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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A CLASS OF FRACTIONAL HAMILTONIAN SYSTEMS WITH SEPARATED VARIABLES

MOHSEN TIMOUMI

Department of Mathematics, Faculty of Sciences, University of Monastir, Monastir 5000, Tunisia

Abstract. In this paper, we study the existence and multiplicity of solutions of a class of fractional Hamiltonian systems with variable separated type nonlinear terms

$$\begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u)(t) + L(t)u(t) = a(t)\nabla G(u(t)),\ t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

where L satisfies a new condition and the potential G satisfies a superquadratic condition weaker than the well-known Ambrosetti-Rabinowitz condition. Moreover, under a new mixed condition, we establish a compact embedding theorem.

Keywords. Fractional Hamiltonian systems; Multiple solutions; Separated variables; Variational methods.

1. Introduction

In this paper, we are interested in the existence and multiplicity of solutions of a class of fractional Hamiltonian systems of the following form

$$(\mathscr{F}\mathscr{H}\mathscr{S}) \qquad \begin{cases} {}_tD^\alpha_\infty({}_{-\infty}D^\alpha_t u)(t) + L(t)u(t) = \nabla W(t,u(t)),\ t \in \mathbb{R}, \\ u \in H^\alpha(\mathbb{R}), \end{cases}$$

where ${}_{-\infty}D^\alpha_t$ and ${}_tD^\alpha_\infty$ are left and right Liouville-Weyl fractional derivatives of order $\frac{1}{2} < \alpha < 1$ on the whole axis respectively, $L \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a symmetric matrix valued function unnecessary coercive, and $W : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is a continuous function, differentiable in the second variable with continuous derivative $\frac{\partial W}{\partial x}(t,x) = \nabla W(t,x)$.

The existence and multiplicity of solutions of fractional differential equations were established by the tools of nonlinear analysis, such as fixed point theory [1, 16], topological degree theory [7], comparison methods [9], and so on. Over the last four decades, the critical point theory has become a basic tool for studying the existence of solutions of differential and partial differential equations with variational methods; see, e.g., [14, 18] and the references therein.

E-mail address: m_timoumi@yahoo.com

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Inspired by the classical works in [14, 18], for the first time, the authors [8] demonstrated that the critical point theory and variational methods are an effective approach to tackle the existence of solutions for the following fractional boundary value problem

$$\begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u)(t) = \nabla W(t,u(t)),\ t \in [0,T], \\ u(0) = u(T), \end{cases}$$

and obtained the existence of at least one nontrivial solution.

In 2013, Torres [25] used the so-called Ambrosetti-Rabinowitz (($\mathscr{AR}$) in short) condition and the Mountain Pass Theorem to obtain the existence of at least one nontrivial solution for problem ($\mathscr{FHS}$):

($\mathscr{AR}$) There exists a constant $\mu > 2$ such that

$$0 < \mu W(t,x) \leq \nabla W(t,x)\cdot x,\ \forall t \in \mathbb{R},\ \forall x \in \mathbb{R}^N \setminus \{0\}.$$

Here and in the following, "." denotes the standard inner product in $\mathbb{R}^N$ and $|.|$ is the induced norm.

Since then, the existence and multiplicity of solutions of ($\mathscr{FHS}$) were studied extensively; see, e.g., [3, 4, 5, 6, 12, 13, 15, 19, 20, 21, 22, 24, 25, 26, 27, 28, 29, 31, 32] and the references listed therein. As we know, condition ($\mathscr{AR}$) is important to achieve mountain pass geometry structure of the energy functional and demonstrates the boundedness of the Palais-Smale sequence. There are many potentials which are superquadratic as $|x| \longrightarrow \infty$, but do not satisfy the ($\mathscr{AR}$)$-$condition. In recent years, authors have paid much attention to weak this condition. In 2014, Chen [4] considered the following generalized superquadratic condition

(1.1) $\widetilde{W}(t,x) = \frac{1}{2}\nabla W(t,x)\cdot x - W(t,x) \geq 0$ and there exist constants $c_0, r_0 > 0$ and $\sigma > 1$ such that

$$|W(t,x)|^{\sigma} \leq c_0 |x|^{2\sigma}\widetilde{W}(t,x),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ |x| \geq r_0.$$

Instead of (1.1), the authors [4] considered also the superquadratic condition

(1.2) There exist constants $\mu > 2$ and $\rho > 0$ such that

$$\mu W(t,x) \leq \nabla W(t,x)\cdot x + \rho |x|^2,\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

In 2018, the author [23] considered the following generalized superquadratic condition

(1.3) There exist constant $c_0, r_0 > 0$ and $\nu \in [0,2]$ such that

$$|W(t,x)| \leq c_0 |x|^{2-\nu}\widetilde{W}(t,x),\ \forall (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ |x| \geq r_0.$$

Besides, in [21], the author considered the following condition

(1.4) There is $\sigma \geq 1$ such that

$$\widetilde{W}(t,sx) \leq \sigma\widetilde{W}(t,x),\ \forall (s,t,x) \in [0,1]\times\mathbb{R}\times\mathbb{R}^N.$$

Conditions (1.1)-(1.4) can be seen as the generalization or supplements of ($\mathscr{AR}$)$-$condition. In [30], Yuan and Zhang considered the classical Hamiltonian system

$$(\mathscr{HS}) \qquad \ddot{u}(t) - L(t)u(t) + \nabla W(t,u(t)) = 0$$

with variable separated type nonlinear terms $W(t,x) = a(t)G(x)$, where $a \in C(\mathbb{R},\mathbb{R}_+^*)$ and $G \in C^1(\mathbb{R}^N,\mathbb{R})$. They studied the existence and multiplicity of homoclinic solutions of system ($\mathscr{HS}$) under suitable assumptions. In particular, they assumed that a satisfies condition $\lim_{|t|\to\infty} a(t) = 0$, and G satisfies ($\mathscr{AR}$)$-$condition. In recent years, Wu et al. [27] considered the fractional Hamiltonian system ($\mathscr{FHS}$) with variable separated type nonlinear terms

$W(t,x)=a(t)G(x)$, where $a\in C(\mathbb{R},\mathbb{R}_+)$ and $G\in C^1(\mathbb{R}^N,\mathbb{R})$. They studied the existence and multiplicity of solutions of system $(\mathscr{FHS})$ under suitable assumptions among which the following two conditions

$$\frac{a(t)}{l(t)}\longrightarrow 0 \text{ as } |t|\longrightarrow\infty, \text{ where } l(t)=\inf_{|\xi|=1}L(t)\xi\cdot\xi. \tag{1.5}$$

(1.6) There are $\mu>0$ and $d_1,\rho_\infty>0$ satisfying

$$\nabla G(x)\cdot x-\mu G(x)\geq -d_1|x|^2,\ \forall |x|\geq\rho_\infty.$$

Condition (1.6) is a generalization of $(\mathscr{AR})-$condition.

In this paper, inspired by the above results, we focus on the existence and multiplicity of solutions of the fractional Hamiltonian system $(\mathscr{FHS})$ with variable separation nonlinear terms $W(t,x)=a(t)G(x)$, where $a\in C(\mathbb{R},\mathbb{R}_+^*)$, L satisfies a new condition, and $G\in C^1(\mathbb{R}^N,\mathbb{R})$ satisfies some kind of superquadratic conditions weaker than well-known Ambresetti-Rabinowitz condition. Precisely, we consider the following assumptions

(L_1) $L\in C(\mathbb{R},\mathbb{R}^{N^2})$ is a symmetric matrix for all $t\in\mathbb{R}$, and there exists a constant $l_0>0$ such that

$$L(t)x\cdot x\geq l_0|x|^2,\ \forall (t,x)\in\mathbb{R}\times\mathbb{R}^N;$$

(L_2) There is a constant $d>0$ verifying

$$a(t)\leq dl(t),\ \forall t\in\mathbb{R};$$

(L_3) There exits a constant $r_0>0$ satisfying

$$\lim_{|s|\longrightarrow\infty} meas\Big(\Big\{t\in[s-r_0,s+r_0]/\frac{l(t)}{a(t)}<b\Big\}\Big)=0,\ \forall b>0;$$

$$G(0)=0, \text{ and } \nabla G(x)=o(|x|) \text{ as } |x|\longrightarrow 0; \tag{G_1}$$

$$\lim_{|x|\longrightarrow\infty}\frac{G(x)}{|x|^2}=+\infty; \tag{G_2}$$

$$\widetilde{G}(t,x)=\frac{1}{2}\nabla G(x)\cdot x-G(x)\geq 0,\ \forall x\in\mathbb{R}^N; \tag{G_3}$$

$$\lim_{|x|\longrightarrow\infty}\frac{\widetilde{G}(x)}{G(x)}|x|^2=+\infty. \tag{G_4}$$

Our first results are the following:

Theorem 1.1. *If $(L_1)-(L_3)$ and $(G_1)-(G_4)$ hold, then fractional Hamiltonian system $(\mathscr{FHS})$ has at least one nontrivial solution.*

Theorem 1.2. *Assume that $(L_1)-(L_3)$ and $(G_1)-(G_4)$ are satisfied and G is even. Then fractional Hamiltonian system $(\mathscr{FHS})$ possesses infinitely many nontrivial solutions.*

Remark 1.3. Consider the following example

$$G(x) = |x|^s + (s-2)\,|x|^{s-\varepsilon}\, sin^2(\frac{|x|^\varepsilon}{\varepsilon}),$$

where $s > 2$ and $\varepsilon \in]0, s-2[$. It is easy to check that G satisfies conditions $(G_1) - (G_4)$, however G satisfies neither condition $(\mathscr{AR})$ nor its generalization (1.6).

Next, consider the assumptions
(G_5) There are $d_0 > 0$ and $\nu > 2$ satisfying

$$|G(x)| \le d_0(|x|^2 + |x|^\nu),\ \forall x \in \mathbb{R}^N;$$

(G_6) There is a constant $\sigma \ge 1$ satisfying

$$\widetilde{G}(sx) \le \sigma \widetilde{G}(x),\ \forall (s,x) \in [0,1] \times \mathbb{R}^N.$$

Now, we give our second results

Theorem 1.4. *Assume that $(L_1) - (L_3)$, (G_1), (G_2), (G_5), and (G_6) are satisfied. Then fractional Hamiltonian system $(\mathscr{FHS})$ admits at least one nontrivial solution.*

Theorem 1.5. *Assume that $(L_1) - (L_3)$, (G_1), (G_2) (G_5), and (G_6) are satisfied and G is even. Then fractional Hamiltonian system $(\mathscr{FHS})$ possesses infinitely many nontrivial solutions.*

Remark 1.6. Let $G(x) = |x|^2 ln(e+|x|) - \frac{1}{2}|x|^2 + e\,|x| - e^2(ln(e+|x|) - 1)$. It is easy to check that (G_1), (G_2), (G_5), and (G_6) hold. However, G satisfies neither condition $(\mathscr{AR})$ nor its generalization (1.6).

Remark 1.7. Let $G(x) = |x|^2 ln(1+|x|^2)$. Then we easily demonstrate that conditions (G_1), (G_2), (G_5), and (G_6) hold. However, G does not satisfy $(\mathscr{AR})-$condition.

The remaining of this paper is organized as follows. In Section 2, we introduce some preliminary results and prove an interesting compact embedding theorem. Section 3 is reserved to the proof of our main results.

2. Preliminaries

2.1. **Liouville-Weyl fractional calculus.** The Liouville-Weyl fractional integrals of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as (see [10, 11, 17])

$$(2.1)\qquad {}_{-\infty}I_t^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^t (t-x)^{\alpha-1} u(x)dx,$$

and

$$(2.2)\qquad {}_tI_\infty^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty (x-t)^{\alpha-1} u(x)dx.$$

The Liouville-Weyl fractional derivatives of order $0 < \alpha < 1$ on the whole axis $\mathbb{R}$ are defined as the left-inverse operators of the corresponding Liouville-Weyl fractional integrals (see [10, 11, 17])

$$(2.3)\qquad {}_{-\infty}D_t^\alpha u(t) = \frac{d}{dt}({}_{-\infty}I_t^{1-\alpha} u)(t),$$

and

$$ {}_tD_{\infty}^{\alpha}u(t) = -\frac{d}{dt}({}_tI_{\infty}^{1-\alpha}u)(t). \tag{2.4} $$

The definitions of (2.3) and (2.4) may be written in an alternative form as follows

$$ {}_{-\infty}D_t^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(t)-u(t-x)}{x^{\alpha+1}}dx, \tag{2.5} $$

and

$$ {}_tD_{\infty}^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^{\infty}\frac{u(t)-u(t+x)}{x^{\alpha+1}}dx. \tag{2.6} $$

2.2. **Fractional derivative space.** For $\alpha > 0$, define the semi-norm

$$ |u|_{I_{-\infty}^{\alpha}} = \|{}_{-\infty}D_t^{\alpha}u\|_{L^2} $$

and the norm

$$ \|u\|_{I_{-\infty}^{\alpha}} = (\|u\|_{L^2} + |u|_{I_{-\infty}^{\alpha}}^2)^{\frac{1}{2}}, $$

and let

$$ I_{-\infty}^{\alpha} = \overline{C_0^{\infty}(\mathbb{R})}^{\|.\|_{I_{-\infty}^{\alpha}}}, $$

where

$$ C_0^{\infty}(\mathbb{R}) = \left\{ u \in C^{\infty}(\mathbb{R},\mathbb{R}^N)/ \lim_{|t|\longrightarrow\infty} u(t) = 0. \right\}. $$

Now, we can define the fractional Sobolev space $H^{\alpha}(\mathbb{R})$ using the Fourier transform $\widehat{u}(s) = \int_{-\infty}^{\infty} e^{-ist}u(t)dt$. Choose $0 < \alpha < 1$, define the semi-norm

$$ |u|_{\alpha} = \left\| |s|^{\alpha}\,\widehat{u} \right\|_{L^2} $$

and the norm

$$ \|u\|_{\alpha} = (\|u\|_{L^2} + |u|_{\alpha}^2)^{\frac{1}{2}}, $$

and let

$$ H^{\alpha}(\mathbb{R}) = \overline{C_0^{\infty}(\mathbb{R})}^{\|.\|_{\alpha}}. $$

Moreover, we notice that a function $u \in L^2(\mathbb{R})$ belongs to $I_{-\infty}^{\alpha}$ if and only if

$$ |s|^{\alpha}\,\widehat{u} \in L^2(\mathbb{R}). $$

Especially, we have

$$ |u|_{I_{-\infty}^{\alpha}} = \left\| |s|^{\alpha}\,\widehat{u} \right\|_{L^2}. $$

Therefore, $I_{-\infty}^{\alpha}$ and $H^{\alpha}(\mathbb{R})$ are isomorphic with equivalent semi-norms and norms.

Let $C(\mathbb{R})$ denote the space of continuous functions from $\mathbb{R}$ into $\mathbb{R}^N$. The following Sobolev lemma is be useful.

Lemma 2.1. [25, Theorem 2.1] *If* $\alpha > \frac{1}{2}$, *then* $H^{\alpha}(\mathbb{R}) \subset C(\mathbb{R})$, *and there exists a constant* C_{α} *such that*

$$ \|u\|_{L^{\infty}} = \sup_{t\in\mathbb{R}} |u(t)| \le C_{\alpha}\,\|u\|_{\alpha}\,, \forall u \in H^{\alpha}(\mathbb{R}). \tag{2.7} $$

Let

$$X^{\alpha}=\left\{u\in H^{\alpha}(\mathbb{R})/\int_{\mathbb{R}}[|_{-\infty}D_t^{\alpha}u(t)|^2+L(t)u(t)\cdot u(t)]dt<\infty\right\},$$

then X^{α} is a reflexive and separable Hilbert space with the inner product

$$<u,v>_{X^{\alpha}}=\int_{\mathbb{R}}[_{-\infty}D_t^{\alpha}u(t)\cdot_{-\infty}D_t^{\alpha}v(t)+L(t)u(t)\cdot v(t)]dt$$

and the corresponding norm $\|u\|_{X^{\alpha}}^2=<u,u>_{X^{\alpha}}$. Evidently, X^{α} is continuously embedded into $H^{\alpha}(\mathbb{R})$. For $p\in[2,\infty[$, define

$$L_a^p(\mathbb{R})=\left\{u:\mathbb{R}\longrightarrow\mathbb{R}^N\ measurable/\int_{\mathbb{R}}a(t)\,|u(t)|^p<\infty\right\}$$

equipped with norm $\|.\|_{L_a^p}=\left(\int_{\mathbb{R}}a(t)\,|u(t)|^p\right)^{\frac{1}{p}}$. By (L_2), we have for $u\in X^{\alpha}$

$$\begin{aligned}\int_{\mathbb{R}}a(t)\,|u(t)|^2\,dt&\leq d\int_{\mathbb{R}}l(t)\,|u(t)|^2\,dt\leq d\int_{\mathbb{R}}L(t)u(t)\cdot u(t)dt\\&\leq d\int_{\mathbb{R}}[|_{-\infty}D_t^{\alpha}u(t)|^2+L(t)u(t)\cdot u(t)]dt\\&=\eta_2^2\,\|u\|_{X^{\alpha}}^2,\end{aligned}$$

where $\eta_2=\sqrt{d}$. For $p\in]2,\infty[$ and $u\in X^{\alpha}$, we have

$$\int_{\mathbb{R}}a(t)\,|u(t)|^p\,dt\leq\|u\|_{L^{\infty}}^{p-2}\int_{\mathbb{R}}a(t)\,|u(t)|^2\,dt\leq C_{\alpha}^{p-2}\,\|u\|_{X^{\alpha}}^{p-2}\,\|u\|_{L_a^2}^2\leq\eta_p^p\,\|u\|_{X^{\alpha}}^p,$$

where $\eta_p^p=dC_{\alpha}^{p-2}$. Hence, for all $p\in[2,\infty]$, X^{α} is continuously embedded in $L_a^p(\mathbb{R})$ and there exists a constant $\eta_p>0$ such that

$$\|u\|_{L_a^p}\leq\eta_p\,\|u\|_{X^{\alpha}},\ \forall u\in X^{\alpha}. \tag{2.8}$$

Lemma 2.2. *Under conditions $(L_1)-(L_3)$, for all $p\in[2,\infty[$, embedding $X^{\alpha}\hookrightarrow L_a^p(\mathbb{R})$ is compact.*

Proof. Let $\{u_n\}\subset X^{\alpha}$ be a bounded sequence. Then there exists $M_0>0$ such that

$$\|u_n\|_{X^{\alpha}}\leq M_0,\forall n\in\mathbb{N}.$$

Taking a subsequence if necessary, we can assume that $u_n\rightharpoonup u_0$ in X^{α}. Setting $v_n=u_n-u_0$, one has $v_n\rightharpoonup 0$ in X^{α}.

Next, we prove that $v_n\to 0$ in $L_a^2(\mathbb{R})$. Choose $\{s_i\}\subset\mathbb{R}$ such that $\mathbb{R}=\cup_{i=1}^{\infty}I_{r_0}(s_i)$ and each $t\in\mathbb{R}$ is contained by two such intervals $I_{r_0}(s_i)=[s_i-r_0,s_i+r_0]$ at most. For $b,r>0$, set

$$A(b,r)=\left\{t\in I_r^c(0):\frac{l(t)}{a(t)}<b\right\}$$

and

$$B(b,r)=\left\{t\in I_r^c(0):\frac{l(t)}{a(t)}\geq b\right\}.$$

We have

$$\int_{B(b,r)}a(t)\,|v_n(t)|^2\,dt\leq\frac{1}{b}\int_{B(b,r)}l(t)\,|v_n(t)|^2\,dt\leq\frac{1}{b}\,\|v_n\|_{X^{\alpha}}^2\leq\frac{4M_0^2}{b}.$$

Letting $\varepsilon > 0$, one has that there exists a constant $b_\varepsilon > 0$ such that

$$\int_{B(b_\varepsilon,r)} a(t)\,|v_n(t)|^2\,dt < \frac{\varepsilon}{4},\ \forall n \in \mathbb{N},\ \forall r > 0. \tag{2.10}$$

Now, let $\beta_r = \sup_{i\in\mathbb{N}} meas_a(A(b_\varepsilon, r)\bigcap I_{r_0}(s_i)$. By (2.8) and H$\ddot{o}$lder's inequality, we obtain

$$\begin{aligned}
&\int_{A(b_\varepsilon,r)} a(t)\,|v_n(t)|^2\,dt\\
&= \int_{A(b_\varepsilon,r)\cap\cup_{i=1}^{\infty} I_{r_0}(s_i)} a(t)\,|v_n(t)|^2\,dt\\
&\le \sum_{i=1}^{\infty}\int_{A(b_\varepsilon,r)\cap I_{r_0}(s_i)} a(t)\,|v_n(t)|^2\,dt\\
&\le \sum_{i=1}^{\infty}(\int_{A(b_\varepsilon,r)\cap I_{r_0}(s_i)} a(t)dt)^{\frac{1}{2}}(\int_{A(b_\varepsilon,r)\cap I_{r_0}(s_i)} a(t)\,|v_n(t)|^4\,dt)^{\frac{1}{2}}\\
&\le \sum_{i=1}^{\infty}(meas_a(A(b_\varepsilon,r)\bigcap I_{r_0(s_i)}))^{\frac{1}{2}}[(\int_{A(b_\varepsilon,r)\cap I_{r_0}(s_i)} a(t)\,|v_n(t)|^4\,dt)^{\frac{1}{4}}]^2\\
&\le \beta_r^{\frac{1}{2}}\sum_{i=1}^{\infty}[(\int_{A(b_\varepsilon,r)\cap I_{r_0}(s_i)} a(t)\,|v_n(t)|^4\,dt)^{\frac{1}{4}}]^2\\
&\le \beta_r^{\frac{1}{2}}\eta_4^2\sum_{i=1}^{\infty}\int_{A(b_\varepsilon,r)\cap B_{r_0}(s_i)}[|{}_tD_\infty^\alpha v_n(t)|^2 + L(t)v_n(t)\cdot v_n(t)]dt\\
&\le 2\eta_4^2\beta_r^{\frac{1}{2}}\,\|v_n\|_{X^\alpha}^2 \le 8\eta_4^2\beta_r^{\frac{1}{2}}M_0^2.
\end{aligned}$$

By (L_3), there is a constant $r_\varepsilon > 0$ satisfying

$$\int_{A(b_\varepsilon,r_\varepsilon)} a(t)\,|v_n(t)|^2\,dt < \frac{\varepsilon}{4},\ \forall n \in \mathbb{N}. \tag{2.11}$$

Combining (2.10) and (2.11) yields

$$\begin{aligned}
\int_{I^c_{r_\varepsilon}(0)} a(t)\,|v_n(t)|^2\,dt &= \int_{A(b_\varepsilon,r_\varepsilon)} a(t)\,|v_n(t)|^2\,dt + \int_{B(b_\varepsilon,r_\varepsilon)} a(t)\,|v_n(t)|^2\,dt\\
&< \frac{\varepsilon}{2},\ \forall n \in \mathbb{N}.
\end{aligned} \tag{2.12}$$

On the other hand, the Sobelev's compact embedding theorem implies that $v_n \to 0$ in $L_a^2(I_{r_\varepsilon}(0))$. Hence, there exists a constant $n_0 \in \mathbb{N}$ such that

$$\int_{I_{r_\varepsilon}(0)} a(t)\,|v_n(t)|^2\,dt < \frac{\varepsilon}{2},\ \forall n \ge n_0, \tag{2.13}$$

which with (2.12) implies that $v_n \longrightarrow 0$ in $L_a^2(\mathbb{R})$. For $p \in]2,\infty[$, we have

$$\begin{aligned}
\int_{\mathbb{R}} a(t)\,|v_n(t)|^p\,dt &\le \|v_n\|_{L^\infty}^{p-2}\int_{\mathbb{R}} a(t)\,|v_n(t)|^2\,dt\\
&\le C_\alpha^{p-2}\,\|v_n\|_{X^\alpha}^{p-2}\,\|v_n\|_{L_a^2}^2\\
&\le C_\alpha^{p-2}(2M_0)^{p-2}\,\|v_n\|_{L_a^2}^2.
\end{aligned}$$

Hence $v_n \longrightarrow 0$ as $n \longrightarrow \infty$, and embedding $X^\alpha \hookrightarrow L_a^p(\mathbb{R})$ is compact. □

The following critical point lemmas is needed in the proof of our results.

Definition 2.3. Let X be a Banach space with norm $\|.\|$. We say that $f \in C^1(X,\mathbb{R})$ satisfies

a) $(PS)-$condition if any sequence $(u_n) \subset X$ satisfying

$$(f(u_n)) \text{ is bounded and } f'(u_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

has a convergent subsequence,

b) $(C)-$condition if any sequence $(u_n) \subset X$ satisfying

$$(f(u_n)) \text{ is bounded and } \left\|f'(u_n)\right\|(1+\|u_n\|) \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

possesses a convergent subsequence.

Lemma 2.4. [18] *Let X be a real Banach space, and let $f \in C^1(X,\mathbb{R})$ satisfy (PS)-condition. Suppose that $f(0) = 0$ and*

(i) there are $\rho, \alpha > 0$, such that $f_{|\partial B_\rho} \geq \alpha$, where $B_\rho = \{u \in X / \|u\| < \rho\}$,

(ii) there is an $e \in X \setminus \overline{B}_\rho$ such that $f(e) < 0$.

Then f has a critical value $c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 2.5. [18] *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional space, and let $f \in C^1(X,\mathbb{R})$ satisfy $(PS)-$condition. Assume that $f(0) = 0$, f is even, and*

(a) There exist constants $\rho, \alpha > 0$ such that $f_{|\partial B_\rho \cap Z} \geq \alpha$;

(b) For any finite dimensional subspace $\widetilde{X} \subset X$, there is $r = r(\widetilde{X}) > 0$ such that $f(u) \leq 0$ on $\widetilde{X} \setminus B_r$.

Then f possesses an unbounded sequence of critical values.

Remark 2.6. As in [2], a deformation lemma can be proved with $(C)-$condition replacing $(PS)-$condition, and it turns out that Lemmas 2.4 and 2.5 still hold true with $(C)-$condition instead of $(PS)-$condition.

3. Proof of Theorem 1.1 and Theorem 1.2

Define the energy functional f associated to the fractional Hamiltonian system $(\mathscr{FHS})$

$$f(u) = \frac{1}{2}\int_{\mathbb{R}}[|_{-\infty}D_t^\alpha u(t)|^2 + L(t)u(t)\cdot u(t)]dt - \int_{\mathbb{R}} W(t,u)dt,\ u \in X^\alpha$$

defined on the space X^α introduced in Section 2. It is known that, under assumption (G_1), $f \in C^1(X^\alpha,\mathbb{R})$ and, for all $u, v \in X^\alpha$,

$$\begin{aligned} f'(u)v &= \int_{\mathbb{R}}[_{-\infty}D_t^\alpha u(t)\cdot {}_{-\infty}D_t^\alpha v(t) + L(t)u(t)\cdot v(t)]dt - \int_{\mathbb{R}} \nabla W(t,u)\cdot v(t)dt \\ &= \frac{1}{2} < u,v >_{X^\alpha} - \int_{\mathbb{R}} \alpha(t)\nabla G(u(t))\cdot v(t)dt. \end{aligned}$$

Moreover, the critical points of f on X^α are solutions to $(\mathscr{F}\mathscr{H}\mathscr{S})$. We shall prove that problem $(\mathscr{F}\mathscr{H}\mathscr{S})$ has mountain pass type solutions. For this purpose, we apply Lemmas 2.4 and 2.5 to functional f on X^α. We claim that, under (G_1) and (G_3),

$$G(x) \geq 0, \ \forall x \in \mathbb{R}. \tag{3.1}$$

In fact, for $x \in \mathbb{R}^N \setminus \{0\}$, set $\varphi(s) = \frac{G(sx)}{s^2}$ for $s > 0$. By (G_3), we have, for $s \in]0, \infty[$,

$$\varphi'(s) = \frac{2}{s^3}[\frac{1}{2}\nabla G(sx) \cdot sx - G(sx)] \geq 0,$$

which means that φ is non-decreasing in $]0, \infty[$. Now, we have by (G_1) and the Hopital's property

$$\lim_{s \longrightarrow 0} |\varphi(s)| = \lim_{s \longrightarrow 0} \frac{|G(sx)|}{s^2} = \lim_{s \longrightarrow 0} \frac{|\nabla G(sx) \cdot x|}{2s} \leq \lim_{s \longrightarrow 0} \frac{1}{2} \frac{|\nabla G(sx)|}{|sx|} |x|^2 = 0.$$

Hence, we have $\varphi(s) \geq 0$ for all $s \in]0, \infty[$. In particular $\varphi(1) \geq 0$, which is (3.1).

Lemma 3.1. *Under conditions $(L_1) - (L_3)$ and (G_1), there are constants $\rho, \nu > 0$ satisfying $f_{|\partial B_\rho(0)} \geq \nu$.*

Proof. By (G_1), there is $r > 0$ verifying

$$|G(x)| \leq \frac{1}{4d} |x|^2, \ \forall |x| \leq r. \tag{3.2}$$

Set $\rho = \frac{r}{\eta_\infty}$ and $\nu = \frac{\rho^2}{4}$. By (3.2) and (L_2), we have, for $\|u\|_{X^\alpha} = \rho$,

$$\begin{aligned} f(u) &= \frac{1}{2} \|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} a(t) G(u(t)) dt \\ &\geq \frac{1}{2} \|u\|_{X^\alpha}^2 - \frac{1}{4d} \int_{\mathbb{R}} a(t) |u(t)|^2 dt \\ &\geq \frac{1}{2} \|u\|_{X^\alpha}^2 - \frac{1}{4} \int_{\mathbb{R}} l(t) |u(t)|^2 dt \\ &\geq \frac{1}{4} \|u\|_{X^\alpha}^2 = \nu. \end{aligned} \tag{3.3}$$

The proof of Lemma 3.1 is completed. □

Lemma 3.2. *Assume that $(L_1) - (L_3)$ and (G_1) are satisfied. Then $\nabla G(u_n) \longrightarrow \nabla G(u)$ in $L^2_a(\mathbb{R})$ if $u_n \rightharpoonup u$ in X^α.*

Proof. Let $u_n \rightharpoonup u$ in X^α. Then there exists $K > 0$ such that

$$\sup_{n \in \mathbb{N}} \|u_n\|_{X^\alpha} \leq K \ \textit{and} \ \|u_n\|_{L^\infty} \leq K, \ \forall n \in \mathbb{N}. \tag{3.4}$$

We claim that $\nabla G(u_n) \longrightarrow \nabla G(u)$ in $L^2_a(\mathbb{R})$. Otherwise, by Lemma 2.2, there is a subsequence (u_{n_k}) satisfying

$$u_{n_k} \longrightarrow u \ \textit{in} \ L^2_a(\mathbb{R}) \ \textit{and} \ u_{n_k}(t) \longrightarrow u(t) \ \textit{a.e.} \tag{3.5}$$

and

$$\int_{\mathbb{R}} a(t) |\nabla G(u_{n_k}(t)) - \nabla G(u(t))| dt \geq \varepsilon_0, \ \forall k \in \mathbb{N} \tag{3.6}$$

for some $\varepsilon_0 > 0$. By (3.5) and going to a subsequence if necessary, we can assume that $\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{L_a^2} < \infty$. Let $v(t) = \sum_{k=1}^{\infty} |u_{n_k}(t) - u(t)|$ for all $t \in \mathbb{R}$. Then $v \in L_a^2(\mathbb{R})$. From (G_1) and (3.4), we can find a constant $K_1 > 0$ such that

$$|\nabla G(u_{n_k}(t))| \leq K_1 |u_{n_k}(t)| \ and \ |\nabla G(u(t))| \leq K_1 |u(t)|, \ \forall k \in \mathbb{N}, \tag{3.7}$$

which implies

$$\begin{aligned} |\nabla G(u_{n_k}(t)) - \nabla G(u(t))|^2 &\leq K_1^2 [|u_n(t)| + |u(t)|]^2 \\ &\leq K_1^2 [|u_n(t) - u(t)| + 2|u(t)|]^2 \\ &\leq 8K_1^2 [|v(t)| + |u(t)|]^2 = h(t). \end{aligned}$$

Since $h \in L_a^2(\mathbb{R})$, then the Lebesgue's dominated convergence theorem implies

$$\lim_{k \longrightarrow \infty} \int_{\mathbb{R}} a(t) |\nabla G(u_{n_k}(t)) - \nabla G(u(t))|^2 dt = \int_{\mathbb{R}} a(t) \lim_{k \longrightarrow \infty} |\nabla G(u_{n_k}(t)) - \nabla G(u(t))|^2 dt = 0,$$

which contradict (3.6). Hence the claim above is true and the proof of Lemma 3.2 is completed. □

Lemma 3.3. *Under assumptions $(L_1) - (L_3)$, (G_1), (G_2), and (G_4), f verifies the (C)-condition.*

Proof. Let $\{u_n\} \subset X^\alpha$ be a (C)-sequence of f, that is, $(f(u_n))$ is bounded and $\|f'(u_n)\|(1 + \|u_n\|) \longrightarrow 0$ as $n \longrightarrow \infty$,. Then there exists a constant $c_1 > 0$ such that

$$|f(u_n)| \leq c_1 \ and \ \|f'(u_n)\| (1 + \|u_n\|_{X^\alpha}) \leq c_1, \forall n \in \mathbb{N}. \tag{3.8}$$

We claim that (u_n) is bounded. Otherwise, we assume that $\|u_n\|_{X^\alpha} \longrightarrow \infty$ as $n \longrightarrow \infty$. Setting $v_n = \frac{u_n}{\|u_n\|_{X^\alpha}}$, one has $\|v_n\|_{X^\alpha} = 1$, which implies that there is a subsequence of (v_n), still denoted by (v_n), such that $v_n \rightharpoonup v_0$ in X^α. We have

$$\left| \int_{\mathbb{R}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt - \frac{1}{2} \right| = \frac{|-f(u_n)|}{\|u_n\|^2} \leq \frac{c_1}{\|u_n\|^2},$$

which implies that

$$\int_{\mathbb{R}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt \longrightarrow \frac{1}{2} \ as \ n \longrightarrow \infty. \tag{3.9}$$

We will discuss two cases.

Case 1: $v_0 \neq 0$.

Let $\Lambda = \{t \in \mathbb{R}/v_0(t) \neq 0\}$. Then we can see that $meas_a(\Lambda) > 0$. So there exists a constant $R > 0$ such that $meas_a(\Omega) > 0$, where $\Omega = \Lambda \cap B_R(0)$. Since $\|u_n\|_{X^\alpha} \longrightarrow \infty$ as $n \longrightarrow \infty$, we have $|u_n(t)| = |v_n(t)| \|u_n\|_{X^\alpha} \longrightarrow +\infty$ as $n \longrightarrow \infty$ for a.e $t \in \Omega$. By (G_2), (3.1), and Fatou's lemma, we have

$$\begin{aligned} \liminf_{n \longrightarrow \infty} \int_{\mathbb{R}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt &\geq \liminf_{n \longrightarrow \infty} \int_{\Omega} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt \\ &\geq \int_{\Omega} \liminf_{n \longrightarrow \infty} \frac{a(t)G(u_n(t))}{|u_n(t)|^2} |v_n(t)|^2 dt \\ &= +\infty, \end{aligned}$$

which can contradicts (3.9). Hence (u_n) is bounded.

Case 2: $v_0 = 0$.

By (G_1), there exists a constant $r > 0$ such that

$$0 \leq G(x) \leq |x|^2, \ \forall |x| \leq r. \tag{3.10}$$

By (G_4), for any $M > 0$, there exists $R > r$ such that

$$\frac{\widetilde{G}(x)}{G(x)} |x|^2 \geq M, \ \forall |x| \geq R. \tag{3.11}$$

Combining (3.10) and (3.11) yields

$$\begin{aligned}
0 &\leq \int_{\mathbb{R}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt \leq \int_{\{t\in\mathbb{R}/|u_n|\leq r\}} \frac{a(t)G(u_n(t))}{|u_n(t)|^2} |v_n(t)|^2 dt \\
&+ \int_{\{t\in\mathbb{R}/r<|u_n|\leq R\}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt + \int_{\{t\in\mathbb{R}/|u_n|\geq R\}} \frac{a(t)G(u_n(t))}{|u_n(t)|^2} |v_n(t)|^2 dt \\
&\leq \int_{\{t\in\mathbb{R}/|u_n|\leq r\}} a(t) |v_n(t)|^2 dt + \int_{\{t\in\mathbb{R}/r<|u_n|\leq R\}} \frac{a(t)G(u_n(t))}{r^2 \|u_n\|^2} |u_n(t)|^2 dt \\
&+ \|v_n\|_{L^\infty} \int_{\{t\in\mathbb{R}/|u_n|\geq R\}} \frac{a(t)G(u_n(t))}{|u_n(t)|^2} dt \\
&\leq \|v_n\|^2_{L^2_a} + \frac{1}{r^2} \max_{|x|\leq r} G(x) \int_{\{t\in\mathbb{R}/r<|u_n|\leq R\}} a(t) |v_n(t)|^2 dt \\
&+ \frac{1}{M} \|v_n\|_{L^\infty} \int_{\{t\in\mathbb{R}/|u_n|\geq R\}} a(t) [\frac{1}{2} \nabla G(u_n(t)) \cdot u_n(t) - G(u_n(t))] dt \\
&\leq (1 + \frac{1}{r^2} \max_{|x|\leq r} G(x)) \|v_n\|^2_{L^2_a} + \frac{1}{M} \|v_n\|_{L^\infty} [f(u_n) - \frac{1}{2} f'(u_n) u_n] \\
&\leq (1 + \frac{1}{r^2} \max_{|x|\leq r} G(x)) \|v_n\|^2_{L^2_a} + \frac{3c_1}{2M} \|v_n\|_{L^\infty}.
\end{aligned}$$

By arbitrariness of M and Lemma 2.2, we obtain

$$\int_{\mathbb{R}} \frac{a(t)G(u_n(t))}{\|u_n\|^2} dt < \frac{1}{3}$$

for n large enough, which contradicts (3.9). Hence (u_n) is bounded in X^α. Up to a subsequence if necessary, we can assume that $u_n \rightharpoonup u$ in X^α, which yields

$$(f'(u_n) - f'(u))(u_n - u) \longrightarrow 0 \ as \ n \longrightarrow \infty,$$

and it follows from $H\ddot{o}$lder's inequality and Lemma 3.2 that

$$\begin{aligned}
&\left| \int_{\mathbb{R}} a(t) (\nabla G(u_n(t)) - \nabla G(u(t))) \cdot (u_n(t) - u(t)) dt \right| \\
&\leq \|\nabla G(u_n) - \nabla G(u)\|_{L^2_a} \|u_n - u\|_{L^2_a} \longrightarrow 0 \ as \ n \to \infty.
\end{aligned}$$

Hence, we deduce that

$$\|u_n - u\|^2_{X^\alpha} = (f'(u_n) - f'(u))(u_n - u) + \int_{\mathbb{R}} a(t) (\nabla G(u_n(t)) - \nabla G(u(t))) \cdot (u_n(t) - u(t)) dt \longrightarrow 0$$

as $n \longrightarrow \infty$. The proof of Lemma 3.3 is completed. □

Proof of Theorem 1.1.

Lemma 3.4. *Suppose that $(L_1)-(L_3)$ and (G_2) hold. Then there exists $e \in X^\alpha$ such that $\|e\|_{X^\alpha} > \rho$ and $f(e) \leq 0$, where ρ is defined in Lemma 3.1.*

Proof. Set $e_0 \in C_0^\infty(]-1,1[)$ with $\|e_0\|_{X^\alpha} = 1$. For $M > (2\int_{-1}^{1} a(t)\,|e_0(t)|^2\,dt)^{-1}$, it follows from (G_2) that there exists a constant $R > 0$ such that

$$G(x) \geq M\,|x|^2\,,\ \forall\,|x| \geq R. \tag{3.12}$$

Let $D = \frac{1}{M}max_{|x|\leq R}G(x)$. Then (3.12) implies

$$G(x) \geq M(|x|^2 - D),\ \forall\,|x| \geq R. \tag{3.13}$$

By (3.13), for every $\xi \in \mathbb{R}$, we have

$$\begin{aligned} f(\xi e_0) &= \frac{\xi^2}{2}\,\|e_0\|_{X^\alpha}^2 - \int_{-1}^{1} a(t)G(\xi e_0(t))dt \\ &\leq \frac{\xi^2}{2} - \int_{-1}^{1} a(t)M(\xi^2\,|e_0(t)|^2 - D)dt \\ &\leq \frac{\xi^2}{2} - M\xi^2\int_{-1}^{1} a(t)\,|e_0(t)|^2\,dt + MD\int_{-1}^{1} a(t)dt \\ &\leq \frac{\xi^2}{2}(1 - 2M\int_{-1}^{1} a(t)\,|e_0(t)|^2\,dt) + MD\int_{-1}^{1} a(t)dt, \end{aligned}$$

which implies that

$$f(\xi e_0) \longrightarrow -\infty \ as\ |\xi| \longrightarrow +\infty.$$

Hence there exists $\xi_0 \in \mathbb{R}$ such that $\|\xi_0 e_0\|_{X^\alpha} > \rho$ and $f(\xi_0 e_0) < 0$. Letting $e(t) = \xi_0(t)e_0(t)$, we finish the proof of Lemma 3.4. □

By Lemmas 2.4, 3.1-3.4, and the fact $f(0) = 0$, we see that f possesses at least one nontrivial critical point u satisfying $f(u) \geq \alpha$. Since $f(0) = 0 < \alpha$, then u is a nontrivial solution of $(\mathscr{F}\mathscr{H}\mathscr{S})$.

Proof of Theorem 1.2.

Lemma 3.5. *Assume that $(L_1)-(L_3)$, (G_1), and (G_4) are satisfied. Then, for each finite-dimensional subspace $\widetilde{X} \subset X^\alpha$, there exists a constant $r = r(\widetilde{X}) > 0$ such that $f \leq 0$ on $\widetilde{X} \setminus B_r(0)$.*

Proof. Let $\widetilde{X} \subset X^\alpha$ be a finite-dimensional subspace. We claim that there is a constant $\varepsilon_0 > 0$ such that

$$meas_a(\{t \in \mathbb{R} / \,|u(t)| \geq \varepsilon_0\,\|u\|_{X^\alpha}\}) < \varepsilon_0,\ \forall u \in \widetilde{X} \setminus \{0\}\,. \tag{3.14}$$

If not, for any $n \in \mathbb{N}$, there is $u_n \in \widetilde{X} \setminus \{0\}$ such that

$$meas_a\Big(\Big\{t \in \mathbb{R} / \,|u_n(t)| \geq \frac{1}{n}\,\|u_n\|_{X^\alpha}\Big\}\Big) < \frac{1}{n}.$$

Let $v_n = \frac{u_n}{\|u_n\|}$. Then $\|v_n\|_{X^\alpha} = 1$ and

$$meas_a\Big(\Big\{t \in \mathbb{R} / \,|v_n(t)| \geq \frac{1}{n}\Big\}\Big) \leq \frac{1}{n},\ \forall n \in \mathbb{N}. \tag{3.15}$$

Since $\widetilde{X}$ is finite-dimensional, then taking a subsequence if necessary, we may assume that $v_n \rightharpoonup v_0$ in $\widetilde{X}$ for some $v_0 \in \widetilde{X}$. Clearly $\|v_0\|_{X^\alpha} = 1$. Note that, up to a subsequence, Lemma 2.2 implies that

$$\int_{\mathbb{R}} a(t)\,|v_n - v_0|^2 \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{3.16}$$

We claim that there is a constant $\delta_0 > 0$ satisfying

$$meas_a(\{t \in \mathbb{R}, |v_0(t)| \geq \delta_0\}) \geq \delta_0. \tag{3.17}$$

If not, for each fixed $n \in \mathbb{N}$ and $m > n$, we have

$$meas_a\Big(\Big\{t \in \mathbb{R}, |v_0(t)| \geq \frac{1}{n}\Big\}\Big) \leq meas_a\Big(\Big\{t \in \mathbb{R}, |v_0(t)| \geq \frac{1}{m}\Big\}\Big) \leq \frac{1}{m}.$$

Letting $m \longrightarrow \infty$, we have $meas_a(\{t \in \mathbb{R}, |v_0(t)| \geq \frac{1}{n}\}) = 0$. Consequently,

$$\begin{aligned} meas_a(\{t \in \mathbb{R}/v_0(t) \neq 0\}) &= meas_a\Big(\bigcup_{n=1}^{\infty}\Big\{t \in \mathbb{R}, |v_0(t)| \geq \frac{1}{n}\Big\}\Big) \\ &\leq \sum_{n=1}^{\infty} meas_a\Big(\Big\{t \in \mathbb{R}, |v_0(t)| \geq \frac{1}{n}\Big\}\Big) = 0 \end{aligned}$$

which implies that $v_0 = 0$ and contradicts $\|v_0\|_{X^\alpha} = 1$. Then (3.17) holds. For any $n \in \mathbb{N}$, let

$$\Lambda_0 = \{t \in \mathbb{R}/\,|v_0(t)| \geq \delta_0\},\ \Lambda_n = \Big\{t \in \mathbb{R}/\,|v_n(t)| < \frac{1}{n}\Big\}.$$

Then, for n large enough, we have from (3.15) and (3.17) that

$$meas_a(\Lambda_0 \cap \Lambda_n) \geq meas_a(\Lambda_0) - meas_a(\Lambda_n^c) \geq \delta_0 - \frac{1}{n} \geq \frac{\delta_0}{2}.$$

Therefore, for n large enough, one obtains

$$\begin{aligned} \int_{\mathbb{R}} a(t)\,|v_n - v_0|^2\,dt &\geq \int_{\Lambda_0 \cap \Lambda_n} a(t)\,|v_n - v_0|^2\,dt \\ &\geq \frac{1}{2}\int_{\Lambda_0 \cap \Lambda_n} a(t)\,|v_0|^2\,dt - \int_{\Lambda_0 \cap \Lambda_n} a(t)\,|v_n|^2\,dt \\ &\geq (\frac{\delta_0^2}{2} - \frac{1}{n^2}) meas_a(\Lambda_0 \cap \Lambda_n) \\ &\geq (\frac{\delta_0^2}{2} - \frac{1}{n^2})\frac{\delta_0}{2} \geq \frac{\delta_0^3}{8}, \end{aligned}$$

which contradicts (3.16). Hence (3.14) holds. For $u \in \widetilde{X} \setminus \{0\}$, set

$$\Lambda_{\varepsilon_0}(u) = \{t \in \mathbb{R}/\,|u(t)| \geq \varepsilon_0\,\|u\|_{X^\alpha}\}.$$

Since $meas_a(\Lambda_{\varepsilon_0}(u)) \geq \varepsilon_0$, $\forall u \in \widetilde{X} \setminus \{0\}$, there exists $\rho > 0$ satisfying

$$meas_a(\Lambda_{\varepsilon_0}(u) \bigcap B_\rho(0)) \geq \frac{\varepsilon_0}{2},\ \forall u \in \widetilde{X} \setminus \{0\}. \tag{3.18}$$

By (G_2), there exists $R > 0$ such that

$$G(u(t)) \geq \frac{2}{\varepsilon_0^3}\,|u(t)|^2 \geq \frac{2}{\varepsilon_0}\,\|u\|^2 \tag{3.19}$$

for all $u \in \widetilde{X} \setminus \{0\}$ and $t \in \Omega_{\varepsilon_0}(u) = \Lambda_{\varepsilon_0}(u) \bigcap B_\rho(0)$ with $\|u\|_{X^\alpha} \geq R$. Then, for any $u \in \widetilde{X} \setminus B_R(0)$, it follows from (3.1), (3.18), and (3.19) that

$$\begin{aligned}
f(u) &= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\mathbb{R}} a(t)G(u(t))dt \\
&= \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\Omega_{\varepsilon_0}(u)} a(t)G(u(t))dt - \int_{\mathbb{R}\setminus\Omega_{\varepsilon_0}(u)} a(t)G(u(t))dt \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \int_{\Omega_{\varepsilon_0}(u)} a(t)G(u(t))dt \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \frac{2}{\varepsilon_0^3}\int_{\Omega_{\varepsilon_0}(u)} a(t)|u(t)|^2 dt \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \frac{2}{\varepsilon_0}\int_{\Omega_{\varepsilon_0}(u)} a(t)\|u\|_{X^\alpha}^2 dt \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \frac{2}{\varepsilon_0} meas_a(\Omega_{\varepsilon_0}(u))\|u\|_{X^\alpha}^2 \\
&\leq \frac{1}{2}\|u\|_{X^\alpha}^2 - \|u\|_{X^\alpha}^2 = -\frac{1}{2}\|u\|_{X^\alpha}^2.
\end{aligned}$$

Thus there exists $r > R$ such that $f_{|\tilde{X}\setminus B_r(0)} \leq 0$. □

The functional f is even and $f(0) = 0$, so Lemmas 3.1, 3.3, and 3.5 imply that f satisfies all the conditions of Lemma 2.5. Consequently, f possesses an unbounded sequence of critical values which proves Theorem 1.2.

4. Proof of Theorem 1.3 and Theorem 1.4

Lemma 4.1. *Assume that $(L_1)-(L_3)$, (G_1), (G_2), (G_5), and (G_6) are satisfied. Then f verifies $(C)-$sequence.*

Proof. Let $(u_n) \subset X^\alpha$ be a $(C)-$sequence. Then there is $c_1 > 0$ satisfying

$$|f(u_n)| \leq c_1 \text{ and } \left\|f'(u_n)\right\|(1 + \|u_n\|_{X^\alpha}) \leq c_1,\ \forall n \in \mathbb{N}. \tag{4.1}$$

We claim that (u_n) is bounded. Assume indirectly that (u_n) is unbounded. Taking a subsequence if necessary, we may assume that

$$\|u_n\|_{X^\alpha} \longrightarrow +\infty \text{ and } v_n = \frac{u_n}{\|u_n\|_{X^\alpha}} \rightharpoonup v_0 \text{ as } n \longrightarrow \infty. \tag{4.2}$$

By Lemma 2.2 and (4.2), without loss of generality, we have

$$v_n \longrightarrow v_0 \text{ both in } L_a^2(\mathbb{R}) \text{ and } L_a^\nu(\mathbb{R}) \text{ and } v_n(t) \longrightarrow v_0(t) \text{ a.e. } t \in \mathbb{R} \tag{4.3}$$

as $n \longrightarrow \infty$.

Case 1. $v_0 \neq 0$ occurs. The proof is similar to the case 1 in the proof of Lemma 3.3.

Case 2. $v_0 = 0$ occurs. Let $(s_n) \subset [0,1]$ be a sequence such that

$$f(s_n u_n) = \max_{s\in[0,1]} f(s u_n).$$

By (G_5) and (4.3), we obtain

$$\begin{aligned}&\left|\int_{\mathbb{R}} a(t)G(4\sqrt{\sigma c_1}v_n(t))dt\right|\\&\leq d_0\Big[16\sigma c_1\int_{\mathbb{R}} a(t)\,|v_n(t)|^2\,dt+(4\sqrt{\sigma c_1})^{\nu}\int_{\mathbb{R}} a(t)\,|v_n(t)|^{\nu}\,dt\Big]\longrightarrow 0 \; as\; n\longrightarrow\infty,\end{aligned}$$

which implies

$$\int_{\mathbb{R}} a(t)G(4\sqrt{\sigma c_1}v_n(t))dt\longrightarrow 0\; as\; n\longrightarrow\infty. \tag{4.4}$$

By the definition of s_n and (4.4), for n large enough, we have

$$\begin{aligned}f(s_nu_n)&\geq f(\frac{4\sqrt{\sigma c_1}}{\|u_n\|^2}u_n)=f\,(4\sqrt{\sigma c_1}v_n)\\&=8\sigma c_1-\int_{\mathbb{R}} a(t)G(4\sqrt{\sigma c_1}v_n(t))dt\geq 4\sigma c_1.\end{aligned} \tag{4.5}$$

Since $f(0)=0$ and $|f(u_n)|\leq c_1$, then $s_n\in]0,1[$. Hence, one has

$$\|s_nu_n\|^2-\int_{\mathbb{R}} a(t)\nabla G(s_nu_n)\cdot s_nu_ndt=f'(s_nu_n)s_nu_n=s_n\frac{d}{ds}(f(su_n)_{|s=s_n}=0. \tag{4.6}$$

It follows from (4.6) and (G_6) that

$$\begin{aligned}\int_{\mathbb{R}} a(t)[\frac{1}{2}\nabla G(u_n)\cdot u_n-G(u_n)]dt&=\int_{\mathbb{R}} a(t)\widetilde{G}(u_n(t))dt\\&\geq\frac{1}{\sigma}\int_{\mathbb{R}} a(t)\widetilde{G}(s_nu_n)dt\\&=\frac{1}{\sigma}\int_{\mathbb{R}} a(t)[\frac{1}{2}\nabla G(s_nu_n)\cdot s_nu_n-G(s_nu_n)]dt\\&=\frac{1}{\sigma}[\frac{1}{2}\,\|s_nu_n\|^2_{X^\alpha}-\int_{\mathbb{R}} a(t)G(s_nu_n)dt\\&=\frac{1}{\sigma}f(s_nu_n),\end{aligned}$$

which together with (4.5) implies that

$$\int_{\mathbb{R}} a(t)[\frac{1}{2}\nabla G(u_n)u_n-G(u_n)]dt\geq 4c_1, \tag{4.7}$$

for n large enough. However, we can deduce from (4.1) that

$$\left|\int_{\mathbb{R}} a(t)[\frac{1}{2}\nabla G(u_n)u_n-G(u_n)]dt\right|=\frac{1}{2}\left|2f(u_n)-f'(u_n)u_n\right|\leq\frac{3}{2}c_1,$$

for all $n\in\mathbb{N}$, which contradicts (4.7). Hence (u_n) is bounded in X^α. Similar to the proof of Lemma 3.3, we can prove that f satisfies $(C)-$condition. The proof of Lemma 4.1 is completed. □

Proof of Theorem 1.3. The condition $f(0)=0$ and Lemmas 3.1, 3.4, and 4.1 imply that functional f verifies all the conditions of Lemma 2.4. Therefore, Lemma 2.4 implies that f possesses a critical point u satisfying $f(u)\geq\alpha>0$. Hence problem $(\mathscr{FHS})$ possesses a nontrivial solution.

Proof of Theorem 1.4. Since f is even, then condition $f(0) = 0$ and Lemmas 3.1, 3.5, and 4.1 imply that functional f verifies all the conditions of Lemma 2.5. Therefore, Lemma 2.5 implies that f has an unbounded sequence of critical values. Hence problem $(\mathscr{F}\mathscr{H}\mathscr{S})$ has infinitely many nontrivial solutions.

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