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COMMON FIXED POINT THEOREMS OF TWO FINITE FAMILIES OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN HYPERBOLIC SPACES

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Abstract. In this paper, we establish strong convergence theorems via an implicit algorithm for two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive maps in hyperbolic spaces. We prove some results concerning Δ-convergence as well as strong convergence of the implicit algorithm. Our results are the generalization of some recent results in CAT(0) spaces, uniformly convex Banach spaces, and hyperbolic spaces.

Keywords. asymptotically quasi-nonexpansive mapping; CAT(0) space, Implicit algorithm; Hyperbolic space.

1. Introduction

Numerous problems in various disciplines of science are nonlinear in nature. Therefore, converting linear structures of a known problem into its corresponding nonlinear structures is becoming more of great interest. Further investigation of numerous problems in spaces without linear structures has its own importance in pure and applied mathematics. A general mathematical framework called a convex structure has been studied and introduced by a number of researchers. One such convex structure is available in a hyperbolic space.

Definition 1.1. Let $\mathscr{U}$ be a nonempty subset of a metric space (X,d), and let $T:\mathscr{U}\to\mathscr{U}$ be a mapping. Denote the set of fixed points of T by $F(T)$

(i) T is nonexpansive if $d(Tx,Ty)\le d(x,y)$ for $x,y\in\mathscr{U}$.
(ii) T is quasi-nonexpansive if $F(T)\neq\emptyset$ and $d(Tx,p)\le d(x,p)$ for $x\in\mathscr{U}$ and $p\in F(T)$.

For an initial $x_0\in\mathscr{U}$, Das and Debata [8] studied the strong convergence of the following iterative sequence via the Ishikawa iteration

$$x_{n+1}=(1-\alpha_n)x_n+\alpha_n T\big((1-\beta_n)x_n+\beta_n Sx_n\big) \tag{1.1}$$

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for two quasi-nonexpansive mappings S,T on a nonempty, closed, and convex subset of a strictly convex Banach space.

Takahashi and Tamura [29] proved the weak convergence of (1.1) to a common fixed point of two nonexpansive mappings in a uniformly convex Banach space which satisfies Opial's condition or whose norm is Fréchet differentiable and the strong convergence in a strictly convex Banach space (see also [13, 30]). Mann and Ishikawa iterative procedures are well-defined in a vector space through its built-in convexity. In the literature, several authors introduced the notion of the convexity in metric spaces. Note that Mann iterative procedure was also investigated in hyperbolic metric spaces; see, e.g., [3, 26] and the references therein.

Recent developments in fixed point theory reflect that the iterative construction of fixed points is vigorously proposed and analyzed for various classes of maps in different spaces. Implicit algorithms provide better approximation of fixed points than explicit algorithms. The number of steps of an algorithm also plays an important role in iterative approximation methods. The case of two maps has a direct link with the minimization problem (see [31]).

In 2001, Xu and Ori [33] investigated te weak convergence of one-step implicit algorithm for a finite family of nonexpansive maps. They also posed an open question on necessary and sufficient conditions required for strong convergence of the algorithm. Since then, numerous authors have considered the weak and strong convergence of various implicit algorithms; see, e.g., [1, 7, 12, 16, 17, 24, 25] and the references therein.

In 2011, Khan et al. [15] proposed and analyzed a general algorithm for strong convergence results in CAT(0) spaces. Later, in 2012, Khan et al. [14] proposed an implicit algorithm for two finite families of nonexpansive maps in a more general setting, hyperbolic spaces. Their results refined and generalized several remarkable results in CAT(0) spaces and uniformly convex Banach spaces.

We recall the following definition.

Definition 1.2. Let (X,d) be a metric space, and let $\mathscr{U}$ be a nonempty subset of X. Let $T : \mathscr{U} \to \mathscr{U}$ be a self-mapping. Recall that

(i). T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x,y),\ \forall x,y \in \mathscr{U} \text{ and } \forall\, n \in \mathbb{N}.$$

(ii). T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \leq k_n d(x,p), \forall x,y \in \mathscr{U}, \forall p \in F(T) \text{ and } \forall n \in \mathbb{N}.$$

(ii). T is said to be uniformly L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x,y),\ \forall x,y \in \mathscr{U} \text{ and } \forall n \in \mathbb{N}.$$

Here $\mathbb{N}$ denotes the set of positive numbers and $F(T) = \{x \in \mathscr{U} : Tx = x\}$.

Remark 1.3. It is easy to see that if $F(T)$ is nonempty, then nonexpansive mapping, quasi-nonexpansive mapping, and asymptotically nonexpansive mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.

The purpose of this paper is to establish common fixed point theorems for two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in the setting of hyperbolic spaces. Our results can be viewed as a generalization of the works of Khan et al. in [14] as well as many related results in the literature.

2. Preliminaries

Let (X,d) be a metric space and $\mathscr{U}$ be a nonempty subset of X. Let T be a selfmap on $\mathscr{U}$. Takahashi [31] introduced a convex structure on a metric space to obtain a nonlinear version of some known fixed point results on Banach spaces.

We now describe another convex structure on a metric space called a Hyperbolic Spaces [18].

Definition 2.1. [18] A hyperbolic space is a triple (X,d,W), where (X,d) is a metric space, $\alpha,\beta \in [0,1]$, and $W : X\times X\times[0,1] \to X$ is such that

(W1). $d(z,W(x,y,\alpha)) \le (1-\alpha)d(z,x)+\alpha d(z,y)$,
(W2). $d(W(x,y,\alpha),\mathbb{W}(x,y,\beta)) = |\alpha-\beta|d(x,y)$,
(W3). $W(x,y,\alpha) = W(y,x,(1-\alpha))$,
(W4). $d(W(x,z,\alpha),W(y,w,\alpha)) \le (1-\alpha)d(x,y)+\alpha d(z,w)$ for all $x,y,z,w \in X$.

It follows from (W1) that, for each $x,y \in X$ and $\alpha \in [0,1]$,

$$d(x,W(x,y,\alpha)) \le \alpha d(x,y),\ d(y,W(x,y,\alpha)) \le (1-\alpha)d(x,y)$$

In fact, we have that (see [23])

$$d(x,W(x,y,\alpha)) = \alpha d(x,y),\ d(y,W(x,y,\alpha)) = (1-\alpha)d(x,y)$$

A subset $\mathscr{U}$ of a hyperbolic space X is convex if $W(x,y,\alpha) \in \mathscr{U}$ for all $x,y \in \mathscr{U}$ and $\alpha \in [0,1]$. Equivalently, a subset U of a hyperbolic space X is said to be convex if $[x,y] \subset \mathscr{U}$, whenever $x,y \in \mathscr{U}$ (see [27]).

It is known that spaces like CAT(0) space and Banach space are special cases of hyperbolic space. The class of hyperbolic spaces also contains Hadamard manifolds [4], R-trees, and Cartesian products of Hilbert balls, the Hilbert open unit ball equipped with the hyperbolic metric [10], as special cases. Some remarkable results in CAT(0) spaces [9, 15, 19, 32] are examples of nonlinear structures which play a major role in recent research in metric fixed point theory.

A map $\eta : (0,\infty)\times(0,2] \to (0,1]$, which provides such a number $\rho = \eta(r,\varepsilon)$ for given $r>0$ and $\varepsilon \in (0,2]$, is called the modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A uniformly convex hyperbolic space is strictly convex (see [19]).

Lemma 2.2. [20] *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. For any $r>0, \varepsilon \in (0,2]$, and for all $x,y,z \in X$,*

$$\left.\begin{array}{l} d(x,z) \le r \\ d(y,z) \le r \\ d(x,y) \ge \varepsilon r \end{array}\right\} \quad \Rightarrow \quad d(W(x,y,\alpha),z) \le (1-2\alpha(1-\alpha)\eta(s,\varepsilon))r,$$

where $\alpha \in [0,1]$ and $s \ge r$.

The concept of Δ-convergence in a metric space was introduced by Lim [21] and its analogue in CAT(0) spaces was investigated by Dhompongsa and Panyanak [9]. Later, Khan et al. [14] continued the investigation of Δ-convergence in the general setting of hyperbolic spaces.

Next, we recall some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, define a continuous functional $r(.,\{x_n\}) : X \to [0,\infty)$ by $r(x,\{x_n\}) = \limsup_{n\to\infty} d(x,x_n)$. The asymptotic radius $\rho = r(\{x_n\})$ of $\{x_n\}$ is given by $\rho = \inf\{r(x,\{x_n\} : x \in X\}$. The asymptotic center of a bounded sequence $\{x_n\}$ with respect to a subset $\mathscr{U}$ of X is defined as follows:

$$A_{\mathscr{U}}(\{x_n\}) = \{x \in X : r(x,\{x_n\}) \le r(y,\{x_n\}) \text{ for any } y \in \mathscr{U}\}.$$

If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that bounded sequences have unique asymptotic centers with respect to closed and convex subsets.

The following lemma is due to Leustean [20] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 2.3. [20] *Let (X,d,W) be a complete uniformly convex hyperbolic space with monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$in X has a unique asymptotic center with respect to any nonempty closed convex subset $\mathscr{U}$ of X. Recall that a sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$ In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = x$ and call x as Δ-limit of $\{x_n\}$.*

Iterative construction by means of classical algorithms like Mann's [22]and Ishikawa's [11]:

(i) $x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, n \ge 0$, (Mann)
(ii) $x_{n+1} = \alpha_n x_n + (1-\alpha_n)T(\beta_n x_n + (1-\beta_n)Tx_n), n \ge 0$, (Ishikawa),

are vigorously applied for approximation of fixed points of various maps under suitable conditions imposed on the control sequences. Algorithm (i) exhibits weak convergence even in the setting of Hilbert space. Moreover, Chidume and Mutangadura [6] constructed an example for Lipschitz pseudocontractive map with a unique fixed point for which the algorithm (i) fails to converge.

From now on, let $\mathscr{L} = \{1,2,3,\ldots,N\}$. In 2001, Xu and Ori [33] obtained a weak convergence result using an implicit algorithm for a finite family of nonexpansive maps

Theorem 2.4. [33] *Let $\{T_i : i \in \mathscr{L}\}$ be a family of nonexpansive selfmaps on a closed convex subset $\mathscr{U}$ of a Hilbert space with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$, let $x_0 \in \mathscr{U}$ and let $\{\alpha_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$. Then the sequence $x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n$, where $n \ge 1$ and $T_n = T_{n(modN)}$ (here the modN function takes values in $\mathscr{L}$), converges weakly to a point in F.*

In 2007, Plubtieng et al. [24] generalized the algorithm of Xu and Ori [33] for two finite families $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ of nonexpansive maps and studied its weak and strong convergence in Banach spaces.

Given x_0 in K (a subset of Banach space), their algorithm is defined as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n[\beta_n x_n + (1-\beta_n)S_n x_n] \tag{2.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0,1)$.

In 2012, Khan et al. [14] investigated Δ-convergence as well as strong convergence through a two-step implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces, a more general setting. The two-step algorithm (2.1) is be defined in a hyperbolic space as:

$$\begin{aligned} x_n &= W(x_{n-1}, T_n y_n, \alpha_n), \\ y_n &= W(x_n, S_n x_n, \beta_n), n \geq 1 \end{aligned} \tag{2.2}$$

where $T_n = T_{n(modN)}$ and $S_n = S_{n(modN)}$. They proved that such implicit algorithm (2.2) is well defined. Khan et al. [14] proved the generalized version of Lemma 1.3 of Schu [28] in a uniformly convex hyperbolic space with monotone modulus of uniform convexity. More precisely, they obtained and proved the following lemmas.

Lemma 2.5. [14] *Let (X,d,W) be a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that*

$$\limsup_{n\to\infty} d(x_n,x) \leq r, \ \limsup_{n\to\infty} d(y_n,x) \leq r$$

and

$$\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Moreover, Khan et al. [14] proved a metric version of a result due to Bose and Laskar [5] which plays a crucial role in proving Δ-convergence of algorithm (2.2)

Lemma 2.6. [14] *Let $\mathscr{U}$ be a nonempty,closed, and convex subset of a uniformly convex hyperbolic space X, and let $\{x_n\}$ be a bounded sequence in $\mathscr{U}$ such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in $\mathscr{U}$ such that $\lim_{n\to\infty} r(y_m,\{x_n\}) = \rho$, then* $\lim_{n\to\infty} y_m = y$.

From now on, for two finite families $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ of maps on $\mathscr{U}$, we set $F(T_i) = \{x : T_i x = x\}$, $F(S_i) = \{x : S_i x = x\}$ and $\mathbb{F} = \cap_{i=1}^N \big(F(T_i) \cap F(S_i)\big) \neq \emptyset$. Recall that a sequence $\{x_n\}$ in a metric space X is said to be Fejér monotone with respect to $\mathscr{U}$ (a subset of X) if $d(x_{n+1},x) \leq d(x_n,x)$ for all $x \in \mathscr{U}$ and for all $n \geq 1$. A map $T : \mathscr{U} \to \mathscr{U}$ is called to be semi-compact if any bounded sequence $\{x_n\}$ satisfying $d(x_n,Tx_n) \to 0$ as $n \to \infty$ has a convergent subsequence.

Let f be a nondecreasing selfmap on $[0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ and let $d(x,A) = \inf\{d(x,y) : y \in A\}$. Then a family $\{T_i : i \in \mathscr{L}\}$ of selfmaps on $\mathscr{U}$ with $\mathscr{F} = \cap_{i=1}^N F(T_i) \neq \emptyset$, satisfies:

(i) condition $(\mathscr{A})$ if $f(d(x,\mathscr{F})) \leq d(x,T_i x)$ for all $x \in \mathscr{U}$, holds for at least one $T_i \in \{T_i : i \in \mathscr{L}\}$ or $f(d(x,\mathscr{F})) \leq \max_{i\in\mathscr{L}} d(x,T_i x)$ holds for all $x \in \mathscr{U}$.

Modifications of condition $(\mathscr{A})$ for two finite families of selfmaps have been made recently in [13, 24] as follows:

Let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of nonexpansive selfmaps on $\mathscr{U}$ with $\mathbb{F} \neq \emptyset$. Then the two families are said to satisfy:

(ii) condition $(\mathscr{B})$ on $\mathscr{U}$ if

$$f(d(x,\mathbb{F}) \leq d(x,T_i x) \text{ or } f(d(x,\mathbb{F})) \leq d(x,S_i x)$$

for all $x \in \mathscr{U}$ holds for at least one $T_i \in \{T_i : i \in \mathscr{L}\}$ or at least one $S_i \in \{S_i : i \in \mathscr{L}\}$, and

(ii) condition $(\mathscr{C})$ on $\mathscr{U}$ if

$$f(d(x,\mathbb{F})) \leq \frac{1}{2}\left[d(x,T_ix)+d(x,S_ix)\right]$$

holds for all $x \in \mathscr{U}$.

Remark 2.7. Condition $(\mathscr{B})$ and condition $(\mathscr{C})$ are equivalent to condition $(\mathscr{A})$ if $T_i = S_i$ for all $i \in \mathscr{L}$.

Lemma 2.8. [2] *Let $\mathscr{U}$ be a nonempty closed subset of a complete metric space (X,d) and $\{x_n\}$ be Fejér monotone with respect to U. Then $\{x_n\}$ converges to some $p \in \mathscr{U}$ if and only if* $\lim_{n\to\infty} d(x_n,\mathscr{U}) = 0$.

We now recollect more results by Khan et al. [14] as follows:

Lemma 2.9. [14] *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a hyperbolic space X and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined implicitly in (2.2). Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathbb{F}$.*

Lemma 2.10. [14] *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of nonexpansive selfmaps of $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined implicitly in (2.2). Then*

$$\lim_{n\to\infty} d(x_n,T_ix_n) = \lim_{n\to\infty} d(x_n,S_ix_n) = 0$$

for each $i = 1,2,3,\ldots,N$.

Theorem 2.11. [14] *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of nonexpansive selfmaps of $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$. Then the sequence $\{x_n\}$ defined implicitly in (2.2) Δ-converges to a common fixed point of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$.*

Theorem 2.12. [14] *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$. Suppose that a pair of maps T_i and S_i in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ respectively, satisfies condition $(\mathscr{B})$. Then the sequence $\{x_n\}$ defined implicitly in (2.2) converges strongly to $p \in \mathbb{F}$.*

Theorem 2.13. [14] *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$. Suppose that one the of maps T_i and S_i in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ is semi-compact. Then the sequence $\{x_n\}$ defined implicitly in (2.2) converges strongly to $p \in \mathbb{F}$.*

The following is a very well known lemma.

Lemma 2.14. *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality $a_{n+1} \leq (1+\delta_n)a_n + b_n, n \geq 1$. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then* *(i).* $\lim_{n\to\infty} a_n$ *exists.*

(ii). In particular, if $\{a_n\}_{n=1}^{\infty}$ *has a subsequence which converges strongly to zero, then* $\lim_{n\to\infty} a_n = 0$.

Inspired and motivated by the works of Khan et al. [14] and some related results, we establish common fixed point theorems for two families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive mappings in hyperbolic spaces. Our results generalize the results obtained by Khan et al. [14], as well as some related results in CAT(0) spaces and uniformly Banach spaces.

3. MAIN RESULTS

In this section, we prove some technical lemmas before proving our main results. Note that we do not need two different sequences $\{k_n^{(T_i)}\}_{n=1}^{\infty} \subset [1,\infty)$ and $\{k_n^{(S_i)}\}_{n=1}^{\infty} \subset [1,\infty)$ satisfying two families of asymptotically quasi-nonexpasive mappings $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$, respectively. We let $k_n = \max\{k_n^{(T_i)}, k_n^{(S_i)}\}$. Thus, from now on, we take only one sequence $\{k_n\}_{n=1}^{\infty} \subset [1,\infty)$ satisfying such two families of asymptotically quasi-nonexpasive mappings. Let L_{T_i} and L_{S_i} be constants satisfying two families of uniformly L-Lipschitzian mappings $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$, respectively. We take $L = \max\{L_{T_i}, L_{S_i}\}$.

Let (X,d,W) be a hyperbolic space. Let $\mathscr{U}$ be a nonempty and convex subset of X. We introduce the following definitions.

Definition 3.1. Let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive self maps on $\mathscr{U}$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. The two-step algorithm can be defined in a hyperbolic space as:

$$\begin{aligned} x_{n+1} &= W(x_n, T_i^n y_n, \alpha_n), \\ y_n &= W(x_n, S_i^n x_n, \beta_n), n \geq 1 \end{aligned} \tag{3.1}$$

where $T_i = T_{i(modN)}$ and $S_i = S_{i(modN)}$ and $i = 1,2,3,\ldots,N$.

We demonstrate that algorithm (3.1) exists, and define a map $\Omega_1 : \mathscr{U} \to \mathscr{U}$ by

$$\Omega_1 x = W(x_0, T_1^n W(x, S_1^n x, \beta_1), \alpha_1).$$

For a given $x_0 \in \mathscr{U}$, the existence of $x_1 = W\big(x_0, T_1^n W(x_1, S_1^n x_1, \beta_1), \alpha_1\big)$ is guaranteed if Ω_1 has a fixed point. Now, for any $u, v \in \mathscr{U}$ and applying (W4), we have

$$\begin{aligned} d(\Omega_1 u, \Omega_1 v) &= d\big(W(x_0, T_1^n W(u, S_1^n u, \beta_1), \alpha_1), W(x_0, T_1^n W(v, S_1^n v, \beta_1), \alpha_1\big) \\ &\leq \alpha_1 d\big(T_1^n W(u, S_1^n u, \beta_1), T_1^n W(v, S_1^n v, \beta_1)\big) \\ &\leq \alpha_1 L_{T_1} d\big(W(u, S_1^n u, \beta_1), W(v, S_1^n v, \beta_1)\big) \\ &\leq \alpha_1 L_{T_1}[(1-\beta_1)d(u,v) + \beta_1 d(S_1^n u, S_1^n v)] \\ &\leq \alpha_1 L_{T_1}[(1-\beta_1)d(u,v) + \beta_1 L_{S_1} d(u,v)] \\ &\leq \alpha_1 L[d(u,v) + Ld(u,v)] \quad (\text{where } L = \max\{L_{T_1}, L_{S_1}\}) \\ &\leq \alpha_1 (L + L^2) d(u,v) \\ &\leq \alpha_1 d(u,v), \ (\text{because } L + L^2 < 1). \end{aligned}$$

Since $\alpha_1 \in (0,1)$, one sees that Ω_1 is a contraction. By the Banach contraction principle, Ω_1 has a unique fixed point. Thus the existence of x_1 is guaranteed. Continuing in this way, we find the existence of x_2, x_3 and so on. Thus implicit algorithm (3.1) is well defined.

We now prove the following lemmas.

Lemma 3.2. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Suppose that the following conditions hold true: (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Then, for the sequence $\{x_n\}$ defined implicitly in (3.1), $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathbb{F}$.*

Proof. For any $p \in \mathbb{F}$, it follows from (3.1) that

$$\begin{aligned}
d(x_{n+1}, p) &= d\big(W(x_n, T_i^n y_n, \alpha_n), p\big) \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n d(T_i^n y_n, p) \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n k_n d(y_n, p) \\
&= (1-\alpha_n) d(x_n, p) + \alpha_n k_n d(W(x_n, S_i^n x_n, \beta_n), p) \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n k_n \big[(1-\beta_n) d(x_n, p) + \beta_n d(S_i^n x_n, p)\big] \\
&\leq (1-\alpha_n) d(x_n, p) + \alpha_n k_n (1-\beta_n) d(x_n, p) + \alpha_n \beta_n k_n^2 d(x_n, p) \\
&= \big[1 - \alpha_n + \alpha_n k_n - \alpha_n \beta_n k_n + \alpha_n \beta_n k_n^2\big] d(x_n, p) \\
&\leq [1 + \alpha_n k_n + \alpha_n \beta_n k_n^2] d(x_n, p) \\
&= (1+\sigma_n) d(x_n, p).
\end{aligned}$$

Since $\sum_{n=1}^{\infty} \sigma_n < \infty$, where $\sigma_n = \alpha_n k_n + \alpha_n \beta_n k_n^2$, it follows from Lemma 2.14 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathbb{F}$. Consequently, $\lim_{n\to\infty} d(x_n, \mathbb{F})$ exists. $\square$

Lemma 3.3. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Suppose that the following conditions hold true: (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Then, for the sequence $\{x_n\}$ defined implicitly in (3.1), we have*

$$\lim_{n to \infty} d(x_n, T_i^n x_n) = \lim_{n\to\infty} d(x_n, S_i^n x_n) = 0$$

for each $i = 1, 2, 3, .., N$. Moreover,

$$\lim_{n to \infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0$$

for each $i = 1, 2, 3, .., N$.

Proof. From Lemma 3.2, it follows that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \mathbb{F}$. Suppose that $\lim_{n\to\infty} d(x_n, p) = c$. If $c = 0$, it is trivial. Next, we consider the case $c > 0$. Observe that

$$\begin{aligned} d(y_n, p) &= d(W(x_n, S_i^n x_n, \beta_n), p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n d(S_i^n x_n, p) \\ &\le (1-\beta_n)d(x_n, p) + \beta_n k_n d(x_n, p) \\ &\le [1-\beta_n + \beta_n k_n] d(x_n, p) \\ &\le (1+\beta_n k_n) d(x_n, p). \end{aligned}$$

Taking the lim sup on both sides of the above inequality, we have

$$\limsup_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(x_n, p) = c.$$

Since T_i is asymptotically quasi-nonexpansive, we have

$$\limsup_{n\to\infty} d(T_i^n y_n, p) \le c. \tag{3.2}$$

We also get

$$\limsup_{n\to\infty} d(x_n, p) \le c. \tag{3.3}$$

Moreover,

$$\limsup_{n\to\infty} d(x_{n+1}, p) = \limsup_{n\to\infty} d(W(x_n, T_i^n y_n, \alpha_n), p) = c. \tag{3.4}$$

From (3.2), (3.3), (3.4), and Lemma 2.5, we have

$$\limsup_{n\to\infty} d(x_n, T_i^n y_n) = 0. \tag{3.5}$$

Next, we note that

$$d(x_{n+1}, x_n) = d(W(x_n, T_i^n y_n, \alpha_n), x_n) \le \alpha_n d(T_i^n y_n, x_n).$$

Taking the lim sup on both sides in the above inequality, we have $\limsup_{n\to\infty} d(x_{n+1}, x_n) \le 0$. Hence,

$$\limsup_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.6}$$

Now,

$$d(x_n, x_{n+i}) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+i-1}, x_{n+i}).$$

Taking the limit as $n \to \infty$ on both sides of the above inequality and using (3.6), we have $\lim_{n\to\infty} d(x_n, x_{n+i}) = 0$ for $i < N$. Furthermore, observe that

$$\begin{aligned} d(x_{n+1}, p) &\le (1-\alpha_n)d(x_n, p) + \alpha_n d(T_i^n y_n, p) \\ &\le (1-\alpha_n)d(x_n, T_i^n y_n) + (1-\alpha_n)d(T_i^n y_n, p) + \alpha_n k_n d(y_n, p) \\ &\le (1-\alpha_n)d(x_n, T_i^n y_n) + (1-\alpha_n)k_n d(y_n, p) + \alpha_n k_n d(y_n, p) \\ &= (1-\alpha_n)d(x_n, T_i^n y_n) + k_n d(y_n, p). \end{aligned}$$

Applying lim inf and lim sup on both sides in the above inequality and using (3.5), we arrive at

$$c \le \liminf_{n\to\infty} d(y_n, p) \le \limsup_{n\to\infty} d(y_n, p) \le c.$$

That is,

$$\lim_{n\to\infty} d\big(W(x_n, S_i^n x_n, \beta_n), p\big) = \lim_{n\to\infty} d(y_n, p) = c. \tag{3.7}$$

Since S_i is asymptotically quasi-nonexpansive, we have

$$\limsup_{n\to\infty} d(S_i^n x_n, p) \le c, \tag{3.8}$$

and

$$\limsup_{n\to\infty} d(x_n, p) \le c. \tag{3.9}$$

From (3.7), (3.8), (3.9), and Lemma 2.5, we obtain $\lim_{n\to\infty} d(x_n, S_i^n x_n) = 0$. Moreover,

$$\begin{aligned} d(x_{n+1}, T_i^n x_n) &\le d(x_{n+1}, T_i^n y_n) + d(T_i^n y_n, T_i^n x_n) \\ &\le d\big(W(x_n, T_i^n y_n, \alpha_n), T_i^n y_n\big) + Ld(y_n, x_n) \\ &\le (1-\alpha_n) d(x_n, T_i^n y_n) + \alpha_n d(T_i^n y_n, T_i^n y_n) + Ld(y_n, x_n) \\ &= (1-\alpha_n) d(x_n, T_i^n y_n) + Ld(W(x_n, S_i^n x_n, \beta_n), x_n) \\ &\le (1-\alpha_n) d(x_n, T_i^n y_n) + L\big[(1-\beta_n) d(x_n, x_n) + \beta_n d(S_i^n x_n, x_n)\big] \\ &= (1-\alpha_n) d(x_n, T_i^n y_n) + L\beta_n d(S_i^n x_n, x_n), \end{aligned}$$

which gives $\lim_{n\to\infty} d(x_{n+1}, T_i^n x_n) = 0$. For each $i \in \mathscr{L}$, we have

$$\begin{aligned} d(x_n, T_{n+i}^n x_n) &\le d(x_n, x_{n+1}) + d(x_{n+1}, T_{n+i}^n x_n) \\ &= d(x_n, x_{n+1}) + d(x_{n+1}, T_i^n x_n). \end{aligned} \tag{3.10}$$

Taking the limit as $n \to \infty$ in above inequality, we have $\lim_{n\to\infty} d(x_n, T_{n+i}^n x_n) = 0$ for each $i \in \mathscr{L}$. Since for each $i \in \mathscr{L}$, the sequence $\{d(x_n, T_i^n x_n)\}$ is a subsequence of $\cup_{i=1}^N \{d(x_n, T_{n+i}^n x_n)\}$ and $\lim_{n\to\infty} d(x_n, T_{n+i}^n x_n) = 0$ for each $i \in \mathscr{L}$. Therefore $\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0$ for each $i \in \mathscr{L}$. Similarly, we have $\lim_{n\to\infty} d(x_n, S_{n+i}^n x_n) = 0$ for each $i \in \mathscr{L}$. Therefore $\lim_{n\to\infty} d(x_n, S_i^n x_n) = 0$ for each $i \in \mathscr{L}$. Now,

$$\begin{aligned} d(x_n, T_{n+i} x_n) &\le d(x_n, T_{n+i}^n x_n) + d(T_{n+i}^n x_n, T_{n+i}^{n+1} x_n) + d(T_{n+i}^{n+1} x_n, T_{n+i} x_n) \\ &\le d(x_n, T_{n+i}^n x_n) + Ld(x_n, T_{n+i} x_n) + Ld(x_n, T_{n+i}^n x_n). \end{aligned}$$

Taking the limit as $n \to \infty$ on both side of the above inequality, we obtain

$$\lim_{n\to\infty} d(x_n, T_{n+i} x_n) \le L \lim_{n\to\infty} d(x_n, T_{n+i} x_n)$$

for each $i \in \mathscr{L}$, which implies $\lim_{n\to\infty} d(x_n, T_{n+i} x_n) = 0$ because $0 < L < 1$. Since, for each $i \in \mathscr{L}$, the sequence $\{d(x_n, T_i x_n)\}$ is a subsequence of $\cup_{i=1}^N \{d(x_n, T_{n+i} x_n)\}$ and $\lim_{n\to\infty} d(x_n, T_{n+i} x_n) = 0$ for each $i \in \mathscr{L}$, one has $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$. Similarly, we can prove $\lim_{n\to\infty} d(x_n, S_{n+i} x_n) = 0$ for each $i \in \mathscr{L}$. Therefore $\lim_{n\to\infty} d(x_n, S_i x_n) = 0$. Hence our proof is finished. □

Now, we are ready to establish Δ-convergence and strong convergence of algorithm (3.1).

Theorem 3.4. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \frac{\sqrt{5}-1}{2}$. Suppose that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Then the sequence $\{x_n\}$ defined by (3.1) Δ-converges to a common fixed point of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$.*

Proof. It follows from Lemma 3.2 that $\{x_n\}$ is bounded. Therefore by Lemma 2.3, $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. Then by Lemma 3.3, we have

$$\lim_{n\to\infty} d(u_n, T_i^n u_n) = 0 = \lim_{n\to\infty} d(u_n, S_i^n u_n)$$

for each $i \in \mathscr{L}$.

We next prove that u is the common fixed point of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$. We define a sequence $\{z_m\}$ in $\mathscr{U}$ by $z_m = T_m^n u$ where $T_m = T_{m(modN)}$. Observe that

$$\begin{aligned} d(z_m, u_n) &\leq d(T_m^n u, T_m^n u_n) + d(T_m^n u_n, T_{m-1}^n u_n) + \cdots + d(T_1^n u_n, u_n) \\ &\leq Ld(u, u_n) + \sum_{i=1}^{m-1} d(T_i^n u_n, u_n) \\ &\leq d(u, u_n) + \sum_{i=1}^{m-1} d(T_i^n u_n, u_n) \text{ (because } 0 < L < 1). \end{aligned}$$

Therefore, we obtain

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

Hence $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 2.6 that $T_{m(modN)} u = u$. Hence u is the common fixed point of $\{T_i : i \in \mathscr{L}\}$. Similarly, we can show that u is the common fixed point of $\{S_i : i \in \mathscr{L}\}$. Therefore u is the common fixed point of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$. For the uniqueness, note that $\lim_{n\to\infty} d(x_n, u)$ exists by Lemma 3.2. Suppose $x \neq u$. By the uniqueness of asymptotic centers, one has

$$\begin{aligned} \limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, x) \\ &\leq \limsup_{n\to\infty} d(x_n, u) \\ &\leq \limsup_{n\to\infty} d(u_n, u), \end{aligned}$$

which is a contradiction. Hence $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, then $A(\{u_n\}) = \{u\}$ for all subsequences $\{u_n\}$ of $\{x_n\}$, which proves that $\{x_n\}$ Δ-converges to a common fixed point u of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$. □

Corollary 3.5. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Suppose that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n \beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Then the sequence $\{x_n\}$ defined by (3.1) Δ-converges to a common fixed point of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$.*

We next use condition $(\mathscr{B})$ to investigate strong convergence of algorithm (3.1). The following technical lemma is crucial.

Lemma 3.6. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η. Let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Suppose that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Then the sequence $\{x_n\}$defined implicitly in (3.1) converges strongly to $p \in \mathbb{F}$ if and only if $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$.*

Proof. Observe from Lemma 3.2 that $d(x_{n+1}, p) \leq d(x_n, p)$. It follows that $\{x_n\}$ is Fejér monotone with respect to $\mathbb{F}$ and $\lim_{n\to\infty} d(x_n, \mathbb{F})$ exists. Hence, the result follows from Lemma 2.8 immediately. □

Based on Lemma 3.6, we now establish the strong convergence of algorithm (3.1).

Theorem 3.7. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Assume that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Suppose that a pair of maps T_i and S_i in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$, respectively, satisfies condition $(\mathscr{B})$. Then the sequence $\{x_n\}$ defined implicitly in (3.1) converges strongly to $p \in \mathbb{F}$.*

Proof. It follows from Lemma 3.2 that $\lim_{n\to\infty} d(x_n, \mathbb{F})$ exists. Moreover, Lemma 3.3 implies that $\lim_{n\to\infty} d(x_n, T_i x_n) = \lim_{n\to\infty} d(x_n, S_i x_n) = 0$ for each $i \in \mathscr{L}$. So condition $(\mathscr{B})$ guarantees that $\lim_{n\to\infty} f(d(x_n, \mathbb{F})) = 0$. Since f is nondecreasing with $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$. Therefore, Lemma 3.6 implies that $\{x_n\}$ converges strongly to a point $p \in \mathbb{F}$. □

Corollary 3.8. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Assume that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Suppose that a pair of maps T_i and S_i in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ respectively, satisfies condition $(\mathscr{B})$. Then the sequence $\{x_n\}$ defined implicitly in (3.1) converges strongly to $p \in \mathbb{F}$.*

As in the proof of Theorem 3.4 in [24], we prove the following result similarly.

Theorem 3.9. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically quasi-nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Assume that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Suppose that one of the map in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ is semi-compact. Then the sequence $\{x_n\}$ defined implicitly in 3.1 converges strongly to $p \in \mathbb{F}$.*

Proof. Suppose that T_{i_0} and S_{j_0} are semi-compact for some $i_0, j_0 \in \mathscr{L}$. It follows from Lemma 3.3 that

$$\lim_{nto\infty} d(x_n, T_{i_0}x_n) = \lim_{n\to\infty} d(x_n, S_{j_0}x_n) = 0.$$

By semi-compactness of the mappings T_{i_0} and S_{j_0}, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{j\to\infty} x_{n_j} = p \in \mathscr{U}$ and $\lim_{nto\infty} d(x_{n_j}, T_{i_0}x_{n_j}) = 0 = \lim_{n\to\infty} d(x_{n_j}, S_{j_0}x_{n_j})$. Now by Lemma 3.3, we have that $\lim_{n\to\infty} d(x_{n_j}, T_i x_{n_j}) = 0$ for all $i \in \mathscr{L}$, which implies that $\lim_{n\to\infty} d(p, T_i p\} = 0$ for all $i \in \mathscr{L}$. Thus $p \in F(T_i)$. Similarly, one can prove that $p \in F(S_i)$. Therefore $p \in \mathbb{F}$, which leads to $\lim_{n\to\infty} d(x_n, \mathbb{F}) = 0$. By Lemma 3.6, one sees that sequence $\{x_n\}$ converges strongly to a common fixed point in $\mathbb{F}$. This completes our proof. □

Corollary 3.10. *Let $\mathscr{U}$ be a nonempty, closed, and convex subset of a complete uniformly convex hyperbolic space X with monotone modulus of uniform convexity η, and let $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ be two finite families of uniformly L-Lipschitzian asymptotically nonexpansive selfmaps on $\mathscr{U}$ such that $\mathbb{F} \neq \emptyset$ with $0 < L < \dfrac{\sqrt{5}-1}{2}$. Assume that the following conditions are satisfied (i) $\alpha_n \to 0, \beta_n \to 0$, (ii) $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$, and (iii) $\sum_{n=1}^{\infty} \alpha_n k_n < \infty, \sum_{n=1}^{\infty} \alpha_n k_n^2 < \infty$. Suppose that one of the map in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ is semi-compact. Then the sequence $\{x_n\}$ defined implicitly in 3.1 converges strongly to $p \in \mathbb{F}$.*

Remark 3.11.
(1) Lemma 3.2 is a refinement and a generalization of Lemma 2.9 ([14] Lemma 2.7).
(2) Lemma 3.3 is a refinement and a generalization of Lemma 2.10 ([14] Lemma 2.8).
(3) Theorem 3.4, which demonstrates that the sequence defined in (3.1) Δ-converges to a common fixed point of $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$, is a refinement and a generalization of Theorem 2.11 ([14]Theorem 3.1).
(4) Lemma 3.6 is a refinement and a generalization of Lemma 3.3 in [14].
(5) Theorem 3.7 demonstrates that if a pair of maps T_i and S_i in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ respectively, satisfies condition ($\mathscr{B}$), then the sequence $\{x_n\}$ defined implicitly in (3.1) converges strongly to $p \in \mathbb{F}$. This Theorem is a refinement and a generalization of Theorem 2.12 ([14]Theorem 3.4).
(6) Theorem 3.9 demonstrates that if one of the map in $\{T_i : i \in \mathscr{L}\}$ and $\{S_i : i \in \mathscr{L}\}$ is semi-compact, then the sequence $\{x_n\}$ defined implicitly in 3.1 converges strongly to $p \in \mathbb{F}$. This theorem is a refinement and a generalization of Theorem 2.13 ([14] Theorem 3.5).

As consequences, we obtain Corollaries 3.5, 3.8, and 3.10. All of our main results hold true for the subclass of asymptotically nonexpansive mappings.

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