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NONLOCAL HILFER PROPORTIONAL SEQUENTIAL FRACTIONAL MULTI-VALUED BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, we study a nonlocal multi-point Hilfer generalized proportional sequential fractional multi-valued boundary value problem. We consider both the cases of convex as well as non-convex valued maps, and we apply standard methods from multi-valued analysis to establish our existence results. In the convex case, we apply the nonlinear alternative of Leray-Schauder for multi-valued maps, while a fixed point theorem of Covitz-Nadler for contractive multi-valued maps is applied in the non convex case. Numerical examples are constructed to illustrate our obtained results.

Keywords. Boundary value problems; Hilfer fractional derivative; Hilfer fractional integral; Proportional fractional derivative; Proportional fractional integral.

1. INTRODUCTION

Fractional calculus has captivated and motivated numerous researchers across a wide spectrum of practical and scientific disciplines. Fractional integrals and derivatives, which can interpolate between operators of integer order, have a long track record and are often employed in real-world applications, such as biology, robotics, physics, ecology, viscoelasticity, control theory, control theory, economics, and so on. For a systematic development of fractional calculus as well as fractional differential equations, we refer the reader to the monographs [1]-[7]. Boundary value problems (BVP, for short) for fractional differential equations and inclusions for different kinds of equations and boundary conditions have been investigated by numerous researchers. Fractional BVP with integral and multi-point boundary conditions were studied in [8], with multi-strip boundary conditions in [9], with dual-antiperiodic boundary conditions in [10], with variable order in [11], fractional sequential BVP with nonlocal integral conditions

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in [12], for systems with integral coupled boundary conditions in [13], and so on. Most of the papers listed in the literature used the classical fractional derivative operators of Riemann-Liouville or Caputo, but these fractional operators are not appropriate to study the appeared models in my cases. To avoid the difficulties, certain modifications were introduced and some new types of fractional order derivative operators, such as Hadamard, Hilfer, Katugampola, to name a few, were proposed. Fractional derivative operators due to Hilfer [14] extend both fractional derivative operators of Riemann–Liouville and Caputo. For applications of Hilfer fractional derivative in nonlocal integro-differential equations, we refer to [15, 16, 17]. The topic of BVP for Hilfer fractional differential inclusions was studied in [18]. The notion of Hilfer generalized proportional fractional derivative operator were introduced in [19], which unifies the both Riemann–Liouville and Caputo generalized proportional fractional derivative.

Recently, the authors in [20] studied a BVP consisting of Hilfer proportional sequential fractional derivative operator of order in $(1,2]$, subject to nonlocal multi-point boundary conditions given by

$$\begin{cases} \left(D^{\alpha,\eta_*,\hat{\sigma}}_{a_0^+}+kD^{\alpha-1,\eta_*,\hat{\sigma}}_{a^+}\right)\pi(w)=f(w,\pi(w)),\ w\in[a_0,b_0],\\ \pi(a_0)=0, \qquad \pi(b_0)=\sum\limits_{j=1}^{m}\theta_j\pi(\xi_j), \end{cases} \tag{1.1}$$

where $D^{\alpha,\eta_*,\hat{\sigma}}_{a^+}$ denotes the Hilfer proportional fractional derivative operator of order $\alpha\in(1,2]$ and parameter $\eta_*\in[0,1]$, $\hat{\sigma}\in(0,1]$, $k\in\mathbb{R}$, $f:[a_0,b_0]\times\mathbb{R}\to\mathbb{R}$ is a continuous function, $a_0\geq 0$, $\theta_j\in\mathbb{R}$, and $\xi_j\in(a_0,b_0)$ for $j=1,2,3,\ldots,m$. The existence and uniqueness results were established in the scalar case with the help of Banach and Krasnosel'skiĭ fixed point theorems and the Leray-Schauder nonlinear alternative. Also an existence result was established when $f:[a_0,b_0]\times\mathbb{E}\to\mathbb{E}$ with $(\mathbb{E},\|\cdot\|_\infty)$ a real Banach space via Mönch's fixed point theorem and the technique of noncompactness measure.

In this paper, we continue the study of problem (1.1) to cover the multi-valued fractional differential equations (inclusions). To be more precisely, we, in this paper, investigate the existence of solutions for the following Hilfer-type sequential fractional proportional BVP for multi-valued fractional differential equations including multi-point nonlocal boundary conditions

$$\begin{cases} \left(D^{\alpha,\eta_*,\hat{\sigma}}_{a^+}+kD^{\alpha-1,\eta_*,\hat{\sigma}}_{a^+}\right)\pi(w)\in\Pi(w,\pi(w)),\ w\in[a_0,b_0],\\ \pi(a_0)=0, \qquad \pi(b_0)=\sum\limits_{j=1}^{m}\theta_j\pi(\xi_j), \end{cases} \tag{1.2}$$

where $\Pi:[a_0,b_0]\times\mathbb{R}\to\mathscr{P}(\mathbb{R})$ is a multi-valued map (here we denote the family of all nonempty subsets of $\mathbb{R}$ by $\mathscr{P}(\mathbb{R})$) and the other notations are the same as in problem (1.1).

Multi-valued differential equations are the generalizations of single-valued differential equations, and have effective and interesting applications in mathematical economics, control theory, sweeping process, optimization, stochastic analysis, and other fields; see, e.g., [21, 22, 23]. Nonlocal boundary conditions can be applied in physics and is more natural than the classical boundary conditions; see the survey paper [24]. They appeared in thermodynamics, petroleum exploitation, wave propagation, elasticity, and so on; see, e.g., [25] and the references therein. Two existence results are proved for the Hilfer sequential inclusion problem (1.2), one via the multi-valued version of Leray-Schauder nonlinear alternative when the multi-valued maps are convex and the other in the case of non-convex multi-valued maps by using Covitz and Nadler fixed point theorem. Numerical examples are also presented illustrating the obtained theoretical results in this paper.

Notice that ψ-Hilfer nonlocal fractional BVP with order in $(0,1]$ was studied in [26], while ψ-Hilfer nonlocal generalized proportional fractional BVP of order in $(1,2]$ was studied recently in [27] for single and multi-valued differential equations. We emphasize that no problems studied in [26] and [27] deals with sequential fractional differential equations. Our results are new and contribute significantly to the new research topic of Hilfer generalized proportional fractional differential inclusions, for which as a new subject the literature is very limited. Our results may be the first results concerning nonlocal fractional BVP of order in $(1,2]$, for sequential fractional differential inclusions involving Hilfer generalized proportional derivative operators. The paper is organized as follows. Some preliminary concepts of fractional calculus and multi-valued analysis are recalled in Section 2. In Section 3, the main results are proved, while in Section 4 some illustrative numerical examples are constructed. Section 5 ends up this paper.

2. Preliminaries

In this section, some basic related concepts are recalled.

2.1. **Fractional calculus.**

Definition 2.1. [2] (i) Let $\bar{\mathfrak{h}} \in L^1([a_0,b_0],\mathbb{R})$. The integral

$$I^{\alpha}_{a_0^+}\bar{\mathfrak{h}}(w) = \frac{1}{\Gamma(\alpha)}\int_{a_0}^{w}(w-\varsigma)^{\alpha-1}\bar{\mathfrak{h}}(\varsigma)d\varsigma,$$

is called Riemann-Liouville fractional integral operator of order $\alpha > 0$.
(ii) Let $\bar{\mathfrak{h}} \in C^n([a_0,b_0],\mathbb{R})$. The fractional derivative operator of Caputo type of order $\alpha > 0$ of the function $\bar{\mathfrak{h}}$ is defined as

$${}^{C}D^{\alpha}_{a_0^+}\bar{\mathfrak{h}}(w) = \frac{1}{\Gamma(n-\alpha)}\int_{a_0}^{w}(w-\varsigma)^{n-\alpha-1}\bar{\mathfrak{h}}^{(n)}(\varsigma)d\varsigma,\ n-1<\alpha<n,\ n\in\mathbb{N}.$$

Definition 2.2. [28] Let $\hat{\sigma} \in (0,1]$ and $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$.
(i) The left-sided generalized fractional proportional integral of order $\alpha > 0$ of the function $\bar{\mathfrak{h}}$ is defined by

$$I^{\alpha,\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}}(w) = \frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\int_{a_0}^{w}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-\varsigma)}(w-\varsigma)^{\alpha-1}\bar{\mathfrak{h}}(\varsigma)d\varsigma,\ w>a_0.$$

(ii) The left Riemann-Liouville generalized fractional proportional derivative of order $\alpha > 0$ of the function $\bar{\mathfrak{h}}$ is defined by

$$D^{\alpha,\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}}(w) = \frac{D^{n,\hat{\sigma}}}{\hat{\sigma}^{n-\alpha}\Gamma(n-\alpha)}\int_{a_0}^{w}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-\varsigma)}(w-\varsigma)^{n-\alpha-1}\bar{\mathfrak{h}}(\varsigma)d\varsigma,$$

where $n=[\alpha]+1$, $[\alpha]$ the integer part of a real number α,

$$D^{n,\hat{\sigma}} = \underbrace{D^{\hat{\sigma}}\cdots D^{\hat{\sigma}}}_{\text{n-times}}, \tag{2.1}$$

and $D^{\hat{\sigma}}\bar{\mathfrak{h}}(w) = (1-\hat{\sigma})\bar{\mathfrak{h}}(v) + \hat{\sigma}\bar{\mathfrak{h}}'(w)$.
(iii) Let $\bar{\mathfrak{h}} \in C^n([a_0,b_0],\mathbb{R})$. Then, the left-sided Caputo type generalized fractional proportional derivative of order $\alpha > 0$ of the function $\bar{\mathfrak{h}}$ is defined by

$$ {}^{C}D^{\alpha,\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}}(w) = \frac{1}{\hat{\sigma}^{n-\alpha}\Gamma(n-\alpha)}\int_{a_0}^{w} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-\varsigma)}(w-\varsigma)^{n-\alpha-1}D^{n,\hat{\sigma}}\bar{\mathfrak{h}}(\varsigma)d\varsigma, $$

provided that the right hand side exists.

Next, some properties of the previous defined operators are summarized.

Lemma 2.3. [28] *Assume that* $\alpha,\overline{\alpha} \in \mathbb{C}$ *satisfy* $\Re(\alpha) \geq 0$, $\Re(\overline{\alpha}) > 0$.
(i) Suppose that $\hat{\sigma} \in (0,1]$. *Then*

$$ (I^{\alpha,\hat{\sigma}}_{a_0^+} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}s}(s-a_0)^{\overline{\alpha}-1})(w) = \frac{\Gamma(\overline{\alpha})}{\hat{\sigma}^{\alpha}\Gamma(\overline{\alpha}+\alpha)} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}t}(w-a_0)^{\overline{\alpha}+\alpha-1}, $$

$$ (D^{\alpha,\hat{\sigma}}_{a_0^+} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}s}(s-a_0)^{\overline{\alpha}-1})(w) = \frac{\hat{\sigma}^{\alpha}\Gamma(\overline{\alpha})}{\Gamma(\overline{\alpha}-\alpha)} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}t}(w-a_0)^{\overline{\alpha}-\alpha-1}. $$

(ii) Suppose that $\hat{\sigma} \in (0,1]$. *If* $\bar{\mathfrak{h}} \in C([a_0,b_0],\mathbb{R})$, *then*

$$ I^{\alpha,\hat{\sigma}}_{a_0^+}(I^{\overline{\alpha},\hat{\sigma}}\bar{\mathfrak{h}})(w) = I^{\overline{\alpha},\hat{\sigma}}_{a_0^+}(I^{\alpha,\hat{\sigma}}\bar{\mathfrak{h}})(w) = (I^{\alpha+\overline{\alpha},\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}})(w), \ w \geq a_0. $$

(iii) Suppose that t $\hat{\sigma} \in (0,1]$ *and* $0 \leq m^* < [\Re(\alpha)]+1$. *If* $\bar{\mathfrak{h}} \in L^1([a_0,b_0],\mathbb{R})$, *then*

$$ D^{m^*,\hat{\sigma}}_{a_0^+}(I^{\alpha,\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}})(w) = \big(I^{\alpha-m^*}_{a_0^+}\bar{\mathfrak{h}}\big)(w), \ w > a_0. $$

Now we introduce the Hilfer generalized proportional fractional derivative.

Definition 2.4. [19] Let $\alpha \in (n-1,n)$, $n \in \mathbb{N}$, $\hat{\sigma} \in (0,1]$ and $\eta_* \in [0,1]$. Then the Hilfer type proportional fractional derivative of the function $\bar{\mathfrak{h}}$ of order α, parameter η_* and proportional number $\hat{\sigma}$ is defined as

$$ (D^{\alpha,\eta_*,\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}})(w) = I^{\eta_*(n-\alpha),\hat{\sigma}}_{a_0^+}[D^{n,\hat{\sigma}}(I^{(1-\eta_*)(n-\alpha),\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}})](w), $$

where $D^{n,\hat{\sigma}}$ is defined in (2.1) and $I^{(\cdot),\hat{\sigma}}$ is the generalized proportional fractional integral defined in Definition 2.2.

The Hilfer type proportional fractional derivative is equivalent to

$$ (D^{\alpha,\eta_*,\hat{\sigma}}_{a^+}\bar{\mathfrak{h}})(w) = I^{\eta_*(n-\alpha),\hat{\sigma}}_{a^+}[D^{n,\hat{\sigma}}](I^{(1-\eta_*)(n-\alpha),\hat{\sigma}}_{a^+}\bar{\mathfrak{h}})(w) = (I^{\eta_*(n-\alpha),\hat{\sigma}}_{a^+}D^{\gamma,\hat{\sigma}}\bar{\mathfrak{h}})(w), $$

where $\gamma = \alpha+\eta_*(n-\alpha)$ and γ satisfies:

$$ 1 < \gamma \leq 2, \ \gamma \geq \alpha, \ \gamma > \eta_*, \ n-\gamma < n-\eta_*(n-\alpha). $$

Lemma 2.5. [19] *Let* $\alpha \in (n-1,n)$, $\hat{\sigma} \in (0,1]$, $\eta_* \in [0,1]$ *and* $\gamma = \alpha+\eta_*(n-\alpha) \in [\alpha,n]$. *If* $\bar{\mathfrak{h}} \in L^1(a_0,b_0)$ *and* $I^{n-\gamma,\hat{\sigma}}_{a_0^+}\bar{\mathfrak{h}} \in C^n([a_0,b_0],\mathbb{R})$, *then*

$$ I^{\alpha,\hat{\sigma}}_{a^+}D^{\alpha,\eta_*,\hat{\sigma}}_{a^+}\bar{\mathfrak{h}}(w) = \bar{\mathfrak{h}}(w) - \sum_{j=1}^{n} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \frac{(w-a_0)^{\gamma-j}}{\hat{\sigma}^{\gamma-j}\Gamma(\gamma+1-j)} \big(I^{j-\gamma,\hat{\sigma}}\bar{\mathfrak{h}}\big)(a_0^+). $$

The following lemma, were proved in [20], is the basic tool to convert the nonlocal BVP (1.2) into a fixed point problem.

Lemma 2.6. *Let* $1<\alpha<2$, $\eta_*\in[0,1]$, $\gamma=\alpha+\eta_*(2-\alpha)\in[\alpha,2]$, $\hat{\sigma}\in(0,1]$, $\hat{\mathfrak{g}}\in C([a_0,b_0],\mathbb{R})$, *and*

$$\Delta=\frac{(b_0-a_0)^{\gamma-1}}{\Gamma(\gamma)}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-a_0)}-\sum_{j=1}^{m}\theta_j\frac{(\xi_j-a_0)^{\gamma-1}}{\Gamma(\gamma)}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-a_0)}\neq 0.$$

Then π is the unique solution to the linear Hilfer-type sequential fractional proportional BVP:

$$\begin{cases}\left(D_{a_+}^{\alpha,\eta_*,\hat{\sigma}}+kD_{a_+}^{\alpha-1,\eta_*,\hat{\sigma}}\right)\pi(w)=\hat{\mathfrak{g}}(w),\ w\in[a_0,b_0],\\ \pi(a_0)=0,\quad \pi(b_0)=\sum_{j=1}^{m}\theta_j\pi(\varsigma_j),\end{cases}\tag{2.2}$$

if and only if

$$\begin{aligned}\pi(w) \;=\;& I_{a^+}^{\alpha,\hat{\sigma}}\hat{\mathfrak{g}}(w)+\frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}\hat{\mathfrak{g}}(s)ds\\ &-\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\int_{a_0}^{b_0}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}\hat{\mathfrak{g}}(s)ds\\ &-\frac{k}{\hat{\sigma}}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds+\frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\Bigg]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)}\\ &-\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds.\end{aligned}\tag{2.3}$$

2.2. **Multi-valued analysis.** Here we recall some basic definitions from multi-valued analysis [21, 22]. Consider a normed space $(X,\|\cdot\|)$. We use the notations $\mathscr{P}_b$, $\mathscr{P}_{cl}$ and $\mathscr{P}_{cp,c}$ to indicate the classes of all bounded, closed, and compact and convex sets, respectively in X. Also, by $S_{\Pi,\pi}:=\{v\in L^1([a_0,b_0],\mathbb{R}):v(w)\in\Pi(w,\pi(w))$ for a.e. $w\in[a_0,b_0]\}$, we define the selections sets of Π, for each $\pi\in C([a_0,b_0],\mathbb{R})$, and the graph of Π by $Gr(\Pi)=\{(x,y)\in X\times Y, y\in\Pi(x)\}$.

3. Existence Results

Let the Banach space $C([a_0,b_0],\mathbb{R})$ of all continuous functions from $[a_0,b_0]$ into $\mathbb{R}$ be equipped with the sup-norm $\|\pi\|:=\sup\{|\pi(w)|:w\in[a_0,b_0]\}$. Also, $L^1([a_0,b_0],\mathbb{R})$ is denoted the space of functions $\pi:[a_0,b_0]\to\mathbb{R}$ such that $\|\pi\|_{L^1}=\int_{a_0}^{b_0}|\pi(\varsigma)|d\varsigma$.

Definition 3.1. A function $\pi\in C([a_0,b_0],\mathbb{R})$ is a solution to problem (1.2) if there exists $v\in L^1([a_0,b_0],\mathbb{R})$ such that $v(w)\in\Pi(w,\pi)$ for a.e. $w\in[a_0,b_0]$ and $\left(D_{a^+}^{\alpha,\eta_*,\hat{\sigma}}+kD_{a^+}^{\alpha-1,\eta_*,\hat{\sigma}}\right)\pi(w)=v(w)$ on $[a_0,b_0]$, $\pi(a_0)=0$, $\pi(b_0)=\sum_{j=1}^{m}\theta_j\pi(\xi_j)$.

For convenience, we set

$$\Phi_1 = \frac{(b_0-a_0)^{\alpha}}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}+\frac{(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}\left[\sum_{j=1}^{m}\theta_j(\xi_j-a_0)^{\alpha}+(b_0-a_0)^{\alpha}\right],\tag{3.1}$$

$$\Phi_2 = \frac{(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{k}{\hat{\sigma}}\left[\sum_{j=1}^{m}\theta_j(\xi_j-a_0)+(b_0-a_0)\right]+\frac{k}{\hat{\sigma}}(b_0-a_0).\tag{3.2}$$

Now, we prove the first existence result for the Hilfer-type nonlocal sequential fractional proportional BVP (1.2) when Π has convex values by applying nonlinear alternative of Leray-Schauder ([29]).

Theorem 3.2. *Assume that:*

- (H_0) $\Phi_2 < 1$, *where* Φ_2 *is defined by (3.2);*
- (H_1) $\Pi : [a_0, b_0] \times \mathbb{R} \to \mathscr{P}_{cp,c}(\mathbb{R})$ *is* L^1*-Carathéodory; (i.e. (i) for each* $\pi \in \mathbb{R}$, $w \longmapsto \Pi(w,\pi)$ *is measurable; (ii) for almost all* $w \in [a_0, b_0]$, $\pi \longmapsto \Pi(w,\pi)$ *is upper semicontinuous (u.s.c.); and (iii) for each* $k_0 > 0$, *there exists* $\varphi_{k_0} \in L^1([a_0,b_0],\mathbb{R}^+)$ *such that* $\|\Pi(w,\pi)\| = \sup\{|v| : v \in \Pi(w,\pi)\} \leq \varphi_{k_0}(w)$ *for all* $\pi \in \mathbb{R}$ *with* $\|\pi\| \leq k_0$ *and for a.e.* $w \in [a_0,b_0]$.*)*
- (H_2) *there exist an increasing function* $\Psi \in C([0,\infty),(0,\infty))$ *and a positive continuous function* Λ *such that*

$$\|\Pi(w,\pi)\|_{\mathscr{P}} := \sup\{|z| : z \in \Pi(w,\pi)\} \leq \Lambda(w)\Psi(\|\pi\|) \text{ for each } (w,\pi) \in [a_0,b_0] \times \mathbb{R};$$

- (H_3) *there exists a constant* $M > 0$ *such that*

$$\frac{(1-\Phi_2)M}{\|\Lambda\|\Psi(M)\Phi_1} > 1.$$

Then, Hilfer generalized sequential proportional BVP (1.2) has at least one solution on $[a_0,b_0]$.

Proof. By introducing an operator $\mathbb{F} : C([a_0,b_0],\mathbb{R}) \longrightarrow \mathscr{P}(C([a_0,b_0],\mathbb{R}))$ by

$$\mathbb{F}(\pi) = \left\{ \begin{array}{l} \varpi \in C([a_0,b_0],\mathbb{R}) : \\ \varpi(w) = \left\{ \begin{array}{l} \dfrac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)} \Bigg[\dfrac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)} \displaystyle\sum_{j=1}^{m} \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} (\xi_j - s)^{\alpha-1} v(s)ds \\ -\dfrac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)} \displaystyle\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} (b_0 - s)^{\alpha-1} v(s)ds \\ -\dfrac{k}{\hat{\sigma}} \displaystyle\sum_{j=1}^{m} \theta_j \int_{a_0}^{\xi_j} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} ds \\ +\dfrac{k}{\hat{\sigma}} \displaystyle\int_{a_0}^{b_0} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} ds \Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ -\dfrac{k}{\hat{\sigma}} \displaystyle\int_{a_0}^{w} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w - s)} ds + I_{a^+}^{\alpha;\hat{\sigma}} v(w), \ v \in S_{\Pi,\pi} \end{array} \right. \end{array} \right\}$$

for $w \in [a_0,b_0]$, we transform the nonlocal BVP (1.2) into a fixed point problem. The fixed points of $\mathbb{F}$ obviously are solutions of the Hilfer generalized proportional BVP (1.2).

We verify in the following steps, the hypotheses of Leray-Schauder multi-valued nonlinear alternative ([29]).

Step 1. For each $\pi \in C([a_0,b_0],\mathbb{R})$, $\mathbb{F}(\pi)$ *is convex.*

Since Π has convex values, it is obvious.

Step 2. Bounded sets in $C([a_0,b_0],\mathbb{R})$ *are mapped by* $\mathbb{F}(\pi)$ *into bounded sets.*

Let $\mathbb{B}_{r_0} = \{\pi \in C([a_0,b_0],\mathbb{R}) : \|\pi\| \leq r_0\}$ be a bounded set in $C([a_0,b_0],\mathbb{R})$. Then, for each $\varpi \in \mathbb{F}(\pi), \pi \in \mathbb{B}_{r_0}$, there exists $v \in S_{\Pi,\pi}$ such that

$$\begin{aligned}\varpi(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}v(s)ds \\ &\quad -\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}v(s)ds \\ &\quad -\frac{k}{\hat{\sigma}}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad -\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds.\end{aligned}$$

Then, for $w \in [a_0,b_0]$, we have

$$\begin{aligned}&|\varpi(w)| \\ &\leq \frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{w} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}(w-s)^{\alpha-1}|v(s)|ds \\ &\quad + \frac{(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\sum_{j=1}^m \theta_j\int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}|v(s)|ds \\ &\quad + \frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}|v(s)|ds \\ &\quad + \frac{k}{\hat{\sigma}}\sum_{j=1}^m \theta_j\int_{a_0}^{\xi_j}|\pi(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}|\pi(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds\Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad + \frac{k}{\hat{\sigma}}\int_{a_0}^{w}|\pi(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds \\ &\leq \frac{\|\Lambda\|\Psi(\|\pi\|)}{\hat{\sigma}^\alpha\Gamma(\alpha+1)}(b_0-a_0)^\alpha \\ &\quad + \frac{\|\Lambda\|\Psi(\|\pi\|)(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha+1)}\sum_{j=1}^m \theta_j(\xi_j-c)^\alpha + \frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha+1)}(b_0-a_0)^\alpha\Bigg] \\ &\quad + \frac{(b_0-a_0)^{\gamma-1}\|\pi\|}{|\Delta|\Gamma(\gamma)}\frac{k}{\hat{\sigma}}\Bigg[\sum_{j=1}^m \theta_j(\xi_j-a_0) + (b_0-a_0)\Bigg] + \frac{k}{\hat{\sigma}}(b_0-a_0)\|\pi\| \\ &\leq \|\Lambda\|\Psi(r_0)\Phi_1 + r_0\Phi_2.\end{aligned}$$

Consequently,

$$\|\varpi\| \leq \|\Lambda\|\Psi(r_0)\Phi_1 + r_0\Phi_2.$$

Step 3. Bounded sets of $C([a_0,b_0],\mathbb{R})$ are mapped by $\mathbb{F}(\pi)$ into equicontinuous sets.

Let $\vartheta_1, \vartheta_2 \in [a_0, b_0]$ with $\vartheta_1 < \vartheta_2$ and $\pi \in \mathbb{B}_{r_0}$. Then, for each $\varpi \in \mathbb{F}(\pi)$, we obtain

$$
\begin{aligned}
&|\varpi(\vartheta_2)-\varpi(\vartheta_1)| \\
&\leq \frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{\vartheta_2}[(\vartheta_2-s)^{\alpha-1}-(\vartheta_1-s)^{\alpha-1}]|v(s)|ds \\
&\quad+\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{\vartheta_1}^{\vartheta_2}(\vartheta_2-s)^{\alpha-1}|v(s)|ds \\
&\quad+\frac{(\vartheta_2-a_0)^{\gamma-1}-(\vartheta_1-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)} \\
&\quad\times\left[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\sum_{j=1}^m\theta_j\int_{a_0}^{\xi_j}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}|v(s)|ds\right. \\
&\quad\left.+\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{b_0}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}|v(s)|ds\right] \\
&\quad+\frac{(\vartheta_2-a_0)^{\gamma-1}-(\vartheta_1-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{k}{\hat{\sigma}}\left[\sum_{j=1}^m\theta_j\int_{a_0}^{\xi_j}|\pi(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds\right. \\
&\quad\left.+\int_{a_0}^{b_0}|\pi(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds\right] \\
&\quad+\frac{k}{\hat{\sigma}}\left[\int_{a_0}^{\vartheta_1}|\pi(s)|\left[e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\vartheta_1-s)}-e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\vartheta_2-s)}\right]ds+\int_{\vartheta_1}^{\vartheta_2}|\pi(s)|\left[e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\vartheta_2-s)}\right]ds\right] \\
&\leq \frac{\|\Lambda\|\Psi(r_0)}{\hat{\sigma}^\alpha\Gamma(\alpha+1)}\left[|(\vartheta_2-a_0)^\alpha-(\vartheta_1-a_0)^\alpha|+2(\vartheta_2-\vartheta_1)^\alpha\right] \\
&\quad+\frac{(\vartheta_2-a_0)^{\gamma-1}-(\vartheta_1-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\|\Lambda\|\Psi(r_0)\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha+1)} \\
&\quad\times\left[\sum_{j=1}^m\theta_j(\xi_j-a_0)^\alpha+(b_0-a_0)^\alpha\right] \\
&\quad+\frac{(\vartheta_2-a_0)^{\gamma-1}-(\vartheta_1-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{kr_0}{\hat{\sigma}}\left[\sum_{j=1}^m\theta_j(\xi_j-a_0)+(b_0-a_0)\right] \\
&\quad+r_0\frac{k}{\hat{\sigma}}\left[\int_{a_0}^{\vartheta_1}\left(e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\vartheta_1-s)}-e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\vartheta_2-s)}\right)ds+\int_{\vartheta_1}^{\vartheta_2}\left(e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\vartheta_2-s)}\right)ds\right]\to 0
\end{aligned}
$$

as $\vartheta_2-\vartheta_1\to 0$, independently of $\pi\in\mathbb{B}_{r_0}$.

By Arzelá-Ascoli theorem the operator $\mathbb{F}: C([a_0,b_0],\mathbb{R})\to\mathscr{P}(C([a_0,b_0],\mathbb{R}))$ is completely continuous.

Next we prove that operator $\mathbb{F}$ is upper semicontinuous (usc). It is enough to prove that the graph of $\mathbb{F}$ is closed since by Proposition 1.2 of [22] it is known that a multi-valued completely continuous map is usc if and only its graph is closed.

Step 4. The graph of $\mathbb{F}$ *is closed.*

Let $\pi_n \to \pi_*, \varpi_n \in \mathbb{F}(\pi_n)$ and $\varpi_n \to \varpi_*$. We prove that $\varpi_* \in \mathbb{F}(\pi_*)$. Let $\varpi_n \in \mathbb{F}(\pi_n)$. Then there exists $v_n \in S_{\Pi,\pi_n}$ such that, for each $w \in [a_0, b_0]$,

$$\begin{aligned}\varpi_n(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v_n(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}v_n(s)ds \\ &\quad -\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}v_n(s)ds \\ &\quad -\frac{k}{\hat{\sigma}}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi_n(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}\pi_n(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\Bigg]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad -\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi_n(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds.\end{aligned}$$

We next prove that there exists $v_* \in S_{\Pi,\pi_*}$ such that, for each $w \in [a_0, b_0]$,

$$\begin{aligned}\varpi_*(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v_*(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}v_*(s)ds \\ &\quad -\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}v_*(s)ds \\ &\quad -\frac{k}{\hat{\sigma}}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi_*(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}\pi_*(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\Bigg]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad -\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi_*(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds.\end{aligned}$$

Let the linear operator $\Theta : L^1([a_0,b_0],\mathbb{R}) \to C([a_0,b_0],\mathbb{R})$ be given by

$$\begin{aligned}&v \mapsto \Theta(v)(w) \\ &= I_{a^+}^{\alpha,\hat{\sigma}} v(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\Bigg[\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}v(s)ds \\ &\quad -\frac{1}{\hat{\sigma}^\alpha\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}v(s)ds \\ &\quad -\frac{k}{\hat{\sigma}}\sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}u(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\Bigg]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad -\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds.\end{aligned}$$

Observe that $\|\varpi_n - \varpi_*\| \to 0$ as $n \to \infty$. Then, by a result on closed graphs due to Lazota-Opial [30], the operator $\Theta \circ S_\Pi$ has a closed graph. Moreover, we have $\varpi_n(w) \in \Theta(S_{\Pi,\pi_n})$. Since

$\pi_n \to \pi_*$, and then

$$\begin{aligned}\varpi_*(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v_*(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\left[\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}v_*(s)ds\right.\\ &\quad -\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}v_*(s)ds\\ &\quad \left.-\frac{k}{\hat{\sigma}}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j}\pi_*(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}\pi_*(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\right]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)}\\ &\quad -\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi_*(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds,\end{aligned}$$

for some $v_* \in S_{\Pi,\pi_*}$.

Step 5. We prove that there exists $U \subseteq C([a_0,b_0],\mathbb{R})$, an open set, such that $\pi \notin \omega\mathbb{F}(\pi)$ for any $\omega \in (0,1)$ and all $\pi \in \partial U$.

Assume that $\omega \in (0,1)$ and $\pi \in \omega\mathbb{F}(\pi)$. Then there exists $v \in L^1([a_0,b_0],\mathbb{R})$ with $v \in S_{\Pi,\pi}$ such that, for $w \in [a_0,b_0]$,

$$\begin{aligned}\varpi(w) &= \omega I_{a^+}^{\alpha,\hat{\sigma}} v(w) + \omega\frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)}\left[\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}v(s)ds\right.\\ &\quad -\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}v(s)ds\\ &\quad \left.-\frac{k}{\hat{\sigma}}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds + \frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\right]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)}\\ &\quad -\omega\frac{k}{\hat{\sigma}}\int_{a_0}^{w}\pi(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds.\end{aligned}$$

As Step 2, we have

$$\begin{aligned}|\pi(w)| &\le \frac{\|\Lambda\|\Psi(\|\pi\|)}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}(b_0-a_0)^{\alpha} + \frac{\|\Lambda\|\Psi(\|\pi\|)(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\\ &\quad \times\left[\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}\sum_{j=1}^{m}\theta_j(\xi_j-c)^{\alpha} + \frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}(b_0-a_0)^{\alpha}\right]\\ &\quad + \frac{(b_0-a_0)^{\gamma-1}\|\pi\|}{|\Delta|\Gamma(\gamma)}\frac{k}{\hat{\sigma}}\left[\sum_{j=1}^{m}\theta_j(\xi_j-a_0)+(b_0-a_0)\right] + \frac{k}{\hat{\sigma}}(b_0-a_0)\|\pi\|\\ &\le \|\Lambda\|\Psi(\|\pi\|)\Phi_1 + \|\pi\|\Phi_2,\end{aligned}$$

or $(1-\Phi_2)\|\pi\| \le \|\Lambda\|\Psi(\|\pi\|)\Phi_1$, which implies that

$$\frac{(1-\Phi_2)\|\pi\|}{\|\Lambda\|\Psi(\|\pi\|)\Phi_1} \le 1.$$

By (A_3), $\|\pi\| \neq M$. Consider the set $U = \{\pi \in C([a_0,b_0],\mathbb{R}) : \|\pi\| < M\}$. We see that $\mathbb{F} : \overline{U} \to \mathscr{P}(C([a_0,b_0],\mathbb{R}))$ is an usc compact multi-valued map, with closed convex values. By the definition of U there does not exists any $\pi \in \partial U$ for some $\omega \in (0,1)$, satisfying $\pi \in \omega\mathbb{F}(\pi)$. By the

Leray-Schauder nonlinear alternative ([29]), operator $\mathbb{F}$ has a fixed point $\pi \in \overline{U}$. Consequently, Hilfer fractional proportional BVP (1.2) has at least one solution on $[a_0, b_0]$, which ends the proof. □

Now, we consider the case that the multi-valued Π is non-convex valued. Our existence result in this case is proved via Covitz and Nadler fixed point theorem [31].

Theorem 3.3. *Suppose that:*

(A_1) $\Pi : [a_0, b_0] \times \mathbb{R} \to \mathscr{P}_{cp}(\mathbb{R})$ *is such that* $\Pi(\cdot, \pi) : [a_0, b_0] \to \mathscr{P}_{cp}(\mathbb{R})$ *is for each* $\pi \in \mathbb{R}$ *measurable;*

(A_2) $H_d(\Pi(w,\pi), \Pi(w,\bar{\pi})) \leq \mu(w)|\pi - \bar{\pi}|$ *for a.a.* $w \in [a_0, b_0]$ *and* $\pi, \bar{\pi} \in \mathbb{R}$ *with* $\mu \in C([a_0, b_0], \mathbb{R}^+)$ *and* $d(0, \Pi(w, 0)) \leq \mu(w)$ *for a.a.* $w \in [a_0, b_0]$.

Then, the Hilfer generalized sequential proportional BVP (1.2) has at least one solution on $[a_0, b_0]$, *provided that* $\|\mu\|\Phi_1 + \Phi_2 < 1$, *where* Φ_1 *and* Φ_2 *are respectively defined by (3.1) and (3.2).*

Proof. We prove that $\mathbb{F} : C([a_0, b_0], \mathbb{R}) \longrightarrow \mathscr{P}(C([a_0, b_0], \mathbb{R}))$, where $\mathbb{F}$ is defined at the beginning of the proof of Theorem 3.2, satisfies the hypotheses of Covitz and Nadler fixed point theorem for multi-valued contractive maps [31]. By the measurable selection (see Theorem III.6 [32]), Π is measurable, and hence it admits a measurable selection. By (A_2), we obtain $|v(w)| \leq \mu(w)(1 + |\pi(w)|)$, which means $v \in L^1([a_0, b_0], \mathbb{R})$. Hence Π is integrably bounded. Thus $S_{\Pi,\pi} \neq \emptyset$. Let $\{u_n\}_{n \geq 0} \in \mathbb{F}(\pi)$ with $u_n \to u$ $(n \to \infty)$ in $C([a_0, b_0], \mathbb{R})$. Then $u \in C([a_0, b_0], \mathbb{R})$ and there exists $v_n \in S_{\Pi, \pi_n}$ such that, for each $w \in [a_0, b_0]$,

$$\begin{aligned} u_n(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v_n(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)} \Bigg[\frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} (\xi_j - s)^{\alpha-1} v_n(s) ds \\ &\quad - \frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} (b_0 - s)^{\alpha-1} v_n(s) ds \\ &\quad - \frac{k}{\hat{\sigma}} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi_n(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} ds + \frac{k}{\hat{\sigma}} \int_{a_0}^{b_0} \pi_n(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} ds \Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w - a_0)} \\ &\quad - \frac{k}{\hat{\sigma}} \int_{a_0}^{w} \pi_n(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)} ds. \end{aligned}$$

Then, as Π has compact values, we can obtain a subsequence (if necessary) v_n converges to v in $L^1([a_0, b_0], \mathbb{R})$. Thus, $v \in S_{\Pi,\pi}$. For each $w \in [a_0, b_0]$, we have

$$\begin{aligned} u_n \to u(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)} \Bigg[\frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} (\xi_j - s)^{\alpha-1} v(s) ds \\ &\quad - \frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} (b_0 - s)^{\alpha-1} v(s) ds \\ &\quad - \frac{k}{\hat{\sigma}} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} ds + \frac{k}{\hat{\sigma}} \int_{a_0}^{b_0} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} ds \Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w - a_0)} \\ &\quad - \frac{k}{\hat{\sigma}} \int_{a_0}^{w} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)} ds. \end{aligned}$$

Thus $u \in \mathbb{F}(\pi)$.

Next, we prove that there exists δ ($\delta := \|\mu\|\Phi_1 + \Phi_2 < 1$) such that

$$H_d(\mathbb{F}(\pi), \mathbb{F}(\bar{\pi})) \leq \delta \|\pi - \bar{\pi}\| \text{ for each } \pi, \bar{\pi} \in C^2([a_0, b_0], \mathbb{R}).$$

Let $\pi, \bar{\pi} \in C^2([a_0, b_0], \mathbb{R})$ and $\varpi_1 \in \mathbb{F}(\pi)$. Then there exists $v_1(w) \in \Pi(w, \pi(w))$ such that, for each $w \in [a_0, b_0]$,

$$\begin{aligned} \varpi_1(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v_1(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)} \Bigg[\frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} (\xi_j - s)^{\alpha-1} v_1(s) ds \\ &\quad - \frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} (b_0 - s)^{\alpha-1} v_1(s) ds \\ &\quad - \frac{k}{\hat{\sigma}} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} ds + \frac{k}{\hat{\sigma}} \int_{a_0}^{b_0} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} ds \Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad - \frac{k}{\hat{\sigma}} \int_{a_0}^{w} \pi(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)} ds. \end{aligned}$$

By (A_2), we have

$$H_d(\Pi(w, \pi), \Pi(w, \bar{\pi})) \leq \mu(w) |\pi(w) - \bar{\pi}(w)|.$$

Hence, there exists $z \in \Pi(w, \bar{\pi}(w))$ such that

$$|v_1(w) - z| \leq \mu(w) |\pi(w) - \bar{\pi}(w)|, \ w \in [a_0, b_0].$$

Define $U : [a_0, b_0] \to \mathscr{P}(\mathbb{R})$ by

$$U(w) = \{ w \in \mathbb{R} : |v_1(w) - z| \leq \mu(w) |\pi(w) - \bar{\pi}(w)| \}.$$

Then, there exists a function $v_2(w)$, which is a measurable selection for U since the multi-valued operator $U(w) \cap \Pi(w, \bar{\pi}(w))$ is measurable (Proposition III.4 [32]). Thus $v_2(w) \in \Pi(w, \bar{\pi}(w))$. For each $w \in [a_0, b_0]$, we have $|v_1(w) - v_2(w)| \leq \mu(w) |\pi(w) - \bar{\pi}(w)|$. For each $w \in [a_0, b_0]$, we define

$$\begin{aligned} \varpi_2(w) &= I_{a^+}^{\alpha,\hat{\sigma}} v_2(w) + \frac{(w-a_0)^{\gamma-1}}{\Delta\Gamma(\gamma)} \Bigg[\frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} (\xi_j - s)^{\alpha-1} v_2(s) ds \\ &\quad - \frac{1}{\hat{\sigma}^\alpha \Gamma(\alpha)} \int_{a_0}^{b_0} e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} (b_0 - s)^{\alpha-1} v_2(s) ds \\ &\quad - \frac{k}{\hat{\sigma}} \sum_{j=1}^m \theta_j \int_{a_0}^{\xi_j} \bar{\pi}(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j - s)} ds + \frac{k}{\hat{\sigma}} \int_{a_0}^{b_0} \bar{\pi}(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0 - s)} ds \Bigg] e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\ &\quad - \frac{k}{\hat{\sigma}} \int_{a_0}^{w} \bar{\pi}(s) e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)} ds. \end{aligned}$$

Thus

$$
\begin{aligned}
& |\varpi_1(w)-\varpi_2(w)| \\
\leq\ & I_{a^+}^{\alpha,\hat{\sigma}}(|v_1(s)-v_2(s)|)(w) \\
& +\frac{(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\left[\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}(\xi_j-s)^{\alpha-1}(|v_1(s)-v_2(s)|)(s)ds\right. \\
& +\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha)}\int_{a_0}^{b_0}e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}(b_0-s)^{\alpha-1}(|v_1(s)-v_2(s)|)(s)ds \\
& \left.+\frac{k}{\hat{\sigma}}\sum_{j=1}^{m}\theta_j\int_{a_0}^{\xi_j}|\pi(s)-\bar{\pi}(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(\xi_j-s)}ds+\frac{k}{\hat{\sigma}}\int_{a_0}^{b_0}|\pi(s)-\bar{\pi}(s)|e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(b_0-s)}ds\right]e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-a_0)} \\
& +\frac{k}{\hat{\sigma}}\int_{a_0}^{w}|\pi(s)-\bar{\pi}(s)|(s)e^{\frac{\hat{\sigma}-1}{\hat{\sigma}}(w-s)}ds \\
\leq\ & \left\{\frac{(b_0-a_0)^{\alpha}}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}+\frac{(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\left[\frac{1}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}\sum_{j=1}^{m}\theta_j(\xi_j-a_0)^{\alpha}+\frac{(b_0-a_0)^{\alpha}}{\hat{\sigma}^{\alpha}\Gamma(\alpha+1)}\right]\right\} \\
& \times\|\mu\|\|\pi-\bar{\pi}\|+\left\{\frac{(b_0-a_0)^{\gamma-1}}{|\Delta|\Gamma(\gamma)}\frac{k}{\hat{\sigma}}\left[\sum_{j=1}^{m}\theta_j(\xi_j-a_0)+(b_0-a_0)\right]+\frac{k}{\hat{\sigma}}(b_0-a_0)\right\}\|\pi-\bar{\pi}\| \\
=\ & \Big(\|\mu\|\Phi_1+\Phi_2\Big)\|\pi-\bar{\pi}\|.
\end{aligned}
$$

Hence

$$
\|\varpi_1-\varpi_2\|\leq\Big(\|\mu\|\Phi_1+\Phi_2\Big)\|\pi-\bar{\pi}\|.
$$

Interchanging the roles of π and $\bar{\pi}$, we have

$$
H_d(\mathbb{F}(\pi),\mathbb{F}(\bar{\pi}))\leq\Big(\|\mu\|\Phi_1+\Phi_2\Big)\|\pi-\bar{\pi}\|.
$$

By $\|\mu\|\Phi_1+\Phi_2<1$, one sees that $\mathbb{F}$ is a contraction. Consequently, $\mathbb{F}$ has a fixed point π by Covitz and Nadler theorem ([31]), which is a solution to Hilfer generalized proportional boundary value problem (1.2). The proof is finished. □

4. Illustrative Examples

Now, we construct some numerical examples to illustrate our theoretical results. Consider the following BVP for Hilfer generalized fractional proportional differential inclusions with nonlocal multi-point boundary conditions

$$
\begin{cases}
\left(D_{\frac{1}{9}}^{\frac{7}{4},\frac{1}{2},\frac{2}{3}}+\frac{1}{15}D_{\frac{1}{9}}^{\frac{3}{4},\frac{1}{2},\frac{2}{3}}\right)\pi(w)\in\Pi(w,\pi(w)), \qquad w\in\left[\frac{1}{9},\frac{10}{9}\right], \\
x\left(\frac{1}{9}\right)=0,\ x\left(\frac{10}{9}\right)=\frac{1}{13}x\left(\frac{2}{9}\right)+\frac{3}{23}x\left(\frac{4}{9}\right)+\frac{5}{33}x\left(\frac{5}{9}\right) \\
+\frac{7}{43}x\left(\frac{7}{9}\right)+\frac{9}{53}x\left(\frac{8}{9}\right).
\end{cases}
\tag{4.1}
$$

Here $\alpha = 7/4$, $\eta_* = 1/2$, $\hat{\sigma} = 2/3$, $k = 1/15$, $a = 1/9$, $b = 10/9$, $m = 5$, $\theta_1 = 1/13$, $\theta_2 = 3/23$, $\theta_3 = 5/33$, $\theta_4 = 7/43$, $\theta_5 = 9/53$, $\xi_1 = 2/9$, $\xi_2 = 4/9$, $\xi_3 = 5/9$, $\xi_4 = 7/9$, and $\xi_5 = 8/9$. Then we can find that $\gamma = 15/8$, $\Delta \approx 0.3354252898$, $\Phi_1 \approx 2.239711391$, and $\Phi_2 \approx 0.5252426845$.

(i) We consider the multifunction $\Pi(w,\pi)$ defined by

$$\Pi(w,\pi) = \left[\frac{1}{9t+10}\left(\frac{x^{28}}{10(1+x^{26})} + \frac{1}{13}e^{-10t}\sin t\right), \frac{1}{9t+1}\left(\frac{x^{28}}{5(1+x^{26})} + \frac{1}{7}e^{-9t}\right)\right], \tag{4.2}$$

for $(w,\pi) \in [1/9, 10/9] \times \mathbb{R}$. Then

$$\|\Pi(w,\pi)\|_{\mathscr{P}} \leq \left(\frac{1}{9t+1}\right)\left(\frac{1}{5}x^2 + \frac{1}{7}\right).$$

If $\Lambda(w) = 1/(9t+1)$ and $\Psi(\pi) = (1/5)x^2 + (1/7)$, then $\|\Lambda\| = 1/2$. Moreover there exists $M \in (0.4203134569,\ 1.699411957)$ satisfying (A_3). By application of Theorem 3.2, the proportional fractional BVP (4.1), with the multifunction Π given by (4.2) has at least one solution on $[1/9, 10/9]$.

(ii) Let $\Pi(w,\pi)$ be defined by

$$\Pi(w,\pi) = \left[0, \frac{1}{9t+4}\left(\frac{x^2+2|x|}{2(1+|x|)} + \frac{1}{2}\right)\right], \tag{4.3}$$

for $(w,\pi) \in [1/9, 10/9] \times \mathbb{R}$. Note that F is measurable for all $\pi \in \mathbb{R}$, and

$$H_d(\Pi(w,\pi), \Pi(w,\bar{\pi})) \leq \frac{1}{9t+4}|x - \bar{\pi}|,$$

for all $\pi, \bar{\pi} \in \mathbb{R}$. Setting $\mu(w) = 1/(9t+4)$, we obtain that $\|\mu\| = 1/5$ and $d(0, \Pi(w,0)) = (1/2)\mu(w) \leq \mu(w)$ for each $w \in [1/9, 10/9]$. Thus

$$\|\mu\|\Phi_1 + \Phi_2 \approx 0.9731849627 < 1.$$

Therefore, by Theorem 3.3, we conclude that the BVP (4.1) with the multi-valued Π given by (4.3) has at least one solution on $[1/9, 10/9]$.

5. Conclusions

In this paper, we initiated the study of BVP consisting of Hilfer generalized proportional sequential fractional differential inclusions operators, supplemented with nonlocal multi-point boundary conditions. We considered both cases, convex-valued and non-convex-valued multi-valued maps and established an existence result in the convex-valued case via multi-valued nonlinear alternative of Leray-Schauder, and in the non-convex-valued case via a fixed point theorem for multi-valued contractive maps due to Covitz and Nadler. Our results, which are new and could enrich the literature on Hilfer generalized proportional sequential fractional BVP, are illustrated by constructed numerical examples.

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