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EXACT NULL CONTROLLABILITY FOR SEMILINEAR DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS IN HILBERT SPACES

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Abstract. We present some sufficient conditions for the exact null controllability of semilinear differential equations with nonlocal conditions in Hilbert spaces. By using operator semigroups and fixed point theorems, we obtain some new results on exact null controllability when the nonlocal item is Lipschitz continuous and is neither Lipschitz nor compact, respectively. The method in this paper can also be applied to other nonlocal differential systems to weaken the compactness of nonlocal item. An example concerning the partial differential equation is presented to illustrate our results.

Keywords. Differential equations; Exact null controllability; Nonlocal conditions; Semilinear differential equations.

1. INTRODUCTION

Controllability is one of the fundamental concepts in mathematical control system. As scientific and engineering problems can be modelled by ordinary differential equations, partial differential equations, and fractional differential equations, the controllability of linear and semilinear systems represented by differential equations has been extensively studied, using operator semigroups and other approaches. Results on exact controllability, approximate controllability, and exact null controllability can be found in [7, 8, 12, 15, 18, 20, 23, 24, 26] and the references therein.

This paper is concerned with the exact null controllability of the following semilinear differential equations with nonlocal conditions:

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t)) + Bu(t), \; t \in J := [0, b], \\ x(0) = g(x), \end{cases} \tag{1.1}$$

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where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a strongly continuous semigroup $T(t),\ t \geq 0$ in a Hilbert space X, B is a linear bounded operator from a Hilbert space U into X, $f : J \times X \to X$, the control function $u(\cdot)$ is given in $L^2(J,U)$, and the nonlocal item $g : C(J,X) \to X$ is an appropriate continuous function to be specified later.

Exact controllability enables us to steer the system to arbitrary final state while exact null controllability means that the system can be steered to zero state; see, e.g., [4, 6, 10, 11, 16]. Balachandran et al. [4] obtained the local null controllability for nonlinear functional differential systems. In [10] Dauer and Balasubramaniam discussed and investigated the sufficient conditions for the exact null controllability of integro-differential systems with infinite delay. The main approach in the papers above is to convert the controllability problem into a fixed point problem with the assumption that linear convolution operator $L_0^b u = \int_0^b T(b-s)Bu(s)\,\mathrm{d}s$ has a bounded inverse operator with values in $L^2(J,U)/\ker(L_0^b)$, but, in view of the result in [11], their conclusions on exact null controllability hold only in finite dimensional spaces when semigroup $T(t)$ is compact. To solve this problem, the authors of [21] replaced the condition of bounded inverse operator by the exact null controllability of the associated linear system with additive term. The new assumption condition does not guarantee the boundedness of $(L_0)^{-1}$, but it guarantees the boundedness of the operator $(L_0)^{-1}(N_0^b)$, see Lemma 2.4.

On the other hand, many types of differential equations with nonlocal initial condtions were studied in the literatures as the nonlocal problems ($x(0) = g(x)$) are found to have better effects in applications than the classical ones($x(0) = x_0$); see, e.g., [1, 2, 3, 15, 19, 24, 25]. When operator semigroups are applied to the existence and controllability of nonlocal problems in Banach spaces, the main difficulty is to deal with the compactness of solution operators under a compact semigroup. Some methods, including approximate solutions and measure of noncompactness, were used to discuss this problem; see [9, 17]. By using approximation methods, Fu and Zhang [14] studied the exact null controllability of nonlocal functional differential systems under the assumption of boundedness of $(L_0)^{-1}(N_0^b)$. In practice, we find compactness conditions to nonlocal items are too strong for applications. Then the initial motivation of this paper is to discuss the exact null controllability of the nonlocal problem without the compactness restriction to nonlocal items. Here under some weaker hypotheses, (H6) and (H7), we obtain the exact null controllability of nonlocal differential system (1.1) without the Lipschitz continuity to nonlinear item f. Firstly the compactness of the Cauchy operator is obtained, and then the nonlocal control system is discussed when the nonlocal items is Lipschitz continuous and is neither Lipschitz nor compact, respectively. The conditions here are more general than the previous results, and the exact null controllability result in [14] can be obtained as corollaries of our results. In fact, the research method of this paper is also applicable to other nonlocal differential problems, such as [9, 19, 24], which can improve the related research results.

The paper is organized as follows. In Section 2, we recall some concepts and facts about the operator semigroups and the exact null controllability. In Section 3, we transform controllability problem (1.1) into a fixed point problem and use some fixed point theorems to establish our results. An example is presented to illustrate the application of our results in Section 4. Finally, Section 5 ends this paper.

2. PRELIMINARIES

Throughout this paper, let X be a Hilbert space with norm $\|\cdot\|$. We denote by $C(J,X)$ the space of X-valued continuous functions on J with norm $\|x\| = \sup\{\|x(t)\|, t \in J\}$ and $L^2(J,X)$ the space of X-valued Bochner integrable functions with norm $\|f\|_{L^2} = (\int_0^b \|f(t)\|^2 \,\mathrm{d}t)^{\frac{1}{2}}$. The exact null controllability for differential equations is connected with the form of solutions to differential equations. So we give the definition of mild solutions to differential system (1.1).

Definition 2.1. A function $x \in C(J,X)$ is called a mild solution to (1.1) if it satisfies

$$x(t) = T(t)g(x) + \int_0^t T(t-s)[f(s,x(s)) + Bu(s)]\,\mathrm{d}s,\ t \in J,$$

for some control function $u \in L^2(J,U)$.

Definition 2.2. System (1.1) is said to be exactly null controllable if there is a function $u \in L^2(J,U)$ such that, under this control, $x(b) = 0$.

Define the operators $L_0^b : L^2(J,U) \to X$ and $N_0^b : X \times L^2(J,U) \to X$ as

$$L_0^b u = \int_0^b T(b-s)Bu(s)\,\mathrm{d}s,\ u \in L^2(J,U)$$

and

$$N_0^b(z_0, f) = T(b)z_0 + \int_0^b T(b-s)f(s)\,\mathrm{d}s,\ (z_0, f) \in X \times L^2(J,X),$$

and consider the linear system

$$\begin{cases} z'(t) = Az(t) + f(t) + Bu(t),\ t \in [0,b], \\ z(0) = z_0, \end{cases} \tag{2.1}$$

associated with system (1.1), where $f \in L^2(J,X)$.

Remark 2.3. It was proved in [8] that linear system (2.1) is exactly null controllable on $[0,b]$ if $\mathrm{Im}\, L_0^b \supset \mathrm{Im}\, N_0^b$, and system (2.1) is exactly null controllable if and only if there exists a number $k > 0$ such that $\|(L_0^b)^* z\| \geq k\|(N_0^b)^* z\|$ for all $z \in X$.

The following lemma is important in the discussion of exact null controllability for differential system 1.1.

Lemma 2.4 ([11])**.** *Suppose that linear system (2.1) is exactly null controllable on $[0,b]$. Then linear operator $H := (L_0)^{-1}(N_0^b) : X \times L^2(J,X) \to L^2(J,U)$ is bounded and the control*

$$u(t) = -(L_0)^{-1}\Big(T(b)z_0 + \int_0^b T(b-s)f(s)\,\mathrm{d}s\Big)(t) = -H(z_0, f)(t)$$

transfers system (2.1) from z_0 to 0, where L_0 is the restriction of L_0^b to $[\mathrm{Ker}\, L_0^b]^{\perp}$.

Next, we introduce the Hausdorff's measure of noncompactness $\beta(\cdot)$ defined by

$$\beta(B) = \inf\{\varepsilon > 0;\ B \text{ has a finite } \varepsilon\text{-net in } X\},$$

for each bounded subset B in Banach space X. We recall the following properties of the Hausdorff's measure of noncompactness β.

Lemma 2.5 ([5]). *Let X be a real Banach space and $B,C \subseteq X$ be bounded. Then the following properties are satisfied:*

(1) B is relatively compact if and only if $\beta(B)=0$;

(2) $\beta(B)=\beta(\overline{B})=\beta(\text{conv } B)$, where $\overline{B}$ and conv B mean the closure and convex hull of B, respectively;

(3) $\beta(B)\le\beta(C)$ when $B\subseteq C$;

(4) $\beta(B+C)\le\beta(B)+\beta(C)$, where $B+C=\{x+y : x\in B, y\in C\}$;

(5) $\beta(B\cup C)\le\max\{\beta(B),\beta(C)\}$;

(6) $\beta(\lambda B)\le|\lambda|\beta(B)$ for any $\lambda\in R$;

(7) If $Q: D(Q)\subseteq X\to Z$ is Lipschitz continuous with constant k, then $\beta_Z(QB)\le k\beta(B)$ for any bounded subset $B\subseteq D(Q)$, where Z is a Banach space.

The map $Q: D\subseteq X\to X$ is said to be a β-contraction if, for any bounded subset $B\subseteq D$, $\beta(QB)<k\beta(B)$ and $k<1$, where X is a Banach space.

Lemma 2.6 (See [5], Darbo-Sadovskii). *If $D\subseteq X$ is bounded closed and convex, and the continuous map $Q: D\to D$ is a β-contraction, then Q has at least one fixed point in D.*

3. Main Results

Define the solution operator $G: C(J,X)\to C(J,X)$ by

$$(Gx)(t)=T(t)g(x)+\int_0^t T(t-s)[f(s,x(s))+Bu(s)]\,\mathrm{d}s, \tag{3.1}$$

$$u(t)=-H(g(x),f)=-(L_0)^{-1}\big(T(b)g(x)+\int_0^b T(b-s)f(s,x(s))\,\mathrm{d}s\big)(t), \tag{3.2}$$

with $(G_1x)(t)=T(t)g(x)$, and $(G_2x)(t)=\int_0^t T(t-s)[f(s,x(s))+Bu(s)]$, $\mathrm{d}s$ for all $t\in[0,b]$. It is easy to see that $u(t)$ is well defined as $g(x)\in X$, $f\in L^2(J,X)$ and this control $u(\cdot)$ steers x_0 to 0. In fact, for mild solution $x(\cdot)$, by using control $u(t)$ defined by (3.2), we have

$$\begin{aligned}
x(b) &= T(b)g(x)+\int_0^b T(b-s)[f(s,x(s))+Bu(s)]\,\mathrm{d}s\\
&= T(b)g(x)+\int_0^b T(b-s)f(s,x(s))\,\mathrm{d}s\\
&\quad -\int_0^b T(b-s)B(L_0)^{-1}\big(T(b)g(x)+\int_0^b T(b-s)f(s,x(s))\,\mathrm{d}s\big)(s)\,\mathrm{d}s\\
&= 0.
\end{aligned}$$

We show that the operator G from $C(J,X)$ into itself has a fixed point, which is just the mild solution to system (1.1).

We give the following hypotheses on differential system (1.1). Let r be a finite positive constant and set $W_r=\{x\in C(J,X): \|x(t)\|\le r,\ t\in J\}$.

(H1) The semigroup $\{T(t): t\ge 0\}$ generated by A is compact, i.e., operator $T(t)$ is compact as $t>0$. Moreover, there exists a positive number M such that $M=\sup_{0\le t\le b}\|T(t)\|$ (see [22]).

(H2) The function $f(t,\cdot): X\to X$ is continuous for a.e. $t\in[0,b]$ and $f(\cdot,x):[0,b]\to X$ is measurable for all $x\in X$. Moreover, for any $r>0$, there exists a function $\rho_r\in L^2(J,R^+)$ such that $\|f(t,x)\|\le\rho_r(t)$ for a.e. $t\in[0,b]$ and $x\in X$ satisfying $\|x\|\le r$.

(H3) $g : C(J,X) \to X$ is a continuous mapping, and there exists a nondecreasing function $\gamma : R^+ \to R^+$ such that $\|g(x)\| \le \gamma(\|x\|_C)\|$.

(H4) Linear system (2.1) is exactly null controllable on $[0,b]$.

Here, we first discuss the compactness property of Cauchy operator to system (1.1).

Lemma 3.1. *Suppose that conditions* (H1) – (H4) *are satisfied. Then the mapping* $G_2 : W_r \to C(J,X)$ *defined by* $(G_2x)(t) = \int_0^t T(t-s)[f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s$ *is compact.*

Proof. It is suffice to prove that G_2W_r is relatively compact in $C(J,X)$. Firstly, we prove that, for each $t \in [0,b]$, set $\{(G_2x)(t) : x \in W_r\}$ is relatively compact in X. If $t = 0$, it is easy to see that $\{(G_2x)(0) : x \in W_r\}$ is relatively compact in X. For $t \in (0,b]$ and $\varepsilon \in (0,t)$, define

$$\begin{aligned} (G_2^{\varepsilon}x)(t) &:= \int_0^{t-\varepsilon} T(t-s)[f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s \\ &= T(\varepsilon)\int_0^{t-\varepsilon} T(t-\varepsilon-s)[f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s. \end{aligned}$$

As $T(\varepsilon)$ is compact, set $(G_2^{\varepsilon}W_r)(t) = \{(G_2^{\varepsilon}x)(t) : x \in W_r\}$ is relatively compact in X. Letting $\varepsilon \to 0^+$, we have $(G_2^{\varepsilon}x)(t) \to (G_2x)(t)$, which infers that, for each $t \in [0,b]$, $(G_2W_r)(t)$ is relatively compact in X by using the total boundedness.

Next, we prove the equicontinuity of G_2W_r. Let $0 \le t_1 < t_2 \le b$, and $x \in W_r$. Then

$$\begin{aligned} & \Big\| \int_0^{t_2} T(t_2-s)[f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s \\ & - \int_0^{t_1} T(t_1-s)[f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s \Big\| \\ = & \Big\| \int_0^{t_1} [T(t_2-s) - T(t_1-s)][f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s \\ & + \int_{t_1}^{t_2} T(t_2-s)[f(s,x(s)) - BH(g(x),f)(s)]\,\mathrm{d}s \Big\| \\ \le & \int_0^{t_1} \|T(t_2-s) - T(t_1-s)\| \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s \\ & + M\int_{t_1}^{t_2} \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s. \end{aligned} \tag{3.3}$$

If $t_1 = 0$, then the right hand of (3.3) can be made small when t_2 is small independent of $x \in W_r$. If $t_1 > 0$, then we can find a small number $\varepsilon > 0$ with $t_1 - \varepsilon > 0$. It follows from (3.3) that

$$\begin{aligned} & \int_0^{t_1} \|T(t_2-s) - T(t_1-s)\| \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s \\ & + M\int_{t_1}^{t_2} \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s \\ \le & \int_0^{t_1-\varepsilon} \|T(t_2-s) - T(t_1-s)\| \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s \\ & + 2M\int_{t_1-\varepsilon}^{t_1} \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s + M\int_{t_1}^{t_2} \|f(s,x(s)) - BH(g(x),f)(s)\|\,\mathrm{d}s \\ := & I_1 + I_2 + I_3. \end{aligned} \tag{3.4}$$

Here, as $T(t)$ is compact for $t>0$, $T(t)$ is continuous by operator norm for $t>0$ and we have $I_1\to 0$ as $t_1\to t_2$, independent of particular choice of $x(\cdot)$. Considering I_3, by the boundedness of linear operator H,

$$\begin{aligned}
&\int_{t_1}^{t_2}\|f(s,x(s))-BH(g(x),f)(s)\|\,\mathrm{d}s\\
\le\;&\int_{t_1}^{t_2}\|f(s,x(s))\|\,\mathrm{d}s+\int_{t_1}^{t_2}\|BH(g(x),f)(s)\|\,\mathrm{d}s\\
\le\;&\sqrt{t_2-t_1}\left(\int_{t_1}^{t_2}\|f(s,x(s))\|^2\,\mathrm{d}s\right)^{\frac12}+\|B\|\sqrt{t_2-t_1}\left(\int_{t_1}^{t_2}\|H(g(x),f)(s)\|^2\,\mathrm{d}s\right)^{\frac12}\\
\le\;&\sqrt{t_2-t_1}\left(\int_{0}^{b}\|f(s,x(s))\|^2\,\mathrm{d}s\right)^{\frac12}+\|B\|\sqrt{t_2-t_1}\left(\int_{0}^{b}\|H(g(x),f)(s)\|^2\,\mathrm{d}s\right)^{\frac12}\\
\le\;&\sqrt{t_2-t_1}\|\rho_r\|_{L^2}+\|B\|\sqrt{t_2-t_1}\|H\|(\|g(x)\|+\|f\|_{L^2})\\
\le\;&\sqrt{t_2-t_1}\|\rho_r\|_{L^2}+\|B\|\sqrt{t_2-t_1}\|H\|(\|\gamma(r)\|+\|\rho_r\|_{L^2}),
\end{aligned}$$

which demonstrates that $I_3\to 0$ as $t_1\to t_2$ independent of x. Similar estimation can ensure that $I_2\to 0$ as $\varepsilon\to 0^+$. Then, from (3.4) and the absolute continuity of integrals, we see that $\{(G_2x)(\cdot): x\in W_r\}$ is equicontinuous on $[0,b]$. By the Ascoli-Arzela theorem, we know that G_2W_r is relatively compact in $C(J,X)$. This completes the proof. □

Now, we give some sufficient conditions for the exact null controllability of (1.1). We make the following assumption.

(H5) $g: C(J,X)\to X$ is Lipschitz continuous with Lipschitz constant k such that $Mk<1$.

Theorem 3.2. *Assume that hypotheses* (H1) – (H5) *are satisfied. Then system* (1.1) *is exactly null controllable on* $[0,b]$*, provided that the condition*

$$\lim_{x\to+\infty}\sup\frac{M\left[\gamma(x)+\sqrt{b}\|\rho_x\|_{L^2}+\|B\|\sqrt{b}\|H\|(\gamma(x)+\|\rho_x\|_{L^2})\right]}{x}<1 \tag{3.5}$$

is satisfied.

Proof. We consider the solution operator $G: C(J,X)\to C(J,X)$ defined in (3.1). It is easy to see that if we can obtain the fixed point of G, then differential system (1.1) is exactly null controllable on $[0,b]$. Subsequently, we prove that G has a fixed point by using Darbo-Sadovskii's fixed point theorem (Lemma 2.6).

Firstly, G is continuous on $C(J,X)$. For this purpose, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $C(J,X)$ with $\lim_{n\to\infty}x_n=x$. By the continuity of f with respect to the second argument, we have that, for each $s\in[0,b]$, $f(s,x_n(s))$ converges to $f(s,x(s))$ in X, and, for $t\in[0,b]$,

$$\begin{aligned}
\|Gx_n(t)-Gx(t)\|\;\le\;&\left\|\int_0^t T(t-s)\left[BH(g(x_n),f(s,x_n(s)))(s)-BH(g(x),f(s,x(s)))(s)\right]\mathrm{d}s\right\|\\
&+\left\|\int_0^t T(t-s)[f(s,x_n(s))-f(s,x(s))]\,\mathrm{d}s\right\|+M\|g(x_n)-g(x)\|.
\end{aligned}$$

It follows that

$$\begin{aligned}\|Gx_n - Gx\| &= \sup_{t\in[0,b]} \|Gx_n(t) - Gx(t)\| \\ &\leq M\|B\| \int_0^b \|H(g(x_n), f(s,x_n(s)))(s) - H(g(x), f(s,x(s)))(s)\| \,\mathrm{d}s \\ &\quad + M\int_0^b \|f(s,x_n(s)) - f(s,x(s))\| \,\mathrm{d}s + M\|g(x_n) - g(x)\|.\end{aligned}$$

Then by the continuity of g, H, $f(t,\cdot)$, and the Lebesgue dominated convergence theorem, we obtain $\lim_{n\to\infty} Gx_n = Gx$ in $C(J,X)$.

Secondly, there exists a number $r > 0$ such that G maps W_r into itself. For $x \in W_r$, from hypotheses (H1)-(H4), we have

$$\begin{aligned}&\|(Gx)(t)\| \\ \leq\ & \|T(t)g(x)\| + \|\int_0^t T(t-s)f(s,x(s))\,\mathrm{d}s\| + \|\int_0^t T(t-s)BH(g(x),f)(s)\,\mathrm{d}s\| \\ \leq\ & M\|g(x)\| + M\int_0^t \|f(s,x(s))\|\,\mathrm{d}s + M\|B\|\int_0^t \|H(g(x),f)(s)\|\,\mathrm{d}s \\ \leq\ & M\gamma(r) + M\sqrt{b}\|\rho_r\|_{L^2} + M\|B\|\sqrt{b}\Big(\int_0^b \|H(g(x),f)(s)\|^2\,\mathrm{d}s\Big)^{\frac{1}{2}} \\ \leq\ & M\gamma(r) + M\sqrt{b}\|\rho_r\|_{L^2} + M\|B\|\sqrt{b}\|H\|(\|g(x)\| + \|f\|_{L^2}) \\ \leq\ & M\big[\gamma(r) + \sqrt{b}\|\rho_r\|_{L^2} + \|B\|\sqrt{b}\|H\|(\gamma(r) + \|\rho_r\|_{L^2})\big],\end{aligned}$$

for $t \in [0,b]$. By condition (3.5), we know that there exists a constant $r > 0$ such that

$$M\big[\gamma(r) + \sqrt{b}\|\rho_r\|_{L^2} + \|B\|\sqrt{b}\|H\|(\gamma(r) + \|\rho_r\|_{L^2})\big] \leq r.$$

Hence $G(W_r) \subseteq W_r$.

Now, according to Lemma 2.6, it remains to prove that G is a $\beta-$contracton in W_r. From assumption (H5), we can obtain that $G_1 : W_r \to C(J,X)$ is Lipschitz continuous with constant Mk. In fact, for $u, v \in W_r$,

$$\|G_1u - G_1v\| \leq M\|g(u) - g(v)\| \leq Mk\|u - v\|.$$

Then it follows from Lemma 2.5 that $\beta(G_1W_r) \leq Mk\beta(W_r)$.

Considering Lemma 3.1, we know the operator G_2 is compact on $C(J,X)$ and hence $\beta(G_2W_r) = 0$. Consequently,

$$\beta(GW_r) \leq \beta(G_1W_r) + \beta(G_2W_r) \leq Mk\beta(W_r).$$

As $Mk < 1$, we have that G is a $\beta-$contraction on W_r. By Darbo-Sadovskii's fixed point theorem, operator G has a fixed point in W_r, which infers that system (1.1) is exactly null controllable on $[0,b]$. This completes the proof. □

Now, We give a new assumption on nonlocal function g, which is neither Lipschitz nor compact.

(H6) For any $r > 0$, the set $g(\overline{\mathrm{conv}}GW_r)$ is relatively compact in X, where $\overline{\mathrm{conv}}B$ denotes the convex closed hull of set $B \subseteq C(J,X)$, G is defined in (3.1).

Remark 3.3. Clearly condition (H6) is weaker than the compactness and convexity of g. In the following, we give some special types of nonlocal item g which are neither Lipschitz continuous nor compact, but satisfy the condition (H6) in the corollaries.

Theorem 3.4. *Assume that hypotheses* (H1) – (H4) *and* (H6) *are satisfied. Then system* (1.1) *is exactly null controllable on* $[0,b]$*, provided that condition* (3.5) *is satisfied.*

Proof. Now we prove that G has a fixed point by using the Schauder fixed point theorem. From the proof of Theorem 3.2, we have known that $G : W_r \to W_r$ is continuous. Next we show that there exists a set $W \subseteq W_r$ such that $G : W \to W$ is compact. For each $t \in (0,b]$, set $\{T(t)g(x) : x \in W_r\}$ is relatively compact in X since $T(t)$ is compact for $t > 0$. Now, we prove that G_1W_r is equicontinuous on $[\eta,b]$ for any small positive number η. As $T(t)$ is operator norm continuous for $t > 0$. Then, for $x \in W_r$ and $\eta \le t_1 < t_2 \le b$, we $\|T(t_2)g(x) - T(t_1)g(x)\| = \|[T(t_2) - T(t_1)]g(x)\| \to 0$ as $t_1 \to t_2$, uniformly for all $x \in W_r$.

Moreover, due to Lemma 3.1, we have that, for each $t \in [0,b]$, set $(G_2W_r)(t)$ is relatively compact in X and G_2W_r is equicontinuous on $[0,b]$. Thus, for the operator $G = G_1 + G_2$, we have proved that $(GW_r)(t)$ is relatively compact for each $t \in (0,b]$ and GW_r is equicontinuous on $[\eta,b]$ for any small positive number η.

Letting $W = \overline{\text{conv}} GW_r$, we see that W is a bounded, closed, and convex subset of $C(J,X)$, satisfying $W \subset W_r$ and $GW \subset W$. It is easy to see that $GW(t)$ is relatively compact in X for every $t \in (0,b]$ and GW is equicontinuous on $[\eta,b]$ for any small positive number η. From hypothesis (H6), we know that $g(W) = g(\overline{\text{conv}} GW_r)$ is relatively compact in X.

Now we claim that $G : W \to W$ is a compact mapping. In fact, $(G_1W)(t)$ is relatively compact in X for every $t \ge 0$ as $g(W) = g(\overline{\text{conv}} GW_r)$ is relatively compact by hypothesis (H6). It remains to prove that G_1W is equicontinuous on $[0,b]$. Letting $x \in W$, and $0 \le t_1 < t_2 \le b$, we have

$$\|(G_1x)(t_1) - (G_1x)(t_2)\| \le \|[T(t_1) - T(t_2)]g(x)\|.$$

In view of the compactness of $g(W)$ and the strong continuity of $T(t)$ on $[0,b]$, we obtain the equicontinuity of G_1W on $[0,b]$. Thus, $G_1 : W \to C(J,X)$ is a compact mapping by Ascoli-Arzela theorem. Hence $G = G_1 + G_2$ is also compact due to Lemma 3.1.

Now G has been proved to be a compact continuous operator on $W \subset C(J,X)$. From the Schauder fixed point theorem, G has a fixed point. This completes the proof. □

Next, we give some special types of nonlocal item g, which is neither Lipschitz nor compact but satisfies condition (H6). We give the following assumptions.

(H7) $g : C(J,X) \to X$ is a continuous mapping, and there is a $\delta = \delta(r) \in (0,b)$ such that $g(x) = g(y)$ for any $x, y \in W_r$, with $x(s) = y(s)$, $s \in [\delta,b]$.

Corollary 3.5. *Assume that hypotheses* (H1) – (H4) *and* (H7) *are satisfied. Then system* (1.1) *is exactly null controllable on* $[0,b]$*, provided that condition* (3.5) *is satisfied.*

Proof. Let

$$(GW_r)_\delta = \{x \in C([0,b],X) : x(t) = y(t) \text{ for } t \in [\delta,b],\ x(t) = y(\delta) \text{ for } t \in [0,\delta), \text{ where } y \in GW_r\}.$$

From the proof of Theorem 3.4, we know that $(GW_r)_\delta$ is relatively compact in $C(J,X)$. As mapping g is continuous, we have that set $g(\overline{\text{conv}}(GW_r)_\delta)$ is relatively compact in X. Moreover, by conditions (H7), $g(\overline{\text{conv}} GW_r) = g(\overline{\text{conv}}(GW_r)_\delta)$ is also relatively compact in X. Thus all the conditions in Theorem 3.4 are satisfied. Therefore system (1.1) is exactly null controllable on $[0,b]$. □

Remark 3.6. Fu and Zhang[14] discussed the exact null controllability of evolution systems with nonlocal conditions under condition (H7) by means of an approximation method. Here by using a different approach, we obtain their results as the corollary of Theorem 3.4. In some studies of nonlocal Cauchy problems, for example [3, 13], the mapping g is given by $g(t_1,\cdots,t_s,x(t_1),\cdots,x(t_s)) = \sum_{i=1}^{s} c_i x(t_i)$ for some given constants c_i. Then the nonlocal item $g(t_1,\cdots,t_s,x(t_1),\cdots,x(t_s))$ allows the measurements at $t = t_1,\cdots,t_s$, rather than just at $t = 0$. It is easy to see that g satisfies condition (H7).

4. Applications

We consider the following partial differential system to illustrate our abstract results.

$$\begin{cases} \frac{\partial}{\partial t}x(t,\theta) = \frac{\partial^2}{\partial\theta^2}x(t,\theta) + f(t,x(t,\theta)) + u(t,\theta),\ 0\le t\le b,\ 0\le\theta\le 1, \\ x(t,0) = x(t,1) = 0, \\ x(0,\theta) = \sum_{j=1}^{q} c_j \sqrt[3]{x(t_j,\theta)}, \end{cases} \tag{4.1}$$

where $t_j \in (0,b)$, $c_j \in R$, $j = 1,\cdots,q$, and $f : R\times R \to R$ is continuous.

Let $X = L^2([0,1])$ and operator $A : D(A) \subseteq X \to X$ be defined by $Az = z''$ with

$$D(A) = \{z \in X : z, z' \text{ are absolutely continuous},\ z'' \in X, z(0) = z(1) = 0\}.$$

From Pazy [22], we know that A is the infinitesimal generator of an analytic semigroup $T(t)$, $t \ge 0$, which implies that A satisfies condition (H1). It is known that A has the eigenvalues $\lambda_n = -n^2\pi^2$, $n \in \mathbb{N}$, and the corresponding eigenvectors $e_n(\theta) = \sqrt{2}\sin(n\pi\theta)$ for $n \ge 1$, $e_0 = 1$, form an orthonormal basis for $L^2([0,1])$. Then $T(t)$ is given by

$$\begin{aligned} T(t)z &= \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < z, e_n > e_n \\ &= \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\sin(n\pi\theta)\int_0^1 z(\alpha)\sin(n\pi\alpha)\,\mathrm{d}\alpha,\ z\in X, \end{aligned}$$

and it is self-adjoint.

Let $u \in L^2([0,b],X)$. Then $B = I$ and $B^* = I$. Firstly, we consider the condition for exact null controllability of the corresponding linear system with additive term $f \in L^2([0,b],X)$

$$\begin{cases} \frac{\partial}{\partial t}x(t,\theta) = \frac{\partial^2}{\partial\theta^2}x(t,\theta) + f(t,\theta) + u(t,\theta),\ 0\le t\le b,\ 0\le\theta\le 1, \\ x(t,0) = x(t,1) = 0, \\ x(0,\theta) = x_0. \end{cases} \tag{4.2}$$

From Remark 2.3, the exact null controllability of linear system (4.2) is equal to the existence of a number $k > 0$ such that

$$\int_0^b \|B^*T^*(b-s)z\|^2\,\mathrm{d}s \ge k\Big(\|T^*(b)z\|^2 + \int_0^b \|T^*(b-s)z\|^2\,\mathrm{d}s\Big),$$

or equivalently

$$\int_0^b \|T(b-s)z\|^2\,\mathrm{d}s \ge k\Big(\|T(b)z\|^2 + \int_0^b \|T(b-s)z\|^2\,\mathrm{d}s\Big).$$

For the linear control system (4.2) with $f=0$, it was demonstrated in [8] that it is exactly null controllable if

$$\int_0^b \|T(b-s)z\|^2\,\mathrm{d}s \geq b\|T(b)z\|^2.$$

It follows that

$$\frac{1}{1+b}\int_0^b \|T(b-s)z\|^2\,\mathrm{d}s \geq \frac{b}{1+b}\|T(b)z\|^2,$$

which infers

$$\int_0^b \|T(b-s)z\|^2\,\mathrm{d}s \geq \frac{b}{1+b}\Big(\|T(b)z\|^2+\int_0^b \|T(b-s)z\|^2\,\mathrm{d}s\Big).$$

Thus linear system (4.2) is exactly null controllable on $[0,b]$ with $k=\frac{b}{1+b}$.

In addition, we may assume that function $f:[0,b]\times X\to X$ is continuous and there exists $\rho\in L^2([0,b],R^+)$ such that $\|f(t,z)\|\leq\rho(t)(\|z\|^{\frac{1}{3}}+1)$ for $(t,z)\in[0,b]\times X$. The function

$$g(x(t,\theta))=\sum_{j=1}^{q}c_j\sqrt[3]{x(t_j,\theta)}$$

satisfies hypotheses (H3), (H6), and (H7). Then all the conditions in Theorem 3.4 are satisfied and system (4.1) is exactly null controllable on $[0,b]$.

5. Conclusions

In this paper, by using operator semigroups and fixed point theorems, we discussed the exact null controllability of nonlocal semilinear differential equations. As the Lipschitz assumption to nonlinear item is completely removed without any more conditions, our work generalizes and improves previous works on exact null controllability of differential equations. It is worth pointing out that the new assumption (H6) on nonlocal item is weaker that the previous results and the method here is also available for other nonlocal differential systems, such as integro-differential equations and fractional differential equations.

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