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VISCOSITY APPROXIMATION WITH MK CONTRACTIONS FOR A COMMON FIXED POINT PROBLEM AND A SPLIT FEASIBILITY PROBLEM

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Abstract. In this paper, a new viscosity approximation method with projections and Meir-Keeler contractive mappings (MK contractions) for solving a common fixed point problem of an infinite family of nonexpansive mappings and a split feasibility problem with a bounded linear mapping is introduce and investigated. A solution theorem of strong convergence is obtained in infinite dimensional spaces.
Keywords. Convergence; Image reconstruction; Inverse problem; Phase retrievals; Split feasibility problem.

1. Introduction

In this paper, H_1 and H_2 are always assumed to be two real Hilbert spaces. They are endowed with inner products and associated induced norms, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. We borrow C to denote a convex and closed set in H_1 and Q to denote a convex and closed set in H_2. In addition, we use H to refer to a Hilbert space.

With convex and closed sets C and Q and a linear and bounded operator $A : H_1 \to H_2$, we consider the celebrated split feasibility problem: find a point $x \in H_1$ such that

$$x \in C, \quad Ax \in Q, \tag{1.1}$$

where $Sol(SFP)$ is denoted the set of solutions of the split feasibility problem. Split feasibility problem (1.1) is said to be consistent iff its solution set is not empty, i.e., $Sol(SFP) \neq \emptyset$. This problem first was initially investigated by Censor and Elfving [4] for a class of inverse problems from image reconstruction and phase retrievals [3] in a finite dimensional space. Here, we also mention split feasibility problem (1.1) are also usually employed to solve the associated problems, such as computer tomograph, image restoration, machine learning, and radiation therapy treatment planning [5, 6].

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On the other hand, a general problem in numerous research fields of applied mathematics, physical science, and computer science consists of finding a point with certain constraints. This problem is commonly referred to as a convex feasibility problem. To be more precise, this problem can be presented as follows: Let $C_1, C_2, C_3, \cdots$ be convex and closed sets with $\cap_{i=1}^{\infty} C_i \neq \emptyset$. Find a point x in $\cap_{i=1}^{\infty} C_i$. Here C_i is referred as the constraint and $\cap_{i=1}^{\infty} C_i$ is referred as the solution set.

It is obvious that the split feasibility problem can be formulated as the following convex feasibility problem: Find a point $x \in C_1 \cap C_2$, where $C_1 = C$ and $C_2 A^{-1}(Q)$. Recently, various efficient numerical algorithms were investigated for the problems in finite and infinite dimensional spaces; see, e.g., [10, 12, 17, 26, 27] and the references therein.

Fixed point problems of nonlinear mappings are core problems in numerous fields in computer science, traffic network, management and financial engineering; see, e.g., [1, 25, 30, 33] and the references therein. Recall that a nonlinear mapping T defined on set C is said to be contractive iff

$$\|Tx - Ty\| \leq \kappa \|x - y\|, \quad \forall x, y \in C.$$

From now on, the fixed point set of T is denoted by $F(T)$. Banach fixed point theorem ensures that each contractive mapping has a unique fixed point in a complete metric space. Picard iteration is efficient to find fixed points of contractive mappings. Indeed, this result plays a significant role in many branches of pure and applied mathematics.

Recall that T is called to be Meir-Keeler contractive (MK contractive) if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon \leq \|x - y\| < \varepsilon + \delta$ implies $\|Tx - Ty\| < \varepsilon$ for each $x, y \in X$. Every MK contractive mapping has a unique fixed point in metric spaces according to Meir and Keeler [18] in 1969. It is obvious that a contractive mapping is a Meir-Keeler contractive mapping, however, a Meir-Keeler contractive mapping is only a conditional contraction. Indeed, it is contractive if some restriction is put on the domain of T.

Next, we further recall that T is nonexpansive iff

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

The theory and applications of nonexpansive mappings are important because their wide applications both in pure and applied mathematics; see, e.g., [9, 20, 21, 28]. Finally, we recall that T is said to be firmly nonexpansive iff

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C.$$

From the viewpoint of mathematical programming computation, the class of firmly nonexpansive mappings is significant. One knows that numerous mathematical programming problems can be solved via its resolvent operators, which are firmly nonexpansive; see, e.g., [7, 8, 16, 22, 23] and the references therein. A trivial example of firmly nonexpansive mappings is nearest point projection, $Proj_C^H$, which is defined by $Proj_C^H(y) := \arg\min\{\|x - y\|, x \in C\}$ for any $y \in H$.

In this paper, we investigate the following problem: find a point x such that

$$x \in \cap_{i=1}^{\infty} F(T_i) \cap Sol(SFP), \tag{1.2}$$

where $F(T_i)$ denotes the fixed point set of T_i and $Sol(SFP)$ is the solution set of 1.1. We consider a new iterative algorithm to solve this convex feasibility problem (both $F(T_i)$ and $Sol(SFP)$ are convex sets) with the aid of Meir-Keeler contractions and nearest point projections.

Recall that Normal Mann iterative method, which is a special Mann iterative method, is an efficient mathemtaical tool to investigate fixed points of nonexpansive mappings and their extensions. Normal Mann iterative method reads as follows. It generates a sequence $\{x_n\}$ in the following manner: $x_1 \in C$ is an initial and

$$x_{n+1} = \alpha_n T x_n + (1-\alpha_n)x_n, \; \forall n \geq 1,$$

where $\{\alpha_n\}$ in $(0,1)$ is viewed as a control sequence.

One knows that the Normal Mann iterative method is only weakly convergent in infinite dimensional spaces. However, many problems in physics, economics, image recovery, and control theory, arise in infinite dimension spaces. To investigate such problems, one strong convergence, i.e., norm convergence, is often desirable than weak convergence, i.e., convergence in weak topology, since it presents the physically tangible property. In addition, the important of strong convergence was presented with the fact that a convex function T is minimized via the proximal-point algorithm in [13] from which one sees that the rate of convergence of the value sequence $\{Tx\}$ is better when sequence $\{x_n\}$ converges strongly that it converges weakly. Such properties have a direct impact when the iterative method is executed in infinite dimensional spaces.

To modify the Normal Mann iterative method, various regularization methods were considered recently. One of most popular method is the Halpern iterative method that was first investigated for fixed points of nonexpansive mappings by Halpern [14] (implicit iteration by Browder [2]). Halpern iterative method reads as follows

$$x_1 \in H, \quad x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n v, \quad \forall n \geq 1,$$

where v is a fixed vector (anchor), T is a nonexpansive mapping on set C, and $\{\alpha_n\}$ is a real sequence in $(0,1)$.

Note that the new mapping, the convex combination of nonexpansive maping T and the anchor v is a contractive mapping. It is known that, to force the convergence, the conditions (c1) $\alpha_n \to 0$ as $n \to \infty$ and (c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are two necessary conditions if the Halpern iteration converges strongly. One wishes that $\alpha_n \to 0$ as fast as possible, however, Halpern iteration may not be a fast iteration due to restriction (c2). The following open question is raised by Haleprn.
Halpern Open Question: Are the conditions (C1) and (C2) are sufficient for the strong convergence of the sequence $\{x_n\}$?

In 1977, Lions [15] improved Halpern's result in Hilbert spaces under the assumptions imposed on the sequence $\{\alpha_n\}$:

(C1) $\lim_{n\to\infty} \alpha_n = 0$
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(C3) $\sum_{n=1}^{\infty} \frac{\alpha_{n+1}-\alpha_n}{\alpha_{n+1}^2} = 0.$

But both Halpern's result and Lions' results exclude the canonical choice $\alpha_n = \frac{1}{n}$.

In 1992, Wittmann [31] proved, still in Hilbert spaces, the strong convergence of the sequence, where $\{\alpha_n\}$ satisfies the following conditions

(C1) $\lim_{n\to\infty} \alpha_n = 0$
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$
(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty.$

In 2000, Moudafi [19] considered a viscosity approximation method, which is known as Moudafi's viscosity method, for a nonexpansive self-mapping in Hilbert spaces

$$x_1 \in C, \quad x_{n+1} = \alpha_n g(x_n) + (1-\alpha_n)Tx_n, \quad \forall n \geq 1,$$

where g is a contractive mapping on C, T is a nonexpansive mapping with fixed points, and $\{\alpha_n\}$ is a real sequence in $(0,1)$. Moudafi demonstrated that $\{x_n\}$ converges in norm to some fixed point of mapping T and the fixed point is also the unique solution to the variational inequality under some assumptions on $\{\alpha_n\}$: $\langle g(x)-x, x-y\rangle \geq 0$ for all $y \in F(T)$.

Recently, many authors investigated the viscosity approximation with various nonlinear operators. In this paper, we consider a Meir-Keeler contractive mapping. Our viscosity method is based on a metric projection and we prove that the vector sequence defined by our viscosity method can converge strongly to a solution to (1.2) with no compact restrictions. This solution also solves some variational inequality with the Meir-Keeler contractive mapping.

2. Preliminaries

Here, one uses $Proj_Q^{H_2}$ to denote the nearest point projection from H_2 onto Q and use $Proj_C^{H_1}$ to denote the nearest point projection from H_1 onto C. Let A^* be the adjoint operator of A. One knows that find a solution of split feasibility problem 1.1 is equivalent to finding a solution to the following fixed point problem if problem (1.1) is consistent

$$x = Proj_C^{H_1}(x - \gamma A^*(Ax - Proj_Q^{H_2}(Ax)), \tag{2.1}$$

where γ is a positive real number.

Recall that a single-valued mapping T from C to H is said to be monotone iff

$$\langle Tx - Ty, x-y\rangle \geq 0, \quad \forall x,y \in C.$$

T is said to be inverse-strongly monotone iff

$$\langle Tx - Ty, x-y\rangle \geq \nu\|Tx-Ty\|^2, \quad \forall x,y \in C,$$

where ν is a constant. One also says that T is ν-inverse-strongly monotone, one knows that if T is ν-inverse-strongly monotone, then it is monotone and $\frac{1}{\nu}$-Lipschitz continuous.

The following properties of nearest point projections are well known.

(a) Given $x \in H$ and $z \in D \subset H$, $z = Proj_D^H x$ iff there holds the inequality: $\langle x-z, y-z\rangle \leq 0$, $y \in D$.
(b) $\|Proj_D^H x - Proj_D^H y\|^2 \leq \|x-y\|^2 - \|(I-Proj_D^H)x - (I-Proj_D^H)y\|^2$, $\forall x,y \in H$.
(c) $\|Proj_D^H x - Proj_D^H y\|^2 \leq \langle Proj_D^H x - Proj_D^H y, x-y\rangle$, $x,y \in H$.
(d) $\|(I-Proj_D^H)x - (I-Proj_D^H)y\|^2 \leq \langle (I-Proj_D^H)x - (I-Proj_D^H)y, x-y\rangle$, $\forall x,y \in H$.

Lemma 2.1. [32] *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-t_n)a_n + b_n + c_n$, $\forall n \geq 0$, where $\{c_n\}$ is a sequence of nonnegative real numbers, $\{t_n\} \subset (0,1)$ and $\{b_n\}$ is a sequence of real numbers. Assume that*
(a) $\limsup_{n\to\infty} \frac{b_n}{t_n} \leq 0$, $\sum_{n=0}^{\infty} t_n = \infty$,
(b) $\sum_{n=0}^{\infty} c_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Let $T_1, T_2, \cdots, T_n, \cdots$ be nonexpansive mappings of C into itself. Focus on the mapping W_n generated by

$$\begin{aligned}
&V_{n,n} = \delta_n T_n V_{n,n+1} + (1-\delta_n)Id,\\
&V_{n,n-1} = \delta_{n-1} T_{n-1} V_{n,n} + (1-\delta_{n-1})Id,\\
&\vdots\\
&V_{n,k} = \delta_k T_k V_{n,k+1} + (1-\delta_k)Id,\\
&V_{n,k-1} = \delta_{k-1} T_{k-1} V_{n,k} + (1-\delta_{k-1})Id,\\
&\vdots\\
&V_{n,2} = \delta_2 T_2 V_{n,3} + (1-\delta_2)Id,\\
&W_n = V_{n,1} = \delta_1 T_1 V_{n,2} + (1-\delta_1)Id,
\end{aligned}$$

where $V_{n,n+1} = Id$, $\delta_1, \delta_2, \cdots$ are real numbers in $(0,1)$.

From [29], if $0 < \delta_n \leq \delta < 1$, where δ is some real number in $(0,1)$ for any $n \geq 1$ and $\cap_{n=1}^{\infty} F(T_n)$, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} V_{n,k}x$ exists. Define a mapping W by

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} V_{n,1}x,$$

$\forall x \in C$. Mapping W is called the W-*mapping* generated by $\delta_1, \delta_2, \cdots$ and $T_1, T_2, \cdots$. From [29], one has $F(W) = \cap_{n=1}^{\infty} F(T_n)$. From now on, one always assumes that $0 < \delta_n \leq \delta < 1$ for all $n \geq 1$ and $\sum \Pi_{i=1}^{n} \delta_i < \infty$.

Lemma 2.2. [34] *Let g be a MK contraction defined on set C. For each $\varepsilon > 0$, it holds that there exists c_ε in $(0,1)$ such that $\|y-x\| \geq \varepsilon$ implies $\|g(y)-g(x)\| \leq c_\varepsilon \|y-x\|$, $\forall y, x \in H$.*

Lemma 2.3. [24] *Let C be a convex and closed set in a Hilbert space H. Let T be a nonexpansive mapping with fixed points on set C. If $x_n \rightharpoonup q$, where $\rightharpoonup$ denotes the weak convergence, and $x_n - Tx_n \to 0$, then q is a fixed point of T, that is, $q = Tq$.*

Lemma 2.4. [11] *Let H be a real Hilbert space, and let C be a closed, convex, and nonempty subset of H. Let $\{T_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} \mathrm{Fix}(T_i) \neq \emptyset$. Then $\lim_{n\to\infty} \sup_{x\in C} \|Wx - W_n x\| = 0$.*

3. Main results

Theorem 3.1. *Let C be a convex and closed set in Hilbert space H_1 and let Q be a convex and clsoed set in Hilbert space H_2. Let $A : H_1 \to H_2$ be a bounded linear operator such that split feasibility problem (1.1) is consistent. Let g be a Meir-Keeler contractive mapping on C and let T_i be a nonexpansive mapping with fixed points on C for each positive integer i. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{x_n\}$ be a sequence defined by the following iterative algorithm: $x_1 \in C$,*

$$x_{n+1} = Proj_C^{H_1}\Big(\alpha_n g(x_n) + \gamma_n(x_n - \mu_n A^*(I - Proj_Q^{H_2})Ax_n) + \beta_n W_n x_n\Big), \tag{3.1}$$

where $Proj_C^{H_1}$ denotes the nearest point projection from H_1 onto C, $Proj_Q^{H_2}$ denotes the nearest point projection from H_2 onto Q, W_n is the mapping defined in Section 2. Assume that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} |\gamma_n - \gamma_{n+1}| < \infty$, $0 < \gamma \leq \gamma_n \leq \gamma' < 1$, and $\{\mu_n\}$ is a real sequence such that $\sum_{n=1}^{\infty} |\mu_n - \mu_{n+1}| < \infty$, $\frac{2}{\|A\|^2} > \mu' \geq \mu_n \geq \mu > 0$, where γ', γ, μ' and

μ are real numbers. If $\cap_{i=1}^{\infty} F(T_i) \cap Sol(SFP) \neq \emptyset$, then $\{x_n\}$ converges strongly to a point $\bar{x}$ in $\cap_{i=1}^{\infty} F(T_i) \cap Sol(SFP)$ and the solution also solves the variational inequality

$$\langle g(\bar{x}) - \bar{x}, x - \bar{x} \rangle \leq 0, \quad \forall x \in \cap_{i=1}^{\infty} F(T_i) \cap Sol(SFP).$$

Proof. Define a mapping M from set C to space H_1 by

$$Mx = A^*(I - Proj_Q^{H_2})Ax, \quad \forall x \in C.$$

On account of the properties of the nearest point projection, we have

$$\begin{aligned}
\langle x - y, Mx - My \rangle &= \langle A^*(I - Proj_Q^{H_2})Ax - A^*(I - Proj_Q^{H_2})Ay, x - y \rangle \\
&= \langle (I - Proj_Q^{H_2})Ax - (I - Proj_Q^{H_2})Ay, Ax - Ay \rangle \\
&\geq \|(I - Proj_Q^{H_2})Ax - (I - Proj_Q^{H_2})Ay\|^2 \\
&\geq \frac{1}{\|A\|^2} \|A^*(I - Proj_Q^{H_2})Ax - A^*(I - Proj_Q^{H_2})Ay\|^2 \\
&= \frac{1}{\|A\|^2} \|Mx - My\|^2,
\end{aligned} \tag{3.2}$$

which demonstrates that M is a $\frac{1}{\|A\|^2}$-inverse-strongly monotone mapping. Observe that (3.1) is equivalent to

$$x_1 \in C, \quad x_{n+1} = Proj_C^{H_1}\Big(\alpha_n g(x_n) + \gamma_n(x_n - \mu_n Mx_n) + \beta_n W_n x_n\Big), \tag{3.3}$$

By the fact that M is $\frac{1}{\|A\|^2}$-inverse-strongly monotone, one asserts that

$$\begin{aligned}
&\|(I - \mu_n M)x - (I - \mu_n M)y\|^2 \\
&= \|x - y\|^2 - 2\mu_n \langle Mx - My, x - y \rangle + \mu_n^2 \|Mx - My\|^2 \\
&\leq \|x - y\|^2 - \frac{2\mu_n}{\|A\|^2} \|Mx - My\|^2 + \mu_n^2 \|Mx - My\|^2 \\
&= \mu_n(\mu_n - \frac{2}{\|A\|^2}) \|Mx - My\|^2 + \|x - y\|^2.
\end{aligned}$$

By the restriction on $\{\mu_n\}$, one sees that

$$\|(I - \mu_n M)x - (I - \mu_n M)y\| \leq \|x - y\|,$$

that is, $(I - \mu_n M)$ is nonexpansive. Hence $Proj_C^{H_1}(x - \gamma A^*(Ax - Proj_Q^{H_2}(Ax))$ is nonexpansive due to the nonexpansivity of nearest point projection $Proj_C^{H_1}$, which further guarantees that $Sol(SFP)$ is convex and closed. Observe that each $F(T_i)$ is convex and closed for each positive integer i. It results that the solution set is convex and closed and then $Proj_{\cap_{i=1}^{\infty} F(T_i) \cap Sol(SFP)}^{H_1} x$ for all $x \in H_1$ is well defined.

Next, we prove $A^{-1}(Q) = M^{-1}(0)$. Fixing $x \in A^{-1}(Q)$, one clearly sees that $x \in T^{-1}(0)$, which demonstrates that $A^{-1}(Q)$ is a subset of $M^{-1}(0)$. Now, fixing $x \in M^{-1}(0)$, one has $Mx = 0$. From the assumption that $\cap_{i=1}^{\infty} F(T_i) \cap Sol(SFP)$ is not empty, one can pick a vector y in it, which sends us to $Wy = y$ and $Ay = Proj_Q^{H_2} Ay$. Hence, $My = 0$. Observe that

$$\|(I - Proj_Q^{H_2})Ax\|^2 = \|(I - Proj_Q^{H_2})Ax - (I - Proj_Q^{H_2})Ay\|^2.$$

It follows from (3.2) that

$$\|(I-Proj_Q^{H_2})Ax-(I-Proj_Q^{H_2})Ay\|^2 \le \langle Mx-My, x-y\rangle,$$

that imply that x is in $A^{-1}(Q)$, that is, $M^{-1}(0)$ is a subset of $A^{-1}(Q)$. It demonstrates that

$$A^{-1}(Q)=M^{-1}(0).$$

Noticing $Proj_C^{H_1}x^* = x^*$, one sees from (3.3) and the nonexpansivity of W_n that

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|Proj_C^{H_1}\big(\alpha_n g(x_n)+\gamma_n(x_n-\mu_n Mx_n)+\beta_n W_n x_n\big)-x^*\| \\
&\le \|\big(\alpha_n g(x_n)+\gamma_n(x_n-\mu_n Mx_n)+\beta_n W_n x_n\big)-x^*\| \\
&\le \gamma_n\|(x_n-\mu_n Mx_n)-x^*\|+\alpha_n\|g(x_n)-x^*\|+\beta_n\|W_n x_n-x^*\| \\
&\le \gamma_n\|(x_n-\mu_n Mx_n)-x^*\|+\alpha_n\|g(x^*)-x^*\|+\alpha_n\|g(x_n)-g(x^*)\|+\beta_n\|x_n-x^*\|.
\end{aligned}$$

Set a positive constant ε. If $\|x_n-x^*\|<\varepsilon$, then it is obvious that vector sequence $\{x_n\}$ is a bounded sequence. If $\|x_n-x^*\|\ge\varepsilon$, we by Lemma 2.2 have that there holds

$$\|g(x)-g(y)\|\le c_\varepsilon\|x-y\|$$

for all $x,y\in C$, where $c_\varepsilon\in(0,1)$ is a real constant. Thus

$$\|x_{n+1}-x^*\|\le\big(1-\alpha_n(1-c_\varepsilon)\big)\|x_n-x^*\|+\alpha_n(1-c_\varepsilon)\frac{\|x^*-g(x^*)\|}{1-c_\varepsilon}.$$

It is easy to see that

$$\|x_{n+1}-x^*\|\le\max\{\|x_1-x^*\|,\frac{\|g(x^*)-x^*\|}{1-c_\varepsilon}\},$$

which demonstrates $\{x_n\}$ is a bounded vector sequence in both cases.

Now one is in a position to demonstrate

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, x_{n+1}-\bar{x}\rangle\le 0.$$

Setting $y_n=x_n-\mu_n Mx_n$, one has

$$\begin{aligned}
\|y_{n-1}-y_n\| &\le \|(x_n-\mu_n Mx_n)-(x_{n-1}-\mu_{n-1}Mx_{n-1})\| \\
&\le |\mu_{n-1}-\mu_n|\|Mx_n\|+\|x_{n-1}-x_n\|,
\end{aligned}$$

which is due to the fact that $I-\mu_n M$ is a nonexpansive mapping. Observe that

$$\begin{aligned}
\|W_n x_n-W_{n-1}x_n\| &= \|(\delta_1T_1V_{n,2}x_n+(1-\delta_1)x_n)-(\delta_1T_1V_{n-1,2}x_n+(1-\delta_1)x_n)\| \\
&\le \|\delta_1T_1V_{n,2}x_n-\delta_1T_1V_{n-1,2}x_n\| \\
&\le \delta_1\|V_{n,2}x_n-V_{n-1,2}x_n\| \\
&= \delta_1\|(\delta_2T_2V_{n,3}x_n+(1-\delta_2)x_n)-(\delta_2T_2V_{n-1,3}x_n+(1-\delta_2)x_n)\| \\
&\le \delta_1\|\delta_2T_2V_{n,3}x_n-\delta_2T_2V_{n-1,3}x_n\| \\
&\le \delta_1\delta_2\|V_{n,3}x_n-V_{n-1,3}x_n\|.
\end{aligned}$$

It follows that

$$\|W_n x_n-W_{n-1}x_n\|\le C\Pi_{i=1}^n\delta_i,$$

where C is some appropriate constant. Furthermore, one has

$$\begin{aligned}\|W_nx_n - W_{n-1}x_{n-1}\| &\leq \|W_{n-1}x_{n-1} - W_{n-1}x_n\| + \|W_{n-1}x_n - W_nx_n\| \\ &\leq \|x_{n-1} - x_n\| + C\Pi_{i=1}^n \delta_i.\end{aligned}$$

Thus

$$\begin{aligned}&\|x_{n+1} - x_n\| \\ &\leq \|\big(\alpha_n g(x_n) + \gamma_n y_n + \beta_n W_n x_n\big) - \big(\alpha_{n-1} g(x_{n-1}) + \gamma_{n-1} y_{n-1} + \beta_{n-1} W_{n-1} x_{n-1}\big)\| \\ &\leq \alpha_n \|g(x_n) - g(x_{n-1})\| + \gamma_n \|y_n - y_{n-1}\| + \beta_n \|W_n x_n - W_{n-1} x_{n-1}\| \\ &\quad + |\alpha_n - \alpha_{n-1}| \|g(x_{n-1})\| + |\beta_n - \beta_{n-1}| \|W_{n-1} x_{n-1}\| + |\gamma_n - \gamma_{n-1}| \|y_{n-1}\| \\ &\leq \big(1 - \alpha_n(1 - c_\varepsilon)\big) \|x_n - x_{n-1}\| + \gamma_n |\mu_n - \mu_{n-1}| \|Mx_n\| + +C\Pi_{i=1}^n \delta_i \\ &\quad + |\alpha_n - \alpha_{n-1}| \|g(x_{n-1})\| + |\gamma_n - \gamma_{n-1}| \|y_{n-1}\| + |\beta_n - \beta_{n-1}| \|W_{n-1} x_{n-1}\|.\end{aligned}$$

By Lemma 2.1, we have

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{3.4}$$

due to the conditions imposed on $\{\alpha_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, and $\{\delta_n\}$. Since $\|\cdot\|^2$ is convex, we find from (3.5) that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|\big(\alpha_n g(x_n) + \gamma_n y_n + \beta_n W_n x_n\big) - x^*\|^2 \\ &\leq \alpha_n \|g(x_n) - x^*\|^2 + \gamma_n \|y_n - x^*\|^2 + \beta_n \|W_n x_n - x^*\|^2.\end{aligned}$$

Observe that

$$\begin{aligned}\|y_n - x^*\|^2 &= \mu_n^2 \|Mx_n - Mx^*\|^2 + \|x_n - x^*\|^2 - 2\mu_n \langle x_n - x^*, Mx_n - Mx^* \rangle \\ &\leq \mu_n^2 \|Mx_n - Mx^*\|^2 + \|x_n - x^*\|^2 - \frac{2\mu_n}{\|A\|^2} \|Mx_n - Mx^*\|^2 \\ &= \|x_n - x^*\|^2 - (\frac{2\mu_n}{\|A\|^2} - \mu_n^2) \|Mx_n\|^2.\end{aligned} \tag{3.5}$$

It follows that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq (1 - \alpha_n) \|x_n - x^*\|^2 + \alpha_n \|g(x_n) - x^*\|^2 + \gamma_n (\mu_n^2 - \frac{2\mu_n}{\|A\|^2}) \|Mx_n\|^2 \\ &\leq \|x_n - x^*\|^2 + \alpha_n \|g(x_n) - x^*\|^2 + \gamma_n (\mu_n^2 - \frac{2\mu_n}{\|A\|^2}) \|Mx_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}&\gamma_n (\frac{2\mu_n}{\|A\|^2} - \mu_n^2) \|Mx_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_n \|g(x_n) - x^*\|^2 \\ &\leq \|x_{n+1} - x_n\| (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\| + \alpha_n \|g(x_n) - x^*\|^2.\end{aligned}$$

Using the conditions imposed on $\{\alpha_n\}$, $\{\gamma_n\}$ $\{\mu_n\}$, we find from (3.4) that $Mx_n \to 0$ as $n \to \infty$.

Next, we prove that $x_n - Wx_n \to 0$ as $n \to \infty$. Note that

$$\begin{aligned}\|W_nx_n - x_n\| &\leq \|Proj_C^{H_1}\Big(\alpha_n g(x_n) + \gamma_n y_n + \beta_n W_n x_n\Big) - W_nx_n\| + \|x_n - x_{n+1}\| \\ &\leq \|\big(\alpha_n g(x_n) + \gamma_n y_n + \beta_n W_n x_n\big) - W_nx_n\| + \|x_n - x_{n+1}\| \\ &\leq \alpha_n\|g(x_n) - W_nx_n\| + \gamma_n\|y_n - W_nx_n\| + \|x_n - x_{n+1}\| \\ &\leq \alpha_n\|g(x_n) - W_nx_n\| + \gamma_n\|x_n - W_nx_n\| + \gamma_n\|y_n - x_n\| + \|x_n - x_{n+1}\|,\end{aligned}$$

which yields

$$\|x_n - W_nx_n\| \leq \frac{\alpha_n}{1-\gamma_n}\|g(x_n) - W_nx_n\| + \frac{\gamma_n}{1-\gamma_n}\|y_n - x_n\| + \frac{1}{1-\gamma_n}\|x_n - x_{n+1}\|.$$

Using the conditions imposed on $\{\alpha_n\}$ and $\{\gamma_n\}$, we find that $\lim_{n\to\infty}\|W_nx_n - x_n\| = 0$. Observe that

$$\|Wx_n - x_n\| \leq \|W_nx_n - Wx_n\| + \|W_nx_n - x_n\|.$$

From Lemma 2.4, we have $x_n - Wx_n \to 0$ as $n \to \infty$.

Now, we demonstrate that

$$\limsup_{n\to\infty}\langle g(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \leq 0, \tag{3.6}$$

We pick a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle g(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle = \lim_{j\to\infty}\langle g(\bar{x}) - \bar{x}, x_{x_{n_j}} - \bar{x}\rangle. \tag{3.7}$$

We may assume, without loss of generality, that $\{x_{n_j}\}$ converges weakly to $\widehat{x} \in H_1$. From 2.3, one has $\widehat{x} \in F(W)$, i.e., $\widehat{x} \in \cap_{i=1}^{\infty} F(T_i)$. From the fact that set C is weakly closed, it results $\widehat{x} \in C$. Since M is $\frac{1}{\|A\|^2}$-inverse-strongly monotone,

$$\frac{1}{\|A\|^2}\|M\widehat{x} - Mx_{n_j}\|^2 \leq \langle x_{n_j} - \widehat{x}, Mx_{n_j} - M\widehat{x}\rangle. \tag{3.8}$$

Letting $j \to \infty$ in (3.8), one obtains

$$0 \leq \frac{1}{\|A\|^2}\|T\widehat{x}\|^2 \leq 0,$$

which demonstrate that $M\widehat{x} = 0$, that is, $\widehat{x} \in M^{-1}(0)$. Observe $\widehat{x} \in C \cap M^{-1}(0) \cap Fix(S) = Sol(SFP) \cap Fix(W)$. Hence, (3.6) holds. Thus

$$\begin{aligned}&\limsup_{n\to\infty}\langle g(\bar{x}) - \bar{x}, x_{n+1} - \bar{x}\rangle \\ &\leq \|g(\bar{x}) - \bar{x}\|\limsup_{n\to\infty}\|x_{n+1} - x_n\| + \limsup_{n\to\infty}\langle g(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle,\end{aligned}$$

which results

$$\limsup_{n\to\infty}\langle x_{n+1} - \bar{x}, g(\bar{x}) - \bar{x}\rangle \leq 0. \tag{3.9}$$

Finally, one proves $x_n \to \bar{x}$ in norm as $n \to \infty$. Assume that the sequence $\{x_n\}$ does not converge to $\bar{x}$ in norm, so there exists $\varepsilon > 0$ and a subsequence $\{x_{n_i}\}$ such that $\|x_{n_i} - \bar{x}\| \geq \varepsilon$ for all i. One may assume that $\|x_n - \bar{x}\| \geq \varepsilon$. By Lemma 2.2, one has that there holds

$$\|g(x_n) - g(\bar{x})\| \leq c_\varepsilon\|x_n - \bar{x}\|$$

for all $x,y\in C$, where $c_\varepsilon\in(0,1)$ is a real constant. Observe that

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \langle \alpha_n(g(x_n)-\bar{x})+\gamma_n(y_n-\bar{x})+\beta_n(W_nx_n-\bar{x}),x_{n+1}-\bar{x}\rangle\\
&\leq \alpha_n\langle g(x_n)-\bar{x},x_{n+1}-\bar{x}\rangle+\gamma_n\langle y_n-\bar{x},x_{n+1}-\bar{x}\rangle+\beta_n\langle W_nx_n-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\leq \alpha_n\langle g(x_n)-g(\bar{x}),x_{n+1}-\bar{x}\rangle+\alpha_n\langle g(\bar{x})-\bar{x},x_{n+1}-\bar{x}\rangle\\
&\quad+\gamma_n\|x_{n+1}-\bar{x}\|\|y_n-\bar{x}\|+\beta_n\|x_{n+1}-\bar{x}\|\|\bar{x}-W_nx_n\|\\
&\leq \big(1-(1-c_\varepsilon)\alpha_n\big)\|x_{n+1}-\bar{x}\|\|x_n-\bar{x}\|\\
&\quad+\alpha_n(1-c_\varepsilon)\frac{\langle x_{n+1}-\bar{x},g(\bar{x})-\bar{x}\rangle}{1-\kappa}.
\end{aligned}$$

Hence,

$$\begin{aligned}
2\|x_{n+1}-\bar{x}\|^2 &\leq \big(1-(1-c_\varepsilon)\alpha_n\big)(\|x_{n+1}-\bar{x}\|^2+\|x_n-\bar{x}\|^2)\\
&\quad+2\alpha_n(1-c_\varepsilon)\frac{\langle x_{n+1}-\bar{x},g(\bar{x})-\bar{x}\rangle}{1-\kappa},
\end{aligned}$$

that is,

$$\begin{aligned}
\|x_{n+1}-\bar{x}\|^2 &\leq \big(1-(1-c_\varepsilon)\alpha_n\big)\|x_{n+1}-\bar{x}\|^2\\
&\quad+2\alpha_n(1-c_\varepsilon)\frac{\langle x_{n+1}-\bar{x},g(\bar{x})-\bar{x}\rangle}{1-\kappa},
\end{aligned}$$

Using Lemma 2.1, we find that $x_n\to\bar{x}$ as $n\to\infty$. This completes the proof. □

From Theorem 3.1, we have the following sub-results.

Corollary 3.2. *Let C be a convex and closed set in Hilbert space H_1 and let Q be a convex and clsoed set in Hilbert space H_2. Let $A: H_1\to H_2$ be a bounded linear operator such that split feasibility problem (1.1) is consistent. Let g be a Meir-Keeler contractive mapping on C and let W be a nonexpansive mapping with fixed points on C. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ with $\alpha_n+\beta_n+\gamma_n=1$. Let $\{x_n\}$ be a sequence defined by the following iterative algorithm: $x_1\in C$,*

$$x_{n+1}=Proj_C^{H_1}\Big(\alpha_ng(x_n)+\gamma_n(x_n-\mu_nA^*(I-Proj_Q^{H_2})Ax_n)+\beta_nWx_n\Big),$$

where $Proj_C^{H_1}$ denotes the nearest point projection from H_1 onto C, $Proj_Q^{H_2}$ denotes the nearest point projection from H_2 onto Q. Assume that $\sum_{n=1}^\infty\alpha_n=\infty$, $\sum_{n=1}^\infty|\alpha_n-\alpha_{n+1}|<\infty$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^\infty|\gamma_n-\gamma_{n+1}|<\infty$, $0<\gamma\leq\gamma_n\leq\gamma'<1$, and $\{\mu_n\}$ is a real sequence such that $\sum_{n=1}^\infty|\mu_n-\mu_{n+1}|<\infty$, $\frac{2}{\|A\|^2}>\mu'\geq\mu_n\geq\mu>0$, where γ',γ,μ' and μ are real numbers. If $Sol(SFP)\cap F(W)\neq\emptyset$, then $\{x_n\}$ converges strongly to a point $\bar{x}$ in $Sol(SFP)\cap F(W)$ and the solution also solves the variational inequality $\langle g(\bar{x})-\bar{x},x-\bar{x}\rangle\leq 0$ for all $x\in Sol(SFP)\cap F(W)$.

Corollary 3.3. *Let C be a convex and closed set in Hilbert space H_1 and let Q be a convex and clsoed set in Hilbert space H_2. Let $A: H_1\to H_2$ be a bounded linear operator such that split feasibility problem (1.1) is consistent. Let g be a Meir-Keeler contractive mapping on C. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be real sequences in $(0,1)$ with $\alpha_n+\beta_n+\gamma_n=1$. Let $\{x_n\}$ be a sequence defined by the following iterative algorithm: $x_1\in C$,*

$$x_{n+1}=Proj_C^{H_1}\Big(\alpha_ng(x_n)+\gamma_n(x_n-\mu_nA^*(I-Proj_Q^{H_2})Ax_n)+\beta_nx_n\Big),$$

where $Proj_C^{H_1}$ denotes the nearest point projection from H_1 onto C, $Proj_Q^{H_2}$ denotes the nearest point projection from H_2 onto Q. Assume that $\sum_{n=1}^{\infty}\alpha_n=\infty$, $\sum_{n=1}^{\infty}|\alpha_n-\alpha_{n+1}|<\infty$, $\lim_{n\to\infty}\alpha_n=0$, $\sum_{n=1}^{\infty}|\gamma_n-\gamma_{n+1}|<\infty$, $0<\gamma\le\gamma_n\le\gamma'<1$, and $\{\mu_n\}$ is a real sequence such that $\sum_{n=1}^{\infty}|\mu_n-\mu_{n+1}|<\infty$, $\frac{2}{\|A\|^2}>\mu'\ge\mu_n\ge\mu>0$, where γ',γ,μ' and μ are real numbers. If $Sol(SFP)\ne\emptyset$, then $\{x_n\}$ converges strongly to a point $\bar{x}$ in $Sol(SFP)$ and the solution also solves the variational inequality $\langle g(\bar{x})-\bar{x},x-\bar{x}\rangle\le 0$ for all $x\in Sol(SFP)$.

Remark 3.4. Corollary 3.3 is the case when each T_i is the identical map on C. In this case $F(T_i)=C$, the formulation in Theorem 3.1 only involves $Sol(SFP)$ rather than $\cap_{i=1}^{\infty}F(T_i)\cap Sol(SFP)$ or $F(W)\cap Sol(SFP)$.

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