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ON THE RELAXED PROJECTION METHOD FOR THE SFPMOS

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Abstract. This paper presents our investigation into the split feasibility problem with multiple output sets (SFPMOS), under the assumption that the corresponding convex subsets are level subsets of convex functionals. To approximate the solutions of this problem, we propose a method that combines relaxed projection with a recently proposed technique. Our approach involves establishing a weak convergence theorem for the fixed stepsize, followed by constructing a variable stepsize that is independent of the norm of the linear operator involved. We also modify these methods to ensure strong convergence. As an application, we develop a new relaxed projection algorithm for solving the split feasibility problem.

Keywords. Multiple output sets; Relaxed projection; Split Feasibility problem; Variable stepsize.

1. Introduction

Since its inception in 1994, the split feasibility problem (SFP) [5] has been under the spotlight due to its real applications in signal processing and image reconstruction [2]. In particular, it has significant progress in intensity-modulated radiation therapy [4]. The SFP involves finding a vector $\hat{x}$ that satisfies two related conditions: it belongs to a nonempty, closed, and convex subset C of a Hilbert space H_0, and its image under a linear bounded operator A belongs to another nonempty, closed, and convex subset Q of a Hilbert space H_1. While the original formulation of the SFP was in finite-dimensional Euclidean spaces, more recent studies focused on the problem in infinite-dimensional Hilbert spaces [10, 16, 22, 23], and even in the setting of Banach spaces [8, 15]. In this paper, we address the SFP in infinite-dimensional Hilbert spaces for the sake of generality.

Assuming that the SFP is consistent, meaning that its solution set is nonempty, iterative methods can be used to solve the problem. One of the most powerful and efficient tools for solving the SFP is the CQ algorithm, which was first proposed by Byrne [3] and has been further studied by numerous researchers; see, e.g., [7, 11, 12, 13] The CQ algorithm generates

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a sequence $\{x_n\}$ recursively in the following formula:

$$x_{n+1} = P_C(x_n - \tau A^*(I - P_Q)Ax_n),$$

where P_C and P_Q are the metric projections onto sets C and Q, and τ is the stepsize chosen in a specific interval. For recent research on the CQ algorithm and its variants, one refer to [26, 27, 17, 18, 24].

However, the CQ algorithm requires that the convex sets C and Q are simple in the sense of projection, which means that their projections onto the respective convex subsets are easy to calculate. In practice, there are numerous sets that do not satisfy this condition, such as level sets of convex functionals. We now focus on the case where C and Q are level sets of convex functionals, which are defined as

$$C = \{x \in H_0 : c(x) \leq 0\};$$
$$Q = \{y \in H_1 : q(y) \leq 0\},$$

where $c : H_0 \to \mathbb{R}$ and $q : H_1 \to \mathbb{R}$ are convex and lower semicontinuous functionals. We assume that ∂c and ∂q are bounded operators, which means that they bounded on bounded sets.

The calculation of a projection onto a level subset is generally difficult. However, Fukushima [9] proposed a method for calculating the projection onto a level set of a convex functionals by computing a sequence of projections onto half-spaces that contain the original level set. This idea was further developed by Yang [26], who introduced a relaxed CQ algorithm:

$$x_{n+1} = P_{C_n}(x_n - \tau A^*(I - P_{Q_n})Ax_n),$$

where C_n and Q_n are given as

$$C_n = \{x \in H_0 : c(x_n) \leq \langle \xi_n, x_n - x\rangle\},$$
$$Q_n = \{y \in H_1 : q(Ax_n) \leq \langle \zeta_n, Ax_n - y\rangle\},$$

where $\xi_n \in \partial c(x_n)$ and $\zeta_n \in \partial q(Ax_n)$. The relaxed CQ algorithm involves a recursive formula that uses the metric projections onto half-spaces C_n and Q_n, which have closed forms and are computationally feasible.

In the literature, there are various generalizations of the SFP, one of which is the split feasibility problem with multiple output sets (SFPMOS). The SFPMOS involves finding a special vector which satisfies a set of constraints involving multiple closed and convex subsets. More precisely, it requires finding $x^\dagger \in H_0$ such that

$$x^\dagger \in C \cap \left(\bigcap_{i=1}^{N} A_i^{-1}(Q_i)\right), \tag{1.1}$$

where $A_i^{-1}(Q_i) = \{x \in H_0 : A_i x \in Q_i\}$ for each $i = 1, 2, \ldots, N$. Reich and Tuyen [14] proposed an iterative method for solving this problem, which involves a recursive formula:

$$x_{n+1} = P_C\left[x_n - \tau \sum_{i=1}^{N} A_i^*(I - P_{Q_i})A_i x_n\right].$$

It has been proved to have the weak convergence under the following condition:

$$0 < \tau < \frac{2}{N \max_{1\leq i\leq N} \|A_i\|^2}.$$

In addition, they further modified the method as follows to obtain an iterative method with strong convergence. For any initial guess y_0, their modified method produces y_n according to the recursion process:

$$y_{n+1} = \gamma_n f(y_n) + (1-\gamma_n) P_C \left[y_n - \tau \sum_{i=1}^{N} A_i^* \left(I - P_{Q_i}\right) A_i y_n \right], \tag{1.2}$$

where $\{\gamma_n\} \subseteq (0,1)$ and f is a contraction. Under certain weak conditions, it has been proven that method (1.2) converges strongly to a solution. However, Wang [19] recently improved the step size condition as follows

$$0 < \tau < \frac{2}{\sum_{1\le i\le N} \|A_i\|^2}.$$

It is worth noting that the above algorithms are only applicable to simple convex sets. To expand the research scope of the SFPMOS, we consider the case where the convex sets Q_i are level sets of convex functionals:

$$Q_i = \{y \in H_i : q_i(y) \le 0\}, \quad i = 0,1,2,\dots,N, \tag{1.3}$$

where $q_i : H_i \to \mathbb{R}$ are convex and lower semicontinuous functionals. We assume that all the functionals q_i are subdifferentiable and that their subdifferential operators are bounded. This allows us to extend the algorithms to more complex convex sets.

After conducting the aforementioned works, we will continue our investigation into the SFPMOS, provided that the corresponding convex subsets fulfill conditions (1.3). To approximate the solution to this problem, we utilize the concept of relaxed projections, along with a recently proposed method. Our approach involves first establishing a weak convergence theorem for the fixed stepsize, followed by constructing a variable stepsize that is not dependent on the norm of linear operators. Additionally, we modify these methods to ensure strong convergence. Ultimately, we apply these techniques to develop a new relaxed projection algorithm to solve the split feasibility problem.

2. Preliminaries

In this paper, one uses the notation $\Lambda = \{0,1,2\dots,N\}$ to denote a set, and H_i to represent a Hilbert space for each $i \in \Lambda$. One also uses I to denote the identity operator. If $f : H_0 \to \mathbb{R}$ is a differentiable functional, one refers to the gradient of f as ∇f. When one has a sequence $\{x_n\}$ in H_0, one uses $\omega_w(x_n)$ and $\omega(x_n)$ to denote the set of cluster points in the weak and strong topology, respectively. One says that "$x_n \to x$" if $\{x_n\}$ converges strongly to x, and "$x_n \rightharpoonup x$" if $\{x_n\}$ converges weakly to x.

Consider an operator $T : H_0 \to H_0$. Recall that T is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in H_0$. One says that T is firmly nonexpansive if, for all $x, y \in H_0$, the following inequality holds:

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I-T)x - (I-T)y\|^2.$$

The above inequality is one of several characterizations of firmly nonexpansive mappings, which can be found in [1].

Lemma 2.1. *The following statements are equivalent.*

(i) T *is firmly nonexpansive,*

(ii) $I - T$ *is firmly nonexpansive,*

(iii) $\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \forall x, y \in H_0$.

Recall that a function $f : H_0 \to \mathbb{R}$ is convex if it satisfies the following inequality for all $\lambda \in (0,1)$ and $x, y \in H_0$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

A differentiable function f is convex if and only if the following relation holds:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle, \quad \forall z \in H_0.$$

One says that an element $g \in H_0$ is a subgradient of f at x if the following inequality holds, for all $z \in H_0$, $f(z) \geq f(x) + \langle g, z - x \rangle$. If f has at least one subgradient at x, then one says that f is subdifferentiable at x. The set of subgradients of f at x is called the subdifferential of f at x, denoted by $\partial f(x)$. A function f is called subdifferentiable if it is subdifferentiable at all $x \in H_0$. If f is convex and differentiable, then its gradient and subgradient coincide. Moving on, one says that a function $f : H_0 \to \mathbb{R}$ is lower semi-continuous (w-lsc) at x if $x_n \to x$ implies the following inequality: $f(x) \leq \liminf_{n\to\infty} f(x_n)$. One says that f is lsc on H_0 if it is lsc at every point $x \in H_0$. Moreover, one says that a function $f : H_0 \to \mathbb{R}$ is weakly lower semi-continuous (w-lsc) at x if $x_n \rightharpoonup x$ implies the following inequality: $f(x) \leq \liminf_{n\to\infty} f(x_n)$. One says that f is w-lsc on H_0 if it is w-lsc at every point $x \in H_0$. For convex functionals, the w-lsc and lsc are the same.

Let us now consider a nonempty, closed, and convex subset $C \subseteq H_0$. One typical example of firmly nonexpansive mappings is the metric projection P_C from H_0 onto C, which is defined as follows:

$$P_C x = \arg\min_{y \in C} \|x - y\|, x \in H_0.$$

In other words, $P_C x$ is the point in C that is closest to x in terms of Euclidean distance.

Lemma 2.2. *Let $x \in H_0$. Then $y = P_C x$ if and only if $y \in C$, and*

$$\langle x - y, z - y \rangle \leq 0, \forall z \in C. \tag{2.1}$$

The concept of Féjer-monotonicity is crucial for the subsequent analysis in this paper. Let us recall the definition: A sequence $\{x_n\} \subseteq H_0$ is said to be Fejér monotone with regard to C if it satisfies the following inequality, for all $n \geq 0$ and $z \in C$, $\|x_{n+1} - z\| \leq \|x_n - z\|$. In other words, the distance between x_{n+1} and any point in C is always smaller than or equal to the distance between x_n and the same point in C.

The following lemmas are significant in the subsequent convergence analysis.

Lemma 2.3. [1] *Suppose that $\{x_n\}$ is Fejér monotone with regard to C. Then, $\{x_n\}$ converges weakly to an element in C if and only if each weak cluster point of $\{x_n\}$ belongs to C.*

Lemma 2.4. [25] *Suppose that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the inequality for all $n \geq 0$: $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n \delta_n$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$. Then, the sequence $\{a_n\}$ converges to 0 if the following conditions hold:*

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \quad \overline{\lim_{n\to\infty}} \delta_n \leq 0.$$

3. Fixed Stepsizes

In 2017, Wang [20] proposed an alternative iterative method for solving the split feasibility problem. This method is defined by the following update rule:

$$x_{n+1} = x_n - \tau[(x_n - P_C x_n) + A^*(Ax_n - P_Q(Ax_n))],$$

where P_C is the metric projection onto a closed convex set C, A is a linear operator from H_0 to H_1, P_Q is the metric projection onto a closed convex set Q, and τ is a positive step size.

Motivated by this method, we aim to construct an iterative method for solving problem (1.1). Specifically, we demonstrate that the SFPMOS is equivalent to finding a fixed point of an appropriate nonlinear operator. For convenience, we define $Q_0 = C$ and A_0 as the identity operator on H_0 in what follows.

Lemma 3.1. *Let $T = I - \tau \sum_{i=0}^{N} A_i^*(I - P_{Q_i})A_i$. For any $(x,z) \in H_0 \times \Omega$, we have the following inequality:*

$$\|Tx - z\|^2 \le \|x - z\|^2 - \tau\left(2 - \tau \sum_{i=0}^{N} \|A_i\|^2\right) \sum_{i=0}^{N} \|A_i x - P_{Q_i}(A_i x)\|^2. \tag{3.1}$$

Furthermore, if $0 < \tau(\sum_{i=0}^{N} \|A_i\|^2) < 2$, then $\Omega = \mathrm{Fix}(T)$, where $\mathrm{Fix}(T)$ denotes the set of fixed points of T.

Proof. For each $i \in \Lambda$, we can use Lemma 2.1 to obtain the following inequalities:

$$\begin{aligned}\langle x - z, A_i^*(A_i x - P_{Q_i}(A_i x))\rangle &= \langle A_i x - A_i z, A_i x - P_{Q_i}(A_i x)\rangle \\ &\ge \|A_i x - P_{Q_i}(A_i x)\|^2 \\ &\ge \|A_i x - P_{Q_i}(A_i x)\|^2.\end{aligned}$$

Additionally, applying the Cauchy-Schwarz inequality yields:

$$\begin{aligned}\left\|\sum_{i=0}^{N} A_i^*(A_i x - P_{Q_i}(A_i x))\right\|^2 &\le \left(\sum_{i=0}^{N} \|A_i\| \|A_i x - P_{Q_i}(A_i x)\|\right)^2 \\ &\le \left(\sum_{i=0}^{N} \|A_i\|^2\right) \sum_{i=0}^{N} \|A_i x - P_{Q_i}(A_i x)\|^2.\end{aligned}$$

Using the definition of T, we can then write:

$$\begin{aligned}\|Tx - z\|^2 &= \left\|x - z - \tau \sum_{i=0}^{N} A_i^*(A_i x - P_{Q_i}(A_i x))\right\|^2 \\ &= \|x - z\|^2 - 2\tau \sum_{i=0}^{N} \langle x - z, A_i^*(A_i x - P_{Q_i}(A_i x))\rangle + \tau^2 \left\|\sum_{i=0}^{N} A_i^*(A_i x - P_{Q_i}(A_i x))\right\|^2 \\ &\le \|x - z\|^2 - \tau\left(2 - \tau \sum_{i=0}^{N} \|A_i\|^2\right) \sum_{i=0}^{N} \|A_i x - P_{Q_i}(A_i x)\|^2,\end{aligned}$$

which gives us the desired inequality.

Next, we need to verify that $\Omega = \mathrm{Fix}(T)$. To do this, we show that $\mathrm{Fix}(T) \subseteq \Omega$ since the converse is trivial. Let $x^\dagger \in \mathrm{Fix}(T)$ and $z \in \Omega$. Using inequality (3.1), we obtain

$$\tau\left(2-\tau\sum_{i=0}^{N}\|A_i\|^2\right)\sum_{i=0}^{N}\|A_ix-P_{Q_i}(A_ix)^\dagger\|^2=0,$$

which implies that $A_ix^\dagger \in P_{Q_i}$ for each $i \in \Lambda$. Therefore, we have $\mathrm{Fix}(T) \subseteq \Omega$, as desired. □

Motivated by Lemma 3.1, we propose the first method to solve problem (1.1).

Algorithm 3.2. Choose $\tau > 0$ and an arbitrary initial $x_0 \in H_0$. Given the current iteration x_n, update the next iteration x_{n+1} via

$$x_{n+1}=x_n-\tau\sum_{i=0}^{N}A_i^*((I-P_{Q_n^i})A_ix_n),$$

where $Q_n^i = \{y \in H_i : q_i(A_ix_n) \le \langle \zeta_n^i, A_ix_n - y\rangle\}$ with $\zeta_n^i \in \partial q_i(A_ix_n)$.

Theorem 3.3. *If $0 < \tau(\sum_{i=0}^{N}\|A_i\|^2) < 2$, then the sequence $\{x_n\}$ produced by Algorithm 3.2 converges weakly to a solution of problem (1.1).*

Proof. Let $z \in \Omega$ be fixed. Then, using Lemma 3.1, we see that

$$\|x_{n+1}-z\|^2 \le \|x_n-z\|^2-\tau\left(2-\tau\sum_{i=0}^{N}\|A_i\|^2\right)\sum_{i=0}^{N}\|(I-P_{Q_n^i})A_ix_n\|^2. \tag{3.2}$$

This implies that $\{x_n\}$ is Fejér monotone with respect to Ω. To complete the proof, we need to demonstrate that every weak cluster point of $\{x_n\}$ belongs to Ω. By induction, we can deduce from (3.2) that

$$\lim_{n\to\infty}\sum_{i=0}^{N}\|(I-P_{Q_n^i})A_ix_n\|=0.$$

Since ∂q_i is bounded on bounded sets, there exists a constant $\delta > 0$ such that $\|\zeta_n^i\| \le \delta$ for all $n \ge 0$ and $i \in \Lambda$. From the definition of Q_n^i, we obtain

$$q_i(A_ix_n) \le \langle \zeta_n, A_ix_n - P_{Q_n^i}(A_ix_n)\rangle \le \delta\|(I-P_{Q_n^i})A_ix_n\| \to 0. \tag{3.3}$$

Now, if $x' \in \omega_w(x_n)$ and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x'$, then the w-lsc and (3.3) imply that

$$q_i(A_ix') \le \liminf_{k\to\infty} q_i(A_ix_{n_k}) \le 0.$$

It follows that $A_ix^\dagger \in Q_i$ for each $i \in \Lambda$. Thus $x^\dagger \in \Omega$ as desired. Consequently, we can use Lemma 2.3 to deduce that the sequence $\{x_n\}$ converges weakly to an element of Ω. □

Remark 3.4. In fact, the following more general step size conditions can be used in the above theorem: $0 < \tau_n(\sum_{i=0}^{N}\|A_i\|^2) < 2$ such that

$$0 < \varliminf_{n\to\infty}\tau_n \le \varlimsup_{n\to\infty}\tau_n < \frac{2}{\sum_{i=0}^{N}\|A_i\|^2}.$$

Since Algorithm 3.2 has only weak convergence, we aim to modify it so that the strong convergence is guaranteed. The idea of our modification is partly taken from the technique for split common fixed point problem [21].

Algorithm 3.5. Start by choosing a fixed element u, $\tau > 0$, a sequence $\{\alpha_n\} \subset [0,1]$, and an arbitrary initial $x_0 \in H_0$. At each iteration, given the current iterate x_n, we update the next iterate x_{n+1} using the formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau \sum_{i=0}^{N} A_i^*((I - P_{Q_n^i})A_i x_n)\right], \tag{3.4}$$

where $Q_n^i = \{y \in H_i : q_i(A_i x_n) \leq \langle \zeta_n^i, A_i x_n - y\rangle\}$ with $\zeta_n^i \in \partial q_i(A_i x_n)$.

Theorem 3.6. *Suppose that the following conditions are true:*

(c1) $\lim_n \alpha_n = 0$;
(c2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(c3) $0 < \tau(\sum_{i=0}^{N} \|A_i\|^2) < 2$.

If problem (1.1) is consistent, then the sequence $\{x_n\}$ produced by Algorithm 3.5 converges strongly to $P_\Omega(u)$.

Proof. To demonstrate that the iterative sequence is bounded, we define

$$y_n = x_n - \tau \sum_{i=0}^{N} A_i^*((I - P_{Q_n^i})A_i x_n)$$

and $\varepsilon = \tau(2 - \tau \sum_{i=0}^{N} \|A_i\|^2)$. Using Lemma 3.1, we obtain the inequality

$$\|y_n - P_\Omega(u)\|^2 \leq \|x_n - P_\Omega(u)\|^2 - \varepsilon \sum_{i=0}^{N} \|(I - P_{Q_n^i})A_i x_n\|^2. \tag{3.5}$$

Since our hypothesis implies that $\|y_n - P_\Omega(u)\| \leq \|x_n - P_\Omega(u)\|$, we can use (3.4) to derive the following chain of inequalities

$$\begin{aligned}
\|x_{n+1} - P_\Omega(u)\| &= \|\alpha_n(u - P_\Omega(u)) + (1-\alpha_n)(y_n - P_\Omega(u))\| \\
&\leq \alpha_n \|u - P_\Omega(u)\| + (1-\alpha_n)\|y_n - P_\Omega(u)\| \\
&\leq \alpha_n \|u - P_\Omega(u)\| + (1-\alpha_n)\|x_n - P_\Omega(u)\| \\
&\leq \max\left\{\|u - P_\Omega(u)\|, \|x_n - P_\Omega(u)\|\right\}.
\end{aligned}$$

Using induction, we can then show that $\|x_n - P_\Omega(u)\| \leq \max\left\{\|x_0 - P_\Omega(u)\|, \|u - P_\Omega(u)\|\right\}$ for all $n \geq 0$, which implies that $\{x_n\}$ is bounded.

Next, we prove

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n, \tag{3.6}$$

where $a_n = \|x_n - P_\Omega(u)\|^2$ and

$$b_n = 2\langle u - P_\Omega(u), x_{n+1} - P_\Omega(u)\rangle - \frac{(1-\alpha_n)\varepsilon}{\alpha_n} \sum_{i=0}^{N} \|(I - P_{Q_n^i})A_i x_n\|^2.$$

Using the inequality in (3.5), we can derive the following inequality:

$$\begin{aligned}
\|x_{n+1} - P_\Omega(u)\|^2 \leq (1-\alpha_n)\|x_n - P_\Omega(u)\|^2 + 2\alpha_n \langle u - P_\Omega(u), x_{n+1} - P_\Omega(u)\rangle \\
- (1-\alpha_n)\varepsilon \sum_{i=0}^{N} \|(I - P_{Q_n^i})A_i x_n\|^2.
\end{aligned}$$

Therefore, we can use the above inequality to prove that (3.6) holds.

Finally, we show the convergence of the iterative sequence. Since $\{b_n\}$ is clearly bounded from above, we can take a subsequence $\{b_{n_k}\}$ such that

$$\begin{aligned}\overline{\lim_{n\to\infty}} b_n &= \lim_{k\to\infty} b_{n_k}\\ &= \lim_{k\to\infty}\left[2\langle u-P_\Omega(u), x_{n_k+1}-P_\Omega(u)\rangle - \frac{\varepsilon}{\alpha_{n_k}}\sum_{i=0}^N \|A_i x_{n_k} - P_{Q^i_{n_k}}(A_i x_{n_k})\|^2\right]. \qquad (3.7)\end{aligned}$$

Since $\langle u-P_\Omega(u), x_{n_k+1}-P_\Omega(u)\rangle$ is bounded, we may assume without loss of generality that its limit exists. Consequently, from (3.7), the following limit

$$\lim_{k\to\infty}\frac{1}{\alpha_{n_k}}\sum_{i=0}^N \|A_i x_{n_k} - P_{Q^i_{n_k}}(A_i x_{n_k})\|^2$$

also exists. This combined with condition (c1) implies that

$$\lim_{k\to\infty}\sum_{i=0}^N \|A_i x_{n_k} - P_{Q^i_{n_k}}(A_i x_{n_k})\| = 0, \qquad (3.8)$$

which implies that any weak cluster point of $\{x_{n_k}\}$ belongs to Ω. Moreover, by the definition of x_n, we have

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &= \|\alpha_{n_k}(u-x_{n_k}) + (1-\alpha_{n_k})(y_{n_k}-x_{n_k})\|\\ &\le \alpha_{n_k}\|u-x_{n_k}\| + \|y_{n_k}-x_{n_k}\|\\ &\le \alpha_{n_k}\|u-x_{n_k}\| + \tau\sum_{i=0}^N \|A_i x_{n_k} - P_{Q^i_{n_k}}(A_i x_{n_k})\|.\end{aligned}$$

From (c1) and (3.8), we see that $\|x_{n_k+1}-x_{n_k}\|\to 0$. Without loss of generality, we may suppose that $\{x_{n_k+1}\}$ converges weakly to $\bar{x}\in\Omega$. Using (3.7) and (2.1), we can prove that

$$\begin{aligned}\overline{\lim_{n\to\infty}} b_n &\le 2\lim_{k\to\infty}\langle u-P_\Omega(u), x_{n_k+1}-P_\Omega(u)\rangle\\ &= 2\langle u-P_\Omega(u), \bar{x}-P_\Omega(u)\rangle \le 0.\end{aligned}$$

Finally, we apply Lemma 2.4 to (3.6) to obtain the desired result. □

Remark 3.7. It is easy to see that the real sequence given by $\alpha_n = 1/(n+1)^p$ with $0<p\le 1$ satisfies (c1)-(c2). Similarly, if we replace "u" by a given contraction "f", we can prove that the above method strongly converges to the unique fixed point of $P_\Omega f$.

4. Variable Stepsizes

It is worth noting that the stepsize selection method mentioned earlier requires prior the knowledge of the value of $\sum_{i=0}^N \|A_i\|^2$, which is often difficult to obtain in practice. However, taking inspiration from our recent work [6], we opted to use a variable stepsize instead. We mention here that the choice of this variable stepsize is not dependent on $\sum_{i=0}^N \|A_i\|^2$, making it a more practical alternative.

Algorithm 4.1. Start by choosing an arbitrary initial $x_0 \in H_0$. At each iteration, given the current iterate x_n, if $\|\sum_{i=0}^N A_i^*((I-P_{Q_n^i})A_ix_n)\| = 0$, we terminate the algorithm. Otherwise, we update the next iterate x_{n+1} using the formula:

$$x_{n+1} = x_n - \tau_n \sum_{i=0}^{N} A_i^*((I-P_{Q_n^i})A_ix_n). \tag{4.1}$$

Here $Q_n^i = \{y \in H_i : q_i(A_ix_n) \leq \langle \zeta_n^i, A_ix_n - y\rangle\}$ with $\zeta_n^i \in \partial q_i(A_ix_n)$ and

$$\tau_n = \frac{\sum_{i=0}^N \|(I-P_{Q_n^i})A_ix_n\|^2}{\|\sum_{i=0}^N A_i^*(I-P_{Q_n^i})A_ix_n\|^2}. \tag{4.2}$$

If our current iteration x_n satisfies $\|\sum_{i=0}^N A_i^*((I-P_{Q_n^i})A_ix_n)\| = 0$, then we can easily verify that it is a solution to problem (1.1). To see this, consider fixing an $z \in \Omega$. Using Lemma 2.1, we can prove that

$$\begin{aligned}\sum_{i=0}^{N} \|(I-P_{Q_n^i})A_ix_n\|^2 &= \sum_{i=0}^{N} \langle A_ix - A_iz, (I-P_{Q_n^i})A_ix_n\rangle \\ &\leq \left\langle x-z, \sum_{i=0}^{N} A_i^*((I-P_{Q_n^i})A_ix_n)\right\rangle \\ &= 0.\end{aligned}$$

This implies that $A_ix_n \in Q_n^i$, which in turn implies from the definition of Q_n^i that $q_i(A_ix_n) \leq 0$. Therefore, we can conclude that $A_ix_n \in Q_i$ for each $i \in \Lambda$. Without losing generality, we may assume that the above iterative sequence is infinite.

Theorem 4.2. *If problem (1.1) is consistent, then the sequence $\{x_n\}$ generated by Algorithm 4.1 weakly converges to a solution of the problem.*

Proof. First, we prove that sequence $\{x_n\}$ satisfies the Fejér monotonicity property with respect to set Ω. Fix any $z \in \Omega$. It then follows from (4.1) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \left\|(x_n - z) - \tau_n \sum_{i=0}^{N} A_i^*((I-P_{Q_n^i})A_ix_n)\right\|^2 \\ &= \|x_n - z\|^2 - 2\tau_n \sum_{i=0}^{N} \langle x_n - z, A_i^*((I-P_{Q_n^i})A_ix_n)\rangle + \tau_n^2 \left\|\sum_{i=0}^{N} A_i^*((I-P_{Q_n^i})A_ix_n)\right\|^2 \\ &\leq \|x_n - z\|^2 - 2\tau_n \sum_{i=0}^{N} \|(I-P_{Q_n^i})A_ix_n\|^2 + \tau_n^2 \left\|\sum_{i=0}^{N} A_i^*((I-P_{Q_n^i})A_ix_n)\right\|^2,\end{aligned}$$

which from (4.2) yields

$$\|x_{n+1} - z\|^2 \leq \|x_n - z\|^2 - \tau_n \sum_{i=0}^{N} \|(I-P_{Q_n^i})A_ix_n\|^2. \tag{4.3}$$

Thus we can prove that the distance between x_{n+1} and a point z is less than or equal to the distance between x_n and z. This implies that the sequence is getting closer to z as n increases.

Next, we prove that any weak cluster point of $\{x_n\}$ belongs to Ω. Let $x^\dagger$ be any weak cluster point of $\{x_n\}$, and let $\{x_{n_k}\}$ be a subsequence that converges weakly to $x^\dagger$. By using equation

(4.3) and induction, we can prove that $\tau_n \sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2$ approaches zero as n goes to infinity. Using the Cauchy-Schwarz inequality, we can then prove that

$$\begin{aligned}
\tau_n &= \frac{\sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2}{\|\sum_{i=0}^N A_i^*((I-P_{Q_n^i})A_i x_n)\|^2} \\
&\geq \frac{\sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2}{(\sum_{i=0}^N \|A_i^*((I-P_{Q_n^i})A_i x_n)\|)^2} \\
&\geq \frac{\sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2}{(\sum_{i=0}^N \|A_i^*\|\|(I-P_{Q_n^i})A_i x_n\|)^2} \\
&\geq \frac{\sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2}{(\sum_{i=0}^N \|A_i\|^2)(\sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2)} \\
&= \frac{1}{\sum_{i=0}^N \|A_i\|^2} > 0,
\end{aligned}$$

This indicates that τ_n is bounded below by a positive constant. Combining this result with $\tau_n \sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2 \to 0$, we can conclude that $\sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|$ approaches zero as n goes to infinity. By a similar argument, we can prove that $A_i x^\dagger \in Q_i$ for each $i \in \Lambda$. Therefore, we can conclude that $x^\dagger \in \Omega$.

Finally, using Lemma 2.3, we can conclude that sequence $\{x_n\}$ converges weakly to a solution of problem (1.1). □

Remark 4.3. We note that the stepsize we have chosen does not require any knowledge of the exact value of $\sum_{i=0}^N \|A_i\|^2$. Moreover, we can use a more general form of the stepsize, given by:

$$\tau_n = \frac{\rho \sum_{i=0}^N \|(I-P_{Q_n^i})A_i x_n\|^2}{\|\sum_{i=0}^N A_i^*((I-P_{Q_n^i})A_i x_n)\|^2}, \text{where } 0 < \rho < 2.$$

Similarly, we can construct another strongly convergent method, as described in Algorithm 4.4.

Algorithm 4.4. Start by choosing a fixed element u, a sequence $\{\alpha_n\} \subset [0,1]$, and an arbitrary initial $x_0 \in H_0$. At each iteration, given the current iterate x_n, if $\|\sum_{i=0}^N A_i^*((I-P_{Q_n^i})A_i x_n)\| = 0$, we terminate the algorithm. Otherwise, we update the next iterate x_{n+1} using the formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau_n \sum_{i=0}^N A_i^*((I-P_{Q_n^i})A_i x_n)\right]. \tag{4.4}$$

Here, $Q_n^i = \{y \in H_i : q_i(A_i x_n) \leq \langle \zeta_n^i, A_i x_n - y\rangle\}$, with $\zeta_n^i \in \partial q_i(A_i x_n)$.

Theorem 4.5. *If problem (1.1) is consistent, then the sequence $\{x_n\}$ generated by Algorithm 4.4 converges strongly to $P_\Omega(u)$.*

5. An application

The SFPMOS is a generalization of the split feasibility problem. Therefore, the convergence theorems discussed above can be applied to develop new iterative techniques for solving the split feasibility problem. For the rest of this discussion, we assume that the solution set of the split feasibility problem (also denoted as Ω) is nonempty.

Algorithm 5.1. Choose an arbitrary initial $x_0 \in H_0$. Given the current iteration x_n, update the next iteration x_{n+1} via

$$x_{n+1} = x_n - \tau \sum_{i=0}^{1} A_i^*(I - P_{Q_n^i})A_i x_n, \tag{5.1}$$

where $Q_n^i = \{y \in H_i : q_i(A_i x_n) \le \langle \zeta_n^i, A_i x_n - y\rangle\}$ with $\zeta_n^i \in \partial q_i(A_i x_n), i = 0, 1$.

Algorithm 5.2. Choose an arbitrary initial $x_0 \in H_0$ and a fixed element u. Given the current iteration x_n, update the next iteration x_{n+1} via

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau \sum_{i=0}^{1} A_i^*(I - P_{Q_n^i})A_i x_n\right], \tag{5.2}$$

where $Q_n^i = \{y \in H_i : q_i(A_i x_n) \le \langle \zeta_n^i, A_i x_n - y\rangle\}$ with $\zeta_n^i \in \partial q_i(A_i x_n), i = 0, 1$.

Corollary 5.3. *If $0 < \tau(\sum_{i=0}^{1} \|A_i\|^2) < 2$, then the iterative sequence generated by Algorithm 5.1 weakly converges to a solution of the SFP. Additionally, if the real sequence $\{\alpha_n\}$ satisfies conditions (c1)-(c2), then the iterative sequence produced by Algorithm 5.2 strongly converge to $P_\Omega(u)$.*

We can obtain similar convergence results for the SFP.

Algorithm 5.4. Start by choosing an arbitrary initial $x_0 \in H_0$. At each iteration, given the current iterate x_n, if $\|\sum_{i=0}^{N} A_i^*(I - P_{Q_n^i})A_i x_n\| = 0$, then stop. Otherwise, update the next iterate x_{n+1} using the formula:

$$x_{n+1} = x_n - \tau_n \sum_{i=0}^{N} A_i^*(I - P_{Q_n^i})A_i x_n. \tag{5.3}$$

Here, $Q_n^i = \{y \in H_i : q_i(A_i x_n) \le \langle \zeta_n^i, A_i x_n - y\rangle\}$, with $\zeta_n^i \in \partial q_i(A_i x_n)$ for $i = 0, 1$. The step size τ_n is defined as:

$$\tau_n = \frac{\sum_{i=0}^{1} \|(I - P_{Q_n^i})A_i x_n\|^2}{\|\sum_{i=0}^{1} A_i^*(I - P_{Q_n^i})A_i x_n\|^2}.$$

Algorithm 5.5. Start by choosing a fixed element u, a sequence $\{\alpha_n\} \subset [0,1]$, and an arbitrary initial $x_0 \in H_0$. At each iteration, given the current iterate x_n, if $\|\sum_{i=0}^{1} A_i^*(I - P_{Q_n^i})A_i x_n\| = 0$, then stop. Otherwise, update the next iterate x_{n+1} using the formula:

$$x_{n+1} = \alpha_n u + (1-\alpha_n)\left[x_n - \tau_n \sum_{i=0}^{N} A_i^*(I - P_{Q_n^i})A_i x_n\right].$$

Here, $Q_n^i = \{y \in H_i : q_i(A_i x_n) \le \langle \zeta_n^i, A_i x_n - y\rangle\}$, with $\zeta_n^i \in \partial q_i(A_i x_n)$ for $i = 0, 1$. The step size τ_n is defined as:

$$\tau_n = \frac{\sum_{i=0}^{1} \|(I - P_{Q_n^i})A_i x_n\|^2}{\|\sum_{i=0}^{1} A_i^*(I - P_{Q_n^i})A_i x_n\|^2}.$$

Corollary 5.6. *If $0 < \tau(\sum_{i=0}^{1} \|A_i\|^2) < 2$, then Algorithm 5.4 generates a sequence that weakly converges to a solution of the SFP. Furthermore, if conditions (c1)-(c2) are satisfied, Algorithm 5.5 produces a sequence that strongly converges to $P_\Omega(u)$.*

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