'(x,y)| + \sup_{x\in X} ||T_1'(x) - T_2'(x)|| + \sup_{x\in X} |f_1'(x) - f_2'(x)|.$$

Remark 4.6. Let D be the weak compact base of C^* and define

$$g'(x,y) + \langle T'(x), y-x\rangle + f'(y) - f'(x) = \max_{z^*\in D} \langle z^*, g(x,y) + \langle T(x), y-x\rangle + f(y) - f(x)\rangle .$$

From the definition of Theorem 4.5, we have

$$g'(x,y) + \langle T'(x), y-x\rangle + f'(y) - f'(x) \geq 0 \Leftrightarrow g(x,y) + \langle T(x), y-x\rangle + f(y) - f(x) \notin -intC.$$

By using this equivalence and the previous proof, we obtain the results of the structural stability and the robustness of the solution set for the *GWMSEP*.

Example 4.7. Let $X = [0,1]$, $g : X\times X \to R$, $T : X \to X$, and $f : X \to R$ be defined by

$$g(x,y) = \frac{1}{3}(x-y)^3,\ \langle T(x), y-x\rangle = \frac{1}{3}(x-y)^3,\ f(y) - f(x) = \frac{1}{3}(x-y)^3.$$

Clearly, $\varphi = (g,T,f)$ satisfies the above conditions of Theorem 4.5.

Next, we construct the bounded rationality model $M = \{\Lambda, X, \Gamma, R\}$ of the above *GWMVEP* as follows.

(i) $\varphi = (g,T,f) \in \Lambda$ and $X = [0,1]$.

(ii) The feasible mapping and behavior mapping for the *GWMSEP* are defined by

$$G(\varphi,x) = X,\ \Gamma(\varphi) = \{x\in X : x \in G(\varphi,x)\} = X.$$

(iii) The rational function of the *GWMVEP* is constructed by

$$R(\varphi,x) = \sup_{y\in X}(-(g(x,y) + \langle T(x), y-x\rangle + f(y) - f(x))) = \sup_{y\in X}(y-x)^3 = (1-x)^3.$$

(iv) The solution set of the *GWMVEP* is expressed by

$$E(\varphi) = \{x\in X : R(\varphi,x) = 0\} = \{x \in X : (1-x)^3 = 0\} = \{1\}.$$

Now, we construct a mapping sequence $\{\varphi_n\}_{n=1}^{\infty}$. Let $\varphi_n = (g_n, T_n, f_n)$, and

$$g_n(x,y) = \frac{1}{3}(x-y)^3,\ \langle T_n(x), y-x\rangle = \frac{1}{3}(x-y)^3 + \frac{(x-y)^3}{2n},\ f_n(y) - f_n(x) = \frac{1}{3}(x-y)^3 + \frac{(x-y)^3}{2n}.$$

It is easy to verify that $\varphi_n \in \Lambda$, and there exists $\varepsilon_n = \frac{1}{n} > 0$ such that $E(\varphi_n, \varepsilon_n) = [1 - \sqrt[3]{\frac{1}{1+n}}, 1]$ since $R(\varphi_n, x) = \sup_{y\in X}(y-x)^3(1+\frac{1}{n}) = (1-x)^3(1+\frac{1}{n})$,

$$E(\varphi_n, \varepsilon_n) = \{x\in X : R(\varphi_n, x) \leq \varepsilon_n\} = \{x \in X : (1-x)^3(1+\frac{1}{n}) \leq \frac{1}{n}\} = [1 - \sqrt[3]{\frac{1}{1+n}}, 1].$$

We have $h(E(\varphi_n, \varepsilon_n), E(\varphi)) \to 0$ $(n\to\infty)$. Therefore, the set of ε_n-equilibrium points $E(\varphi_n, \varepsilon_n)$ obtained by bounded rationality can replace the set of equilibrium points $E(\varphi)$ obtained by full rationality. Then M is robust to the ε-equilibrium at φ.

5. Conclusion

We first prove the existence of the GWMVEP solutions under a class of weak conditions by using the KKMF lemma. We also discuss the stability of the GWMVEP solutions in a bounded rationality framework. For $\varphi \in \Lambda$, M is structurally stable at φ and robust to the ε-equilibrium in the sense of Baire classification. Compared with previous results, this paper reduces the compactness of the space and the continuity of constraints, and adds the weak vector optimization problem to the generalized weakly-mixed vector equilibrium problem, which derive stable results for the GWMVEP solution set. The results proved in this paper improve and extend some results of [13].

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