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AN APPROXIMATE APPROACH FOR SOLVING FIXED POINT AND SYSTEMS OF VARIATIONAL INEQUALITY PROBLEMS WITHOUT CERTAIN CONSTRAINTS

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Abstract. This paper constructs a new iterative method for identifying a common solution of a general system of variational inequalities and a fixed point problem of a nonexpansive mapping. Furthermore, this paper establishes some necessary and sufficient conditions of strong convergence of iterative sequences without any assumption that the solution set of the problem is nonempty in Hilbert spaces. Finally, some applications and examples are provided to support the main results.

Keywords. Approximate Approach; Convergence analysis; System of variational inequalities; Split feasibility problem.

1. Introduction

The *Fixed Point Problem* is a problem in mathematics that involves finding a special point in a set, that is, finding a point, which is mapped to itself by a given operator. This problem arises in many areas of pure and applied mathematics, such as in nonlinear functional analysis, topology, dynamical systems, optimization, physics, and engineering. The fixed point problem is important because it provides a way to study the behavior of an operator or a system, and it also helps to develop numerical methods for approximating solutions of certain types of operator equations; see, e.g., [15, 21, 22] and the references therein.

In this paper, we consider H as a real Hilbert space, equipped with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $|\cdot|$. Additionally, we take C to be a non-empty, closed, and convex subset of H.

In this context, we consider a nonlinear mapping $T : H \to H$ and set its fixed point as $F(T)$, that is, $F(T) = \{x \in H : Tx = x\}$. Recall the following definitions.

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i) T is said to be *nonexpansive* if the following inequality holds for all $x,z \in H$: $\|Tx - Tz\| \leq \|x - z\|$.

There are numerous applications of nonexpansive operators in operator equations, optimization theory, and data science. In the last decade, many new results on fixed points of nonexpansive operators were established in various spaces; see, e.g., [1, 2, 9, 17] and the references therein.

Let A be a mapping defined on C to H. Recall the following definitions.

ii) A is said to be *monotone* if the following inequality holds for all $x,z \in H$: $\langle Ax - Az, x - z\rangle \geq 0$.

iii) A is said to be $\eta-$*strongly monotone* with a constant $\eta > 0$ if the following inequality holds for all $x,z \in H$: $\langle Ax - Az, x - z\rangle \geq \eta \|x - z\|^2$.

iv) A is said to be $\beta-$*inverse-strongly monotone* with a constant $\beta > 0$ if the following inequality holds for all $x,z \in H$: $\langle Ax - Az, x - z\rangle \geq \beta \|Ax - Az\|^2$.

v) A is said to be *L-Lipschitz* with a constant $L > 0$ if the following inequality holds for all $x,z \in C$: $\|Ax - Az\| \leq L\|x - z\|$. If $L < 1$, then A is called *L-contraction* or a *contraction*. It is obvious that inverse-strongly monotone operators are Lipschitz continuous.

Recall the classical monotone variational inequality, denote by $VI(C,A)$, is to

$$\text{find } x^* \in C \text{ such that } \langle x - x^*, Ax^*\rangle \geq 0$$

for all x in set C, where A is a monotone operator from C to H. Many problems in engineering, economic, management, and computer sciences can modelled into a provides a variational inequality. One of the simplest methods for solving the inequality is the projected-gradient method: Given the current iterate x_n, calculate the next iterate x_{n+1} as $x_{n+1} = P_C(x_n - \lambda_n Ax_n)$, where $P_C : H \to C$ denotes the metric (nearest point) projection from H onto C, characterized by $P_C(x) := \arg\min\{\|x - y\|, y \in C\}$ and $\{\lambda_n\}$ is a sequence of positive real numbers.

Recently, a number of researchers proposed various iterative methods for solving the problems via fixed point algorithms; see, e.g., [10, 11, 13, 19, 20] and the references therein. Note that if A is not inverse-strongly monotone, then the sequence defined by the projected-gradient method above may fail to converge to its solution.

A widely recognized method for resolving the variational inequality problem in the Euclidean space $\mathbb{R}^n$ is the extragradient method, which was developed by Korpelevich [8]. The extragradient method reads as follows: select an initial point x_0 in C, and, for each $n \geq 0$, compute

$$\begin{aligned} y_n &= P_C(x_n - \lambda_n Ax_n), \\ x_{n+1} &= P_C(x_n - \lambda_n Ay_n), \end{aligned}$$

where A is a Lipschitz monotone operator with the constant L and $\{\lambda_n\}$ is a sequence in $\left(0, \frac{1}{L}\right)$. If $VI(C,A)$ is not empty, the sequence, $\{x_n\}$, generated by this method converges weakly to an element in $VI(C,A)$.

Recently, the convex feasibility problems with fixed point problems and variational inequality problems attracted much attention in the fields of nonlinear optimization. Nadezhkina and Takahashi [12], Yao and Yao [22], and Zeng and Yao [23] proposed several iterative methods for finding solutions in the intersection of $F(T)$ and $VI(C,A)$, where T is a nonexpansive mapping and A is a monotone operator. In particular, in [22], Yao and Yao developed a strong convergence theorem in a real Hilbert space under certain suitable conditions.

The results in [12], [22], and [23] were established under the assumption that the intersection of $F(T)$ and $VI(C,A)$ is not empty.

In 2008, Ceng et al. [3] presented a system and an iteration method, known as the relaxed extragradient method for fixed point problems and variational inequality problems. They investigated the problem of finding a point $(x^*,y^*) \in C\times C$ for all $x\in C$ that satisfies the following constraints

$$\begin{cases} \langle x-x^*, \alpha A_1 y^* + x^* - y^*\rangle \geq 0, \\ \langle x-y^*, \beta A_2 x^* + y^* - x^*\rangle \geq 0, \end{cases} \tag{1.1}$$

where operators $A_1, A_2 : C \to H$ are inverse-strongly monotone and parameters α and β are nonnegative.

In 2019, Siriyan and Kangtanyakarn [18] introduced a new system of variational inequalities in a real Hilbert space by modifying (1.1). Find a point (x^*,y^*,z^*) in $C\times C\times C$ such that, for all $x\in C$,

$$\begin{cases} \langle x-x^*, x^* - (I-\alpha_1 A_1)(bx^* + (1-b)y^*)\rangle \geq 0, \\ \langle x-y^*, y^* - (I-\alpha_2 A_2)(bx^* + (1-b)z^*)\rangle \geq 0, \\ \langle x-z^*, z^* - (I-\alpha_3 A_3)x^*\rangle \geq 0, \end{cases} \tag{1.2}$$

where mappings $A_1, A_2, A_3 : C \to H$ are inverse-strongly monotone, and $\alpha_1, \alpha_2, \alpha_3 > 0$ and $b \in [0,1]$ are constants.

By setting $b = 0$ in (1.2), we have

$$\begin{cases} \langle x-x^*, x^* - (I-\alpha_1 A_1)y^*\rangle \geq 0, \\ \langle x-y^*, y^* - (I-\alpha_2 A_2)z^*\rangle \geq 0, \\ \langle x-z^*, z^* - (I-\alpha_3 A_3)x^*\rangle \geq 0. \end{cases} \tag{1.3}$$

By setting $x^* = z^*$ and $A_3 = 0$, the new system problem (1.2) can be reduced to general system problem (1.1). Additionally, a new iteration process was introduced to find solutions of problem (1.2). Researchers developed strong convergence theorems for solutions of the variational inequality problem by using the condition on the fixed point set and the solution set of variational inequalities, which is non-empty according to [3, 23].

In 2013, Kangtanyakarn [7] proposed a strong convergence theorem for finding solutions to both the fixed point problem of a nonexpansive mapping T and the solution set of the variational inequality problem associated with a mapping $A : C \to H$ in the framework of Hilbert space without assuming that $VI(C,A)\cap F(T) \neq \emptyset$. This was done by utilizing necessary and sufficient conditions. In this paper, we present a new iteration method for finding approximate solutions to both (1.2) and the fixed point problem of a nonexpansive mapping T in a real hilbert space. Unlike previous methods, this approach does not require the condition $F(T)\cap F(G) \neq \emptyset$, where G is a mapping defined in Lemma 2.1. Our method is inspired by the prior research of Siriyan and Kangtanyakarn [18] and Kangtanyakarn [7]. In addition, we use our main theorem to investigate the General Split Feasibility Problem (GSFP) and provide numerical examples to validate our results.

2. Preliminaries

Let P_C be the metric projection of H onto C, that is, for every $x \in H$, there is a unique nearest point $P_C x$ in C such that $\|P_C x - x\| = \min_{y \in C} \|y - x\|$. It is known that P_C has the following properties:

i) P_C is firmly nonexpansive, i.e.,

$$\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2 \text{ for all } x, y \in H.$$

ii) For each $x \in H$,

$$\langle x - z, z - y \rangle \geq 0 \Leftrightarrow z = P_C x \text{ for all } y \in C.$$

The following known results are valid in a real Hilbert space H:

i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$ for all $x, y \in H$.

ii) $\|\alpha x + (1-\alpha) y\|^2 = \alpha \|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ for all $x, y \in H$ and $\alpha \in [0,1]$.

iii) $\|\lambda_1 x + \lambda_2 y + \lambda_3 z\|^2 = \lambda_1 \|x\|^2 + \lambda_2 \|y\|^2 + \lambda_3 \|z\|^2 - \lambda_1 \lambda_2 \|x-y\|^2 - \lambda_1 \lambda_3 \|x-z\|^2 - \lambda_2 \lambda_3 \|y - z\|^2$ for all $x, y, z \in H$ and $\lambda_1, \lambda_2, \lambda_3 \in [0,1]$ with $\lambda_1 + \lambda_2 + \lambda_3 = 1$.

Lemma 2.1. [18] *Let $A_1, A_2, A_3 : C \to H$ be three mappings. For $\alpha_1, \alpha_2, \alpha_3 > 0$ and $b \in [0,1]$, the following assertions are equivalent*

i) The solution of problem (1.2) is $(x^*, y^*, z^*) \in C \times C \times C$,

ii) $x^* \in F(G)$, where defined $G : C \to C$ for all $x \in C$ by

$$G(x) = P_C(I - \alpha_1 A_1)\Big(bx + (1-b)P_C(I - \alpha_2 A_2)\big(bx + (1-b)P_C(I - \alpha_3 A_3)x\big)\Big),$$

where $y^* = P_C(I - \alpha_2 A_2)\big(bx^* + (1-b)z^*\big)$ and $z^* = P_C(I - \alpha_3 A_3)x^*$.

We denote strong convergence of a sequence by symbol $\to$, and weak convergence by symbol $\rightharpoonup$.

Lemma 2.2. [14]. *For any sequence $\{x_n\}$ that converges weakly to x, $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n \to \infty} \|x_n - x\| < \liminf_{n \to \infty} \|x_n - y\|, \forall y \in H$$

holds for all $y \neq x$.

We remark that the Opial's condition holds in Hilbert spaces.

3. Main Results

Theorem 3.1. *Let C be a nonempty, convex and closed subset of a real Hilbert space H and T be a nonexpansive mapping defined on C. Let $A_1 : C \to H$ be L-lipshitz γ_1-strongly monotone with $L \leq \gamma_1 < \frac{1}{2}$ and let $A_2, A_3 : C \to H$ be γ_2, γ_3-inverse-strongly monotone. For every $\alpha_1 \in (0, \gamma_1)$, $\alpha_2 \in (0, 2\gamma_2)$, $\alpha_3 \in (0, 2\gamma_3)$, define $G : C \to C$ by*

$$G(x) = P_C(I - \alpha_1 A_1)\Big(bx + (1-b)P_C(I - \alpha_2 A_2)\big(bx + (1-b)P_C(I - \alpha_3 A_3)x\big)\Big), \ \forall x \in C,$$

where $b \in [0,1]$. Suppose that $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = G\big(\lambda x_n + (1-\lambda)T x_n\big), \tag{3.1}$$

for all $n \geq 1$ and $\lambda \in (0,1)$. Then these statements are equivalent:

i) the sequence $\{x_n\}$ defined by (3.1) converges strongly to $x^* \in F(T) \cap F(G)$,

ii) $\lim_{n \to \infty} \|x_n - T x_n\| = 0$.

Proof. Assume that $\{x_n\}$ converges strongly to $x^* \in F(T)\cap F(G)$. Since $x^* \in F(T)\cap F(G)$, then

$$\|x_n - Tx_n\| \leq \|x_n - x^*\| + \|Tx^* - Tx_n\| \leq 2\|x_n - x^*\|,$$

which implies that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Let $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. First, we demonstrate that G is a contraction. From the fact that $A_1 : C \to H$ is L-lipshitz and γ_1 strongly monotone, we have

$$\begin{aligned}
&\|(I-\alpha_1 A_1)x - (I-\alpha_1 A_1)y\|^2 \\
&= \|x-y\|^2 + \alpha_1^2\|A_1x - A_1y\|^2 - 2\alpha_1\langle x-y, A_1x - A_1y\rangle \\
&\leq (1-\alpha_1\gamma_1)^2\|x-y\|^2,
\end{aligned}$$

that is,

$$\|(I-\alpha_1 A_1)x - (I-\alpha_1 A_1)y\| \leq (1-\alpha_1\gamma_1)\|x-y\|.$$

Since A_2 is γ_2-inverse strongly monotone and $\alpha_2 \in (0, 2\gamma_2)$, we have

$$\begin{aligned}
&\|(I-\alpha_2 A_2)x - (I-\alpha_2 A_2)y\|^2 \\
&= \|x-y\|^2 + \alpha_2^2\|A_2x - A_2y\|^2 - 2\alpha_2\langle x-y, A_2x - A_2y\rangle \\
&\leq \|x-y\|^2 - \alpha_2(2\gamma_2 - \alpha_2)\|A_2x - A_2y\|^2,
\end{aligned}$$

that is,

$$\|(I-\alpha_2 A_2)x - (I-\alpha_2 A_2)y\| \leq \|x-y\|.$$

Hence $P_C(I-\alpha_2 A_2)$ is a nonexpansive mapping. By employing the method used above, we have $(I-\alpha_3 A_3)$ and $P_C(I-\alpha_3 A_3)$ are also nonexpansive. Let $x, y \in C$. Then,

$$\begin{aligned}
&\|Gx - Gy\| \\
&= (1-\alpha_1\gamma_1)\Big\|b(x-y) + (1-a)\Big(P_C(I-\alpha_2 A_2)\big(bx + (1-b)P_C(I-\alpha_3 A_3)x\big) \\
&- P_C(I-\alpha_2 A_2)\big(by + (1-b)P_C(I-\alpha_3 A_3)y\big)\Big)\Big\| \\
&\leq (1-\alpha_1\gamma_1)\Bigg(b\|x-y\| + (1-b)\Big\|P_C(I-\alpha_2 A_2)\big(bx + (1-b)P_C(I-\alpha_3 A_3)x\big) \\
&- P_C(I-\alpha_2 A_2)\big(by + (1-b)P_C(I-\alpha_3 A_3)y\big)\Big\|\Bigg) \\
&\leq (1-\alpha_1\gamma_1)\Bigg(b\|x-y\| + (1-b)\Big(b\|x-y\| + (1-b)\big\|P_C(I-\alpha_3 A_3)x \\
&- P_C(I-\alpha_3 A_3)y\big\|\Big)\Bigg) \\
&\leq (1-\alpha_1\gamma_1)\|x-y\|,
\end{aligned}$$

which demonstrates that G is a $(1-\alpha_1\gamma_1)$ contraction. Since G is a $(1-\alpha_1\gamma_1)$ contraction and T is a nonexpansive mapping, we conclude

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq (1-\alpha_1\gamma_1)\Big\|\big(\lambda x_n+(1-\lambda)Tx_n\big)-\big(\lambda x_{n-1}+(1-\lambda)Tx_{n-1}\big)\Big\| \\
&\leq (1-\alpha_1\gamma_1)\Big(\lambda\|x_n-x_{n-1}\|+(1-\lambda)\|Tx_n-Tx_{n-1}\|\Big) \\
&\leq (1-\alpha_1\gamma_1)\|x_n-x_{n-1}\| \\
&\leq (1-\alpha_1\gamma_1)\big((1-\alpha_1\gamma_1)\|x_{n-1}-x_{n-2}\|\big) \\
&\leq (1-\alpha_1\gamma_1)^3\|x_{n-2}-x_{n-3}\| \\
&\cdots \\
&\leq (1-\alpha_1\gamma_1)^n\|x_1-x_0\|.
\end{aligned}$$

For any number $n,k\in\mathbb{N}$, we conclude

$$\begin{aligned}
\|x_{n+k}-x_n\| &\leq \sum_{j=n}^{n+k-1}\|x_{j+1}-x_j\| \\
&\leq \sum_{j=n}^{n+k-1}(1-\alpha_1\gamma_1)^j\|x_1-x_0\| \\
&\leq \frac{(1-\alpha_1\gamma_1)^n}{1-(1-\alpha_1\gamma_1)}\|x_1-x_0\|.
\end{aligned}$$

Since $\lim_{n\to\infty}(1-\alpha_1\gamma_1)^n=0$, we have that $\{x_n\}$ is a Cauchy sequence. Thus there is $x^*\in C$ such that $\lim_{n\to\infty}x_n=x^*$. In view of $\lim_{n\to\infty}\|x_n-Tx_n\|=0$, and the Opial's property, we have $x^*\in F(T)$. From definition of x_n and $x^*\in F(T)$, we have

$$\Big\|G\big(\lambda x_n+(1-\lambda)Tx_n\big)-Gx^*\Big\| \leq \lambda\|x_n-x^*\|+(1-\lambda)\|Tx_n-x^*\| \leq \|x_n-x^*\|,$$

which demonstrates that

$$x^*=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}G\big(\lambda x_n+(1-\lambda)Tx_n\big)=Gx^*.$$

It follows that $x^*\in F(G)$, so $x^*\in F(T)\cap F(G)$. Therefore $\{x_n\}$ converges strongly to $x^*\in F(T)\cap F(G)$. This completes our proof. □

By using Theorem 3.1, we obtain the method for solving a general system of variational inequalities (1.2) without assumption $F(T)\cap F(G)\neq\emptyset$.

Finally, we give the following example.

Example 3.2. Let $\mathbb{R}$ be the set of real number, $C=\{x\in\mathbb{R}|\langle x,u\rangle=\xi\}$, and

$$P_Cx=x+\frac{\xi-\langle x,u\rangle}{\|u\|^2}u$$

with $u=2$ and $\xi=5$. Let the sequence $\{x_n\}$ be generated by $x_1\in C$, $x_{n+1}=P_C(I-\frac{1}{3}A_1)x_n$ for all $n\in\mathbb{N}$, where A_1 is mapping on $\mathbb{R}$ defined by $A_1x=\frac{x}{4}$. Then $\{x_n\}$ convergence strongly to $\frac{5}{2}$.

Observe that the sequence can be rewritten as

$$\begin{aligned} x_{n+1} &= P_C(I-\frac{1}{3}A_1)x_n \\ &= P_C(I-\frac{1}{3}A_1)\Big(bI+(1-b)P_C(I-2A_2)\big(bI+(1-b)P_C(I-3A_3)I\big)\Big)\big(\frac{1}{4}x_n+\frac{3}{4}Ix_n\big), \end{aligned}$$

where $A_2 \equiv A_3 \equiv 0$, $b = 0$. Putting

$$G \equiv P_C(I-\frac{1}{3}A_1)\Big(bI+(1-b)P_C(I-2A_2)\big(bI+(1-b)P_C(I-3A_3)I\big)\Big).$$

It is obvious that I is a nonexpansive mapping, and $\lim_{n\to\infty}\|x_n - Ix_n\| = 0$. From theorem 3.1, we have that $\{x_n\}$ converges strongly to $\frac{5}{2} \in F(T)\cap F(G)$.

4. Applications

In this section, we apply our main results to the general system of variational inequality and fixed points of a nonexpansive mapping provided that $F(T)\cap F(G)$ is non-empty.

Theorem 4.1. *Let G,T,A_1,A_2,A_3, be mappings and $\alpha_1,\alpha_2,\alpha_3,\gamma_1,\gamma_2,\gamma_3,b,\lambda$ be defined as in Theorem 3.1. If $F(T)\cap F(G)\neq\emptyset$, then the sequence $\{x_n\}$ define as in (3.1) converges strongly to $x^*\in F(T)\cap F(G)$.*

Proof. From Theorem 3.1, we see that $\{x_n\}$ is a Cauchy sequence with $\{x_n\}$ converging strongly to $x^*\in C$, which implies that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{4.1}$$

Fixing $p\in F(T)\cap F(G)$, we have

$$\begin{aligned} \|x_{n+1}-p\|^2 &= \|G\big(\lambda x_n+(1-\lambda)Tx_n\big)-p\|^2 \\ &\le \|\lambda(x_n-p)+(1-\lambda)(Tx_n-p)\|^2 \\ &\le \|x_n-p\|^2-\lambda(1-\lambda)\|x_n-Tx_n\|^2, \end{aligned}$$

which yields that

$$\lambda(1-\lambda)\|x_n-Tx_n\|^2 \le \big(\|x_n-p\|+\|x_{n+1}-p\|\big)\|x_{n+1}-x_n\|.$$

From (4.1), we have $\lim_{n\to\infty}\|Tx_n-x_n\| = 0$. By Theorem 3.1, ii) $\to$ i), we conclude that sequence $\{x_n\}$ converges strongly to $x^*\in F(T)\cap F(G)$. Thus the proof is complete. □

Split Feasibility Problem (SFP) is a type of optimization problem whose goal is to find a solution that simultaneously satisfies two or more constraints that are given in the form of separate feasibility problems. The SFP is often encountered in a variety of fields, including signal processing, control systems, and image processing.

Let C and Q be convex and closed subsets in Hilbert spaces H_1 and H_2, respectively. Censor and Elfving [4] defined this problem is to find x^* in C $Dx^*\in Q$, where D is a linear and bounded operator from H_1 to H_2. Notice that the SFP can be transformed into constrained minimization problems, fixed point problems, and variational inequality problems. Recently, various iterative algorithms were introduced to investigate solutions of the SFP; see, e.g., [5, 6, 16] and the references therein.

In 2019, Kangtanyakarn [6] introduced the Generalized Split Feasibility Problem (GSFP), which is to find a point x^* in C such that $D_1x^*,\ D_2x^* \in Q$, with $D_2 : H_1 \to H_2$ being a bounded linear operator. The set of all solutions to the GSFP is denoted by $\Gamma = \{x \in C : D_1x, D_2x \in Q\}$. If $D_1 \equiv D_2$, the GSFP is reduced to the SFP. Kangtanyakarn presented the following results to solve the GSFP.

Lemma 4.2. [6]. *Let C,Q be closed and convex subset of two real Hilbert spaces H_1 and H_2, respectively. Let $D_1,D_2 : H_1 \to H_2$ be bounded linear operators with D_1^*,D_2^* are adjoint of D_1 and D_2, respectively. Let Γ be a nonempty set. Then, the followings are equivalent:*

i) $x^* \in \Gamma$,

ii) $x^* = P_C\left(I - b\Big(\frac{D_1^*(I-P_Q)D_1}{2} + \frac{D_2^*(I-P_Q)D_2}{2}\Big)\right)x^*$, *for all* $b > 0$,

where L_{D_1},L_{D_2} are spectal redius of $D_1^*D_1$ and $D_2^*D_2$, respectively with $b \in (0,\frac{2}{L})$ and $L = \max\{L_{D_1},L_{D_2}\}$.

Theorem 4.3. *Let C,Q be closed and convex subsets of two real Hilbert spaces H_1 and H_2. Let D_1,D_2 be bounded linear operators from H_1 to H_2 with adjoints D_1^*,D_2^* and $\Gamma \neq \emptyset$. The mappings A_1,A_2,A_3,G and constants $\alpha_1,\alpha_2,\alpha_3,\gamma_1,\gamma_2,\gamma_3,a,\lambda$ are defined as in Theorem 3.1. The mapping T in equation (3.1) is defined as*

$$T = P_C\left(I - \bar{b}\Big(\frac{D_1^*(I-P_Q)D_1}{2} + \frac{D_2^*(I-P_Q)D_2}{2}\Big)\right),$$

where L_{D_1} and L_{D_2} are spectal redius of $D_1^*D_1$ and $D_2^*D_2$, respectively with $\bar{b} \in (0,\frac{2}{L})$ and $L = \max\{L_{D_1},L_{D_2}\}$. If $F(\Gamma)\cap F(G) \neq \emptyset$, then the sequence $\{x_n\}$ define as in (3.1) converges strongly to $x^* \in F(\Gamma)\cap F(G)$.

Proof. The conclusion can be derived from Theorem 4.1 and Lemma 4.2 immediately. □

Example 4.4. Let $\mathbb{R}$ be the set of real number and A_1,A_2,A_3 be the mappings defined from $[0,30]$ to $\mathbb{R}$ by $A_1x = \frac{x-1}{2}$, $A_2x = \frac{x-1}{3}$, and $A_3x = \frac{x-1}{4}$. Let T be a mapping from $[0,30]$ into itself defined by $Tx = \frac{x+3}{4}$ for all $x \in [0,30]$. Let $x_1 \in \mathbb{R}$ and $\{x_n\}$ be generated by (3.1), where $b = 0.5$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = 1$, $\alpha_3 = 1.5$, and $\lambda = \frac{1}{2}$. By the definition A_1,A_2,A_3,and T, $\{1\} \in F(T)\cap F(G)$. Theorem 4.1 implies that sequence $\{x_n\}$ converges strongly to 1.

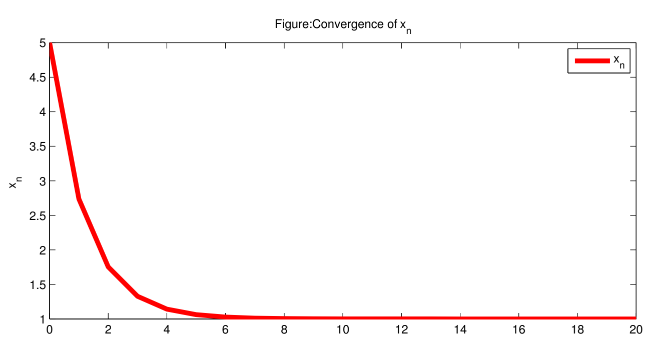

FIGURE 1. The convergence of the sequence $\{x_n\}$.

TABLE 1. The sequence $\{x_n\}$ starting with $x_1 = 5$

n	x_n	n	x_n
1	5	11	1.0009
2	2.7361	12	1.0004
3	1.7535	13	1.005
4	1.327	14	1.0022
5	5	15	1.0009
6	2.7361	16	1.0004
7	1.7535	17	1.0002
8	1.327	18	1.0001
9	1.005	19	1
10	1.0022	20	1

Example 4.5. Let $\mathbb{R}$ be the set of real number,

$$C = \left\{(x_1,x_2) \in \mathbb{R}^2 | x_1u_1 + x_2u_2 = \eta\right\}$$

and

$$P_C(x_1,x_2) = (x_1,x_2) + \frac{\eta - x_1u_1 - x_2u_2}{\sqrt{u_1^2+u_2^2}}(u_1,u_2)$$

for all $(x_1,x_2) \in \mathbb{R}^2$. Let A_1,A_2,A_3 be the mappings on C defined by $A_1(x_1,x_2) = \frac{1}{2}(x_1-2,x_2-3)$, $A_2(x_1,x_2) = \frac{1}{3}(x_1-2,x_2-3)$, and $A_3(x_1,x_2) = \frac{1}{4}(x_1-2,x_2-3)$ for all $(x_1,x_2) \in C$. Let T be a mapping from C into itself defined by $T(x_1,x_2) = \frac{1}{4}(x_1+6,x_2+9)$ for all $(x_1,x_2) \in C$. Let $\{(x_1^n,x_2^n)\}$ be a sequence generated by (3.1) and $(x_1^1,x_2^1) \in \mathbb{R}^2$, where $u = (2,1)$, $\eta = 7$, $b = \frac{1}{2}$, $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{3}{2}$, $\alpha_3 = 2$, and $\lambda = \frac{1}{2}$. By the definition A_1,A_2,A_3, and T, $(2,3) \in F(T)\cap F(G)$. Theorem 4.1 implies that sequence $\{(x_1^n,x_2^n)\}$ converges strongly to $(2,3)$.

TABLE 2. The sequence $\{(x_1^n,x_2^n)\}$ starting with $(x_1^1,x_2^1) = (5,5)$.

n	$(x_1^n$	,	$x_2^n)$
1	(5	,	5)
2	(3.875	,	4.25)
3	(1.3874	,	2.8056)
4	(2.1452	,	3.1127)
5	(2.1452	,	2.9886)
6	(2.0109	,	3.0106)
7	(1.9956	,	2.9996)
8	(2.0008	,	3.0011)
9	(1.9996	,	3)
10	(2.0001	,	3.0001)

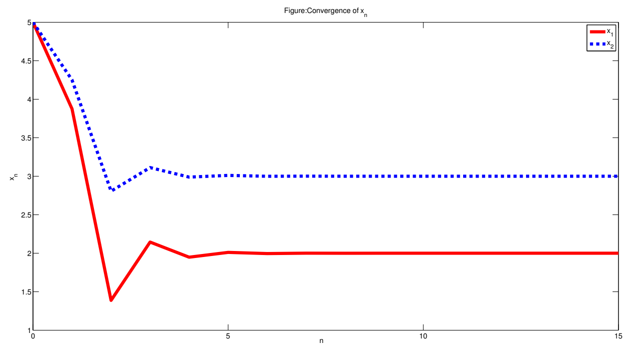

FIGURE 2. The convergence of the sequence $\{(x_1^n, x_2^n)\}$.

5. CONCLUSIONS

In Theorem 3.1, we proposed a new iteration process for solving both fixed point problems and the general systems of variational inequalities. We also proved the strong convergence of this process without assuming the existence of solutions for these problems. By letting $b = 0$ and $A_2 \equiv A_3 \equiv 0$ in Theorem 3.1, we can reduce $F(G) = VI(C, A_1)$. The sequence $\{x_n\}$ defined by modifying (3.1) converges strongly to an element $x^* \in VI(C, A_1) \cap F(T)$ without the assumption of $F(T) \cap VI(C, A_1) \neq \emptyset$. Using Theorem 3.1, we also proved the convergence for the general split feasibility problem. To further support our findings, we provided three numerical examples, Example 3.2, Example 4.4, and Example 4.5.

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