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LYAPUNOV-TYPE INEQUALITIES FOR FRACTIONAL MULTI-POINT BOUNDARY VALUE PROBLEMS USING A NEW GENERALIZED FRACTIONAL DERIVATIVE

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Abstract. In this paper, we present a new class of fractional derivative (bi-order Hilfer-Katugampola fractional derivative). In the framework of this type of fractional calculus, we discuss a class of fractional multi-point boundary value problems (BVPs) and obtain some new fractional Lyapunov-type inequalities. Based on the generality of the definition of bi-order Hilfer-Katugampola fractional derivative, we provide a series of corollaries, which demonstrate that our results unify and generalize some known results in the existing literature.

Keywords. bi-order Hilfer-Katugampola fractional derivative; Lyapunov-type inequality; Fractional multi-point BVP.

1. Introduction

In [1], Lyapunov discussed the following second-order Dirichlet problem

$$\begin{cases} x''(t)+q(t)x(t)=0,\ t\in(a,b), \\ x(a)=x(b)=0, \end{cases} \tag{1.1}$$

where $q(t)\in C([a,b],\mathbb{R})$. If BVP (1.1) has a nontrivial solution $x(t)$, then the following inequality

$$\int_a^b |q(s)|ds > \frac{4}{b-a} \tag{1.2}$$

holds. The Lyapunov inequality and its generalisations are powerful and efficient tools in investigating differential and difference equations, including differential stability, oscillation theory, prior estimation, and eigenvalue problems; see, e.g, [2, 3, 4]. Because fractional calculus is more effective and powerful in illustrating numerous practical phenomena than integer calculus, more and more researchers are paying attention to this research topic. In 2013, Ferreira

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[5] first extended inequality (1.2) to the fractional case, and considered the following Dirichlet problem of fractional differential equation

$$\begin{cases} (D_{a^+}^{\alpha}x)(t)+q(t)x(t)=0,\ t\in(a,b),\ 1<a\leq 2, \\ x(a)=x(b)=0, \end{cases}$$

where $q(t)\in C([a,b],\mathbb{R})$, and $D_{a^+}^{\alpha}$ is the Riemann-Liouville fractional derivative of order α. If the BVP has a nontrivial solution $x(t)$, then the following inequality $\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}$ holds. One year later, the same author, in [6] analyzed (1.1) involving Caputo fractional derivative in a similar way and obtained the following inequality

$$\int_a^b \left|q(s)\right|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}.$$

Lyapunov inequality (1.2) has applications in various problems related to fractional differential equations. Due to the applications, it has been generalized in many forms. The existing literature can be roughly divided into two categories. First, based on the definitions of fractional calculus, inequality (1.2) has been extended to various forms with different fractional derivatives, such as Caputo [7, 8], Hadamard [9, 10], Katugampola [11], Hilfer [12, 13], Caputo-Fabrizio [14], Hilfer-Katugampola [15, 16], and so on. Second, under different boundary conditions, scholars studied fractional Lyapunov-type inequalities for nonlocal BVPs (multi-point BVPs [17, 18] and integral BVPs [19, 20]). To the best of the authors' knowledge, there are only few papers on Lyapunov-type inequalities for fractional differential equations with m-point boundary value problems in the literature; see [12, 15, 18]. In [12], Wang et al. proposed a new Lyapunov-type inequality for Hilfer fractional differential equation with mult-ipoint boundary conditions. Later, a Lyapunov type inequalities for multi-point boundary problems with Caputo-Hadamard fractional derivative was established by the same author [18]. Recently, Zhang et al. [15] analysed the m-point BVPs with Hilfer-Katugampola fractional differential equation

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha,\beta}x(t)+q(t)x(t)=0,\ t\in(a,b),\ 1<\alpha<2,\ \rho>0, \\ x(a)=0,\ x(b)=\sum\limits_{i=1}^{m-2}\omega_i x(\varphi_i), \end{cases}$$

where $q(t)\in C([a,b],\mathbb{R})$, ${}^{\rho}D_{a^+}^{\alpha,\beta}$ denotes the Hilfer-Katugampola fractional derivative of order α and type β $(0\leq\beta\leq 1)$, $\omega_i\geq 0$, $a<\varphi_i<b$, $a<\varphi_1<\varphi_2<\cdots<\varphi_{m-2}<b$, with $\sum\limits_{i=1}^{m-2}\omega_i(\varphi_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}<(b^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}$. If the BVPs has a nontrivial solution $x(t)$, then

$$\int_a^b |q(s)|ds \geq \frac{[2(\alpha-1)+\beta(2-\alpha)]^{2(\alpha-1)+\beta(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+Q(b)\sum_{i=1}^{m-2}\omega_i]\max\{a^{\rho-1},b^{\rho-1}\}},$$

where,

$$\Delta_1=(\alpha-1)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(b^{\rho}-a^{\rho})^{\alpha-1}$$

and

$$Q(b)=\frac{(b^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}{(b^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}-\sum_{i=1}^{m-2}\omega_i(\varphi_i^{\rho}-a^{\rho})^{1-(2-\alpha)(1-\beta)}}.$$

In [21], Bulavatsky presented a new generalized Hilfer's derivative in the form

$$D_{a^+}^{(\alpha,\beta)\mu}x(t)=I_{0^+}^{\mu(1-\alpha)}\frac{d}{dt}I_{0^+}^{\mu(1-\alpha)(1-\beta)}x(t),$$

where $0<\alpha,\beta<1$ and $0<\mu<1$. In [22], Karimov provided the higher order definition of bi-order Hilfer fractional derivative of orders α,β $(n-1<\alpha,\beta<n)$ and type μ $(0\le\mu\le1)$ in the following form

$$D_{a^+}^{(\alpha,\beta)\mu}x(t)=I_{0^+}^{\mu(n-\alpha)}\left(\frac{d}{dt}\right)^nI_{0^+}^{\mu(1-\mu)(n-\beta)}x(t).$$

In this paper, motivated by the above works, we propose a new class of fractional derivative (bi-order Hilfer-Katugampola fractional derivative). Under this new framework of fractional calculus, we establish fractional Lyapunov-type inequalities with multi-point boundary conditions

$$\begin{cases} {}^\rho D_{a^+}^{(\alpha,\beta)\mu}x(t)+q(t)x(t)=0,\ 0<a<t<b,\ 1<\alpha,\beta<2,\ \rho>0,\\ x(a)=0,\ x(b)=\sum\limits_{i=1}^{m-2}\omega_ix(\varphi_i),\end{cases}\tag{1.3}$$

and

$$\begin{cases} {}^\rho D_{a^+}^{(\alpha,\beta)\mu}x(t)+q(t)x(t)=0,\ 0<a<t<b,\ 1<\alpha,\beta<2,\ \rho>0,\\ x(a)=0,\ t^{1-\rho}\frac{d}{dt}x(t)|_{t=b}=\sum\limits_{i=1}^{m-2}\lambda_ix(\eta_i),\end{cases}\tag{1.4}$$

where $q(t)\in C([a,b],\mathbb{R})$, ${}^\rho D_{a^+}^{(\alpha,\beta)\mu}$ is bi-order Hilfer-Katugampola fractional derivative of order α,β and type μ $(0\le\mu\le1)$ (see Sectiom 3), $\omega_i,\lambda_i\ge0$, $a<\varphi_i,\eta_i<b$ $(i=1,2,\cdots,m-2)$, $a<\varphi_1<\varphi_2<\cdots<\varphi_{m-2}<b$, and $a<\eta_1<\eta_2<\cdots<\eta_{m-2}<b$. Let $\gamma=\beta+\mu(2-\beta)$ and $\delta=\beta+\mu(\alpha-\beta)$. The interest of this article is to derive Lyapunov-type inequalities for two different types of differential equations involving a new fractional derivative. To illustrate the main results of this paper, we assume that the following conditions:

($\mathbb{A}$) $\sum_{i=1}^{m-2}\omega_i(\varphi_i^\rho-a^\rho)^{\gamma-1}<(b^\rho-a^\rho)^{\gamma-1}$;
($\mathbb{B}$) $\sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho-a^\rho)^{\gamma-1}<(\gamma-1)\rho(b^\rho-a^\rho)^{\gamma-2}$.

The main contributions of our work are summarized as follows:

• We present a new definition of the bi-order Hilfer-Katugampola fractional derivative and prove its property.

• We obtain the fractional Lyapunov-type inequalities for the m-point BVPs (1.3) and (1.4).

• There are two special cases of BVPs (1.3) and BVPs (1.4). One is limit case $\alpha=\beta,\rho\to0$. Our results are an extension of that obtained recently in [12]; the other is the limit case $\alpha=\beta$, so BVPs (1.3) and BVPs (1.4) can be degenerated to the results presented in [15] if we choose $\alpha=\beta,\rho\to1$. On the other hand, BVPs (1.3) can be degenerated to the result presented in [18]. In addition, if we set $\alpha=\beta,\mu=0$, the BVPs (1.3) is an extension of that obtained recently in [23].

The rest of this paper is structured as follows: In Section 2, we recall some definitions of fractional calculus and offer the related properties that will be used in the following. In Section 3, we give a new definition of the bi-order Hilfer-Katugampola fractional derivative and prove its property. Finally, the main results and a series of corollaries are given in Section 4. The concluding remark is presented in the last section, Section 5.

2. Preliminaries

In this section, we present some basic definitions and lemmas, which are useful for establishing our results. Let $c \in \mathbb{R}$, $p \geq 1$ and $X_c^p(a,b)$ be the space of all complex valued Lebesgue measurable functions x on (a,b) with $\|x\|_{X_c^p} < \infty$, where the norm is defined by

$$\|x\|_{X_c^p} = \Big(\int_a^b \left|t^c x(t)\right|^p \frac{d}{dt}\Big)^{1/p} < \infty.$$

Definition 2.1. [24, 25] Let $0 < a < t < b < \infty$ and $f \in X_c^p(a,b)$. The left-side Katugampola fractional integral of order $\alpha > 0$ and $\rho > 0$ is defined by

$$({}^{\rho}I_{a^+}^{\alpha} f)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^{\rho} - s^{\rho})^{\alpha-1} s^{\rho-1} f(s) ds, \; t \in [a,b].$$

Definition 2.2. [24, 25] Let $\alpha > 0$, $n = [\alpha] + 1$ and $\rho > 0$ for $0 < a < t < b < \infty$. The left-side Katugampola fractional derivative ${}^{\rho}D_{a^+}^{\alpha} f$ of order α is defined by

$$({}^{\rho}D_{a^+}^{\alpha} f)(t) = \zeta_{\rho}^{n}(I_{a^+}^{n-\alpha} f)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \big(t^{1-\rho}\frac{d}{dt}\big)^n \int_a^t \frac{s^{\rho-1} f(s)}{(t^{\rho} - s^{\rho})^{\alpha-n+1}} ds,$$

where $\zeta_{\rho}^{n} = \big(t^{1-\rho}\frac{d}{dt}\big)^n$.

Definition 2.3. [26] Let $\alpha > 0$, $n = [\alpha] + 1$, and $\rho > 0$. The left-side Hilfer-Katugampola fractional derivative ${}^{\rho}D_{a^+}^{\alpha,\beta} f$ of order α and type β $(0 \leq \beta \leq 1)$ of a function f is defined by

$$({}^{\rho}D_{a^+}^{\alpha,\beta} f)(t) = ({}^{\rho}I_{a^+}^{\beta(n-\alpha)} \big(t^{1-\rho}\frac{d}{dt}\big)^n {}^{\rho}I_{a^+}^{(1-\beta)(n-\alpha)} f)(t).$$

Lemma 2.4. [24, 25] *Let $\alpha, \beta > 0$, $0 < a < b < \infty$, $\rho, c \in \mathbb{R}$, $1 \leq \rho \leq \infty$, and $\rho \geq c$. Then, for $f \in X_c^p(a,b)$, the semi group property is valid, that is, $({}^{\rho}I_{a^+}^{\alpha}\, {}^{\rho}I_{a^+}^{\beta} f)(t) = ({}^{\rho}I_{a^+}^{\alpha+\beta} f)(t)$.*

Lemma 2.5. [27] *Let $\alpha > 0$, $n = [\alpha] + 1$, $f \in X_c^p(a,b)$, and ${}^{\rho}I_{a^+}^{\alpha} f \in AC_{\zeta_{\rho}}^{n}[a,b]$. Then*

$$({}^{\rho}I_{a^+}^{\alpha}\, {}^{\rho}D_{a^+}^{\alpha} f)(t) = f(t) - \sum_{j=1}^{n} \frac{(\zeta_{\rho}^{n-j}({}^{\rho}I_{a^+}^{n-\alpha} f))(a)}{\Gamma(\alpha - j + 1)} \big(\frac{t^{\rho} - a^{\rho}}{\rho}\big)^{\alpha-j},$$

where

$$AC_{\zeta_{\rho}}^{n}[a,b] = \{f : [a,b] \to R |\, \zeta_{\rho}^{n-1} f \in AC[a,b]\}.$$

3. New Definition and Property of Bi-Order Hilfer-Katugampola Fractional Calculus

Definition 3.1. Let $\alpha > 0$, $n - 1 < \alpha, \beta \leq n$, and $\rho > 0$. The bi-order Hilfer-Katugampola fractional derivative ${}^{\rho}D_{a^+}^{(\alpha,\beta)\mu} f$ of order α, β and type μ $(0 \leq \mu \leq 1)$ of a function f is defined by $({}^{\rho}D_{a^+}^{(\alpha,\beta)\mu} f)(t) = ({}^{\rho}I_{a^+}^{\mu(n-\alpha)} (t^{1-\rho}\frac{d}{dt})^n {}^{\rho}I_{a^+}^{(1-\mu)(n-\beta)} f)(t)$.

Lemma 3.2. *Let $n - 1 < \alpha, \beta < n$, $n = [\alpha] + 1$, $\rho > 0$, $f \in X_c^p(a,b)$, ${}^{\rho}I_{a^+}^{\alpha}, {}^{\rho}I_{a^+}^{\beta} \in AC_{\zeta_{\rho}}^{n}[a,b]$. Then*

$$({}^{\rho}I_{a^+}^{\delta}\, {}^{\rho}D_{a^+}^{(\alpha,\beta)\mu} f)(t) = ({}^{\rho}I_{a^+}^{\delta}\, {}^{\rho}I_{a^+}^{\gamma-\delta}\, {}^{\rho}D_{a^+}^{\gamma} f)(t) = f(t) - \sum_{j=1}^{n} \frac{(\zeta_{\rho}^{n-j}({}^{\rho}I_{a^+}^{n-\gamma} x))(a)}{\Gamma(\gamma - j + 1)} \big(\frac{t^{\rho} - a^{\rho}}{\rho}\big)^{\gamma-j},$$

where $\gamma = \beta + \mu(n-\beta)$, $\delta = \beta + \mu(\alpha - \beta)$, $\gamma > \delta$, *and* $\zeta_\rho^n = (t^{1-\rho}\frac{d}{dt})^n$.

Proof. Let $\gamma = \beta + \mu(n-\beta)$ and $\delta = \beta + \mu(\alpha-\beta)$. According to the definitions and Lemma 2.4, we have

$$\begin{aligned}\left({}^\rho D_{a^+}^{(\alpha,\beta)\mu} f\right)(t) &= \left({}^\rho I_{a^+}^{\mu(n-\alpha)} (t^{1-\rho}\frac{d}{dt})^n {}^\rho I_{a^+}^{(1-\mu)(n-\beta)} f\right)(t) \\ &= \left({}^\rho I_{a^+}^{\gamma-\delta} (t^{1-\rho}\frac{d}{dt})^n {}^\rho I_{a^+}^{n-\gamma} f\right)(t) \\ &= \left({}^\rho I_{a^+}^{\gamma-\delta}\, {}^\rho D_{a^+}^{\gamma} f\right)(t),\end{aligned}$$

It follows form Lemma 2.5 that

$$\left({}^\rho I_{a^+}^{\delta}\, {}^\rho D_{a^+}^{(\alpha,\beta)\mu} f\right)(t) = \left({}^\rho I_{a^+}^{\delta}\, {}^\rho I_{a^+}^{\gamma-\delta}\, {}^\rho D_{a^+}^{\gamma} f\right)(t) = f(t) - \sum_{j=1}^{n} \frac{\left(\zeta_\rho^{n-j}({}^\rho I_{a^+}^{n-\gamma} f)\right)(a)}{\Gamma(\gamma-j+1)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-j},$$

which completes the proof. □

4. Main Results

4.1. Green's functions of BVPs (1.3) and (1.4).

Lemma 4.1. *Assume that* $(\mathbb{A})$ *holds. A function* $x(t) \in C[a,b]$ *is a solution to BVPs* (1.3) *if and only if it satisfies the integral equation*

$$x(t) = \int_a^b G(t,s)q(s)x(s)ds + M(t)\sum_{i=1}^{m-2}\omega_i \int_a^b G(\varphi_i,s)q(s)x(s)ds,\ t \in [a,b],$$

where $M(t)$ *and Green's function* $G(t,s)$ *are defined as*

$$M(t) = \frac{(t^\rho - a^\rho)^{\gamma-1}}{(b^\rho - a^\rho)^{\gamma-1} - \sum_{i=1}^{m-2}\omega_i(\varphi_i^\rho - a^\rho)^{\gamma-1}},\ t \in [a,b],$$

$$G(t,s) = \frac{\rho^{1-\delta}s^{\rho-1}}{\Gamma(\delta)(b^\rho - a^\rho)^{\gamma-1}} \begin{cases} g_1(t,s),\ a \le s \le t \le b, \\ g_2(t,s),\ a \le t \le s \le b,\end{cases}$$

and

$$\begin{aligned} g_1(t,s) &= (t^\rho - a^\rho)^{\gamma-1}(b^\rho - s^\rho)^{\delta-1} - (b^\rho - a^\rho)^{\gamma-1}(t^\rho - s^\rho)^{\delta-1}, \\ g_2(t,s) &= (t^\rho - a^\rho)^{\gamma-1}(b^\rho - s^\rho)^{\delta-1}.\end{aligned}$$

Proof. Applying operator ${}^\rho I_{a^+}^{\delta}$ to both sides of (1.3) and combining Lemma 3.2, we obtain

$$x(t) = -{}^\rho I_{a^+}^{\delta} q(t)x(t) + c_0\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + c_1\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-2},$$

where $c_0, c_1 \in \mathbb{R}$, $\gamma = \beta + \mu(2-\beta)$, and $\delta = \beta + \mu(\alpha - \beta)$. Since $x(a) = 0$, we immediately obtain $c_1 = 0$. Thus

$$x(t) = -{}^\rho I_{a^+}^{\delta} q(t)x(t) + c_0\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}. \tag{4.1}$$

Boundary condition $x(b)=\sum_{i=1}^{m-2}\omega_i x(\varphi_i)$ implies that

$$-{}^{\rho}I_{a^+}^{\delta}q(t)x(t)|_{t=b}+c_0\Big(\frac{b^\rho-a^\rho}{\rho}\Big)^{\gamma-1}=\sum_{i=1}^{m-2}\omega_i x(\varphi_i),$$

and hence

$$c_0=\Big(\frac{\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\Big(\sum_{i=1}^{m-2}\omega_i x(\varphi_i)+{}^{\rho}I_{a^+}^{\delta}q(t)x(t)|_{t=b}\Big).$$

Substituting the values c_0 into (4.1), we arrive at

$$\begin{aligned}
x(t)&=-{}^{\rho}I_{a^+}^{\delta}q(t)x(t)+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\Big(\sum_{i=1}^{m-2}\omega_i x(\varphi_i)+{}^{\rho}I_{a^+}^{\delta}q(t)x(t)|_{t=b}\Big)\\
&=-{}^{\rho}I_{a^+}^{\delta}q(t)x(t)+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}{}^{\rho}I_{a^+}^{\delta}q(t)x(t)|_{t=b}+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\sum_{i=1}^{m-2}\omega_i x(\varphi_i)\\
&=-\frac{\rho^{1-s}}{\Gamma(\delta)}\int_a^t\big(t^\rho-s^\rho\big)^{\delta-1}s^{\rho-1}q(s)x(s)ds\\
&\quad+\frac{\rho^{1-s}}{\Gamma(\delta)}\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\int_a^b\big(b^\rho-s^\rho\big)^{\delta-1}s^{\rho-1}q(s)x(s)ds+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\sum_{i=1}^{m-2}\omega_i x(\varphi_i)\\
&=\int_a^b G(t,s)q(s)x(s)ds+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\sum_{i=1}^{m-2}\omega_i x(\varphi_i).
\end{aligned}\tag{4.2}$$

It follows that

$$\sum_{i=1}^{m-2}\omega_i x(\varphi_i)=\sum_{i=1}^{m-2}\omega_i\int_a^b G(\varphi_i,s)q(s)x(s)ds+\sum_{i=1}^{m-2}\omega_i x(\varphi_i)\sum_{i=1}^{m-2}\omega_i\Big(\frac{\varphi_i^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1},$$

that is

$$\sum_{i=1}^{m-2}\omega_i x(\varphi_i)=\frac{\sum_{i=1}^{m-2}\omega_i\int_a^b G(\varphi_i,s)q(s)x(s)ds(b^\rho-a^\rho)^{\gamma-1}}{(b^\rho-a^\rho)^{\gamma-1}-\sum_{i=1}^{m-2}\omega_i(\varphi_i^\rho-a^\rho)^{\gamma-1}},\ t\in[a,b].\tag{4.3}$$

By substituting (4.3) into (4.2), one has

$$\begin{aligned}
x(t)&=\int_a^b G(t,s)q(s)x(s)ds+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\sum_{i=1}^{m-2}\omega_i x(\varphi_i)\\
&=\int_a^b G(t,s)q(s)x(s)ds+\Big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\Big)^{\gamma-1}\frac{(b^\rho-a^\rho)^{\gamma-1}\sum_{i=1}^{m-2}\omega_i\int_a^b G(\varphi_i,s)q(s)x(s)ds}{\big(b^\rho-a^\rho\big)^{\gamma-1}-\sum_{i=1}^{m-2}\omega_i\big(\varphi_i^\rho-a^\rho\big)^{\gamma-1}}\\
&=\int_a^b G(t,s)q(s)x(s)ds+\frac{\big(t^\rho-a^\rho\big)^{\gamma-1}\sum_{i=1}^{m-2}\omega_i\int_a^b G(\varphi_i,s)q(s)x(s)ds}{\big(b^\rho-a^\rho\big)^{\gamma-1}-\sum_{i=1}^{m-2}\omega_i\big(\varphi_i^\rho-a^\rho\big)^{\gamma-1}}.
\end{aligned}$$

This proof is completed. □

Lemma 4.2. *Assume that* $(\mathbb{B})$ *holds. A function* $x(t)\in C[a,b]$ *is a solution to BVPs* (1.4) *if and only if it satisfies the integral equation*

$$x(t)=\int_a^b Y(t,s)q(s)x(s)ds+L(t)\sum_{i=1}^{m-2}\lambda_i\int_a^b Y(\eta_i,s)q(s)x(s)ds,\ t\in[a,b],$$

where $L(t)$ and Green's function $Y(t,s)$ are given as

$$L(t)=\frac{(t^\rho-a^\rho)^{\gamma-1}}{(\gamma-1)\rho(b^\rho-a^\rho)^{\gamma-2}-\sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho-a^\rho)^{\gamma-1}},\ t\in[a,b],$$

$$Y(t,s)=\frac{(b^\rho-s^\rho)^{\delta-2}\rho^{1-\delta}s^{\rho-1}}{(\gamma-1)\Gamma(\alpha)}\begin{cases} y_1(t,s),\ a\le s\le t\le b,\\ y_2(t,s),\ a\le t\le s\le b,\end{cases}$$

and

$$y_1(t,s)=(\delta-1)(b^\rho-a^\rho)^{2-\gamma}(t^\rho-a^\rho)^{\gamma-1}-(\gamma-1)\frac{(t^\rho-s^\rho)^{\delta-1}}{(b^\rho-s^\rho)^{\delta-2}},$$
$$y_2(t,s)=(\delta-1)(b^\rho-a^\rho)^{2-\gamma}(t^\rho-a^\rho)^{\gamma-1}.$$

Proof. The proof of this lemma follows by a similar method employed in Lemma 4.1, we obtain

$$x(t)=-{}^\rho I_{a^+}^{\delta}q(t)x(t)+\tilde{c}_0\big(\frac{t^\rho-a^\rho}{\rho}\big)^{\gamma-1}+\tilde{c}_1\big(\frac{t^\rho-a^\rho}{\rho}\big)^{\gamma-2},$$

where $\tilde{c}_0,\tilde{c}_1\in\mathbb{R}$, $\gamma=\beta+\mu(2-\beta)$, and $\delta=\beta+\mu(\alpha-\beta)$. By using boundary condition $x(a)=0$, we immediately obtain that $\tilde{c}_1=0$, which finds

$$x(t)=-{}^\rho I_{a^+}^{\delta}q(t)x(t)+\tilde{c}_0\big(\frac{t^\rho-a^\rho}{\rho}\big)^{\gamma-1}.$$

Taking derivative to the equality above with respect to t, and multiplying the both sides of the results obtained by $t^{1-\rho}$, we arrive at

$$t^{1-\rho}\frac{d}{dt}x(t)=-{}^\rho I_{a^+}^{\delta-1}q(t)x(t)+\tilde{c}_0(\gamma-1)\big(\frac{t^\rho-a^\rho}{\rho}\big)^{\gamma-2},$$

which together with boundary condition $t^{1-\rho}\frac{d}{dt}x(t)|_{t=b}=\sum\limits_{i=1}^{m-2}\lambda_i x(\eta_i)$ yields

$$\tilde{c}_0=\frac{\rho^{\gamma-2}\big(\sum_{i=1}^{m-2}\lambda_i x(\eta_i)+{}^\rho I_{a^+}^{\delta-1}q(t)x(t)|_{t=b}\big)}{(\gamma-1)(b^\rho-a^\rho)^{\gamma-2}}.$$

Thus

$$\begin{aligned}x(t)&=-{}^\rho I_{a^+}^{\delta}q(t)x(t)+\frac{(t^\rho-a^\rho)^{\gamma-1}\big(\sum_{i=1}^{m-2}\lambda_i x(\eta_i)+{}^\rho I_{a^+}^{\delta-1}q(t)x(t)|_{t=b}\big)}{\rho(\gamma-1)(b^\rho-a^\rho)^{\gamma-2}}\\&=\int_a^b Y(t,s)q(s)x(s)ds+\frac{(t^\rho-a^\rho)^{\gamma-1}}{\rho(\gamma-1)(b^\rho-a^\rho)^{\gamma-2}}\sum_{i=1}^{m-2}\lambda_i x(\eta_i).\end{aligned}\tag{4.4}$$

Moreover,

$$\sum_{i=1}^{m-2}\lambda_i x(\eta_i)=\sum_{i=1}^{m-2}\lambda_i\int_a^b Y(\eta_i,s)q(s)x(s)ds+\sum_{i=1}^{m-2}\lambda_i x(\eta_i)\frac{\sum_{i=1}^{m-2}\lambda_i(\eta_i{}^\rho-a^\rho)^{\gamma-1}}{\rho(\gamma-1)(b^\rho-a^\rho)^{\gamma-2}},$$

which implies

$$\sum_{i=1}^{m-2}\lambda_i x(\eta_i)=\frac{\rho(\gamma-1)(b^\rho-a^\rho)^{\gamma-2}\sum_{i=1}^{m-2}\lambda_i\int_a^b Y(\eta_i,s)q(s)x(s)ds}{\rho(\gamma-1)(b^\rho-a^\rho)^{\gamma-2}-\sum_{i=1}^{m-2}\lambda_i(\eta_i{}^\rho-a^\rho)^{\gamma-1}}.\tag{4.5}$$

Substitute (4.5) into (4.4) finishes the proof. □

The following lemma is from 4.14.

Lemma 4.3. *If* $1<\ell<2$, *then* $\frac{2-\ell}{(\ell-1)^{\ell-1/\ell-2}}\le\frac{(\ell-1)^{\ell-1}}{\ell^{\ell}}$.

Lemma 4.4. *The Green's functions* $G(t,s)$ *and* $Y(t,s)$ *defined in Lemmas 4.1 and 4.2, separately, satisfy the following properties:*
(i) $G(t,s)$ *and* $Y(t,s)$ *are two continuous functions for any* $(t,s)\in[a,b]\times[a,b]$,
(ii) for any $(t,s)\in[a,b]\times[a,b]$, *then*

$$|G(t,s)|\le\frac{(\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}}{\Gamma(\alpha)(\delta+\gamma-2)^{\delta+\gamma-2}}\rho^{1-\delta}s^{\rho-1}(b^{\rho}-a^{\rho})^{\delta-1},$$

(iii) for any $(t,s)\in[a,b]\times[a,b]$,

$$|Y(t,s)|\le\frac{\rho^{1-\delta}s^{\rho-1}(b^{\rho}-s^{\rho})^{\delta-2}}{\Gamma(\alpha)(\gamma-1)}\max\{\gamma-\delta,\delta-1\}.$$

Proof. (i) is obvious. We now demonstrate that (ii) holds. From the function $g_2(t,s)$, we see that $0\le g_2(t,s)\le g_2(s,s)$, $(t,s)\in[a,b]\times[a,b]$. Differentiating $g_1(t,s)$ with respect to s, we obtain

$$\frac{\partial g_1(t,s)}{\partial s}=(\delta-1)\rho s^{\rho-1}(b^{\rho}-a^{\rho})^{\gamma-1}(t^{\rho}-s^{\rho})^{\delta-2}\Big(1-\big(\frac{t^{\rho}-s^{\rho}}{b^{\rho}-s^{\rho}}\big)^{2-\delta}\big(\frac{t^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\big)^{\gamma-1}\Big)\ge0,$$

which indicate that $g_1(t,s)$ is increasing with respect to $s\in[a,t]$. It follows that $g_1(t,a)\le g_1(t,s)\le g_1(t,t)$, $a\le s\le t\le b$. On account of

$$g_1(t,a)=(t^{\rho}-a^{\rho})^{\gamma-1}(b^{\rho}-a^{\rho})^{\delta-1}\Big(1-\big(\frac{b^{\rho}-a^{\rho}}{t^{\rho}-a^{\rho}}\big)^{\gamma-\delta}\Big)\le0,$$

we have $|g_1(t,s)|\le\max\{\max_{t\in[a,b]}g_1(t,t),\max_{t\in[a,b]}-g_1(t,a)\}$. Observe

$$h_1(t)=g_1(t,t)=(t^{\rho}-a^{\rho})^{\gamma-1}(b^{\rho}-t^{\rho})^{\delta-1},\ t\in[a,b],$$

and

$$h_2(t)=-g_1(t,a)=(b^{\rho}-a^{\rho})^{\gamma-1}(t^{\rho}-a^{\rho})^{\delta-1}\Big(1-\big(\frac{t^{\rho}-a^{\rho}}{b^{\rho}-a^{\rho}}\big)^{\gamma-\delta}\Big),\ t\in[a,b].$$

Now, differentiating $h_1(t)$ on (a,b), one has

$$h_1'(t)=\rho t^{\rho-1}(t^{\rho}-a^{\rho})^{\gamma-2}(b^{\rho}-t^{\rho})^{\delta-2}\Big(1-\big(\frac{\delta-1}{\gamma-1}\big)\big(\frac{t^{\rho}-a^{\rho}}{b^{\rho}-t^{\rho}}\big)\Big).$$

Observe that $h_1'(t)=0$ have a unique solution $\tilde{t}$, which is given as

$$t=\tilde{t}=\left(a^{\rho}+\frac{(b^{\rho}-a^{\rho})(\gamma-1)}{(\delta+\gamma-2)}\right)^{1/\rho}=\left(b^{\rho}-\frac{(b^{\rho}-a^{\rho})(\delta-1)}{(\delta+\gamma-2)}\right)^{1/\rho}.$$

Since $h_1'(\tilde{t})>0$ on $(a,\tilde{t})$ and $h_1'(\tilde{t})<0$ on $(\tilde{t},b)$, we conclude that

$$\begin{aligned}\max_{t\in[a,b]}h_1(t)=h_1(\tilde{t})=&\left(\frac{(b^{\rho}-a^{\rho})(\gamma-1)}{\delta+\gamma-2}\right)^{\gamma-1}\left(\frac{(b^{\rho}-a^{\rho})(\delta-1)}{\delta+\gamma-2}\right)^{\delta-1}\\=&\frac{(\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}}{(\delta+\gamma-2)^{\delta+\gamma-2}}(b^{\rho}-a^{\rho})^{\delta+\gamma-2}.\end{aligned}$$

Next, we need to prove that

$$\max_{t\in[a,b]} h_2(t) \le \max_{t\in[a,b]} h_1(t). \tag{4.6}$$

If $\gamma-\delta=0$, inequality (4.6) is obvious. If $\gamma-\delta\neq 0$, then

$$h_2'(t)=\rho t^{\rho-1}(b^\rho-a^\rho)^{\delta-1}(t^\rho-a^\rho)^{\delta-2}\times\Big(1-\big(\frac{\gamma-1}{\delta-1}\big)\big(\frac{t^\rho-a^\rho}{b^\rho-a^\rho}\big)^{\gamma-\delta}\Big).$$

Observe that $h_2'(t)=0$ have a unique solution $\hat{t}$ given as

$$t=\hat{t}=\bigg(a^\rho+\Big(\frac{\delta-1}{\gamma-1}\Big)^{1/\gamma-\delta}(b^\rho-a^\rho)\bigg)^{1/\rho}.$$

Since $h_2'(\hat{t})>0$ on $(a,\hat{t})$ and $h_2'(\hat{t})<0$ on $(\hat{t},b)$, we conclude that

$$\max_{t\in[a,b]} h_2(t)=h_2(\hat{t})=(b^\rho-a^\rho)^{\delta+\gamma-2}\frac{\gamma-\delta}{\gamma-1}\Big(\frac{\delta-1}{\gamma-1}\Big)^{\delta-1/\gamma-\delta}.$$

To prove that $h_2(\hat{t})\le h_1(\tilde{t})$, we set $\ell=\frac{\delta+r-2}{r-1}(1<\ell<2)$. By using Lemma 4.3, we obtain

$$\begin{aligned}h_2(\hat{t})\le&\bigg(\frac{(\delta-1)^{\delta-1}(\gamma-1)^{\gamma-1}}{(\delta+\gamma-2)^{\delta+\gamma-2}}\bigg)^{1/\gamma-1}(b^\rho-a^\rho)^{\delta+\gamma-2}\\ \le&\bigg(\frac{(\delta-1)^{\delta-1}(\gamma-1)^{\gamma-1}}{(\delta+\gamma-2)^{\delta+\gamma-2}}\bigg)(b^\rho-a^\rho)^{\delta+\gamma-2}\\ =&h_1(\tilde{t}).\end{aligned}$$

Thus

$$\begin{aligned}|g_1(t,s)|&\le\max\{\max_{t\in[a,b]} g_1(t,t),\max_{t\in[a,b]}-g_1(t,a)\}\\ &=\max\{\max_{t\in[a,b]} h_1(t),\max_{t\in[a,b]} h_2(t)\}\\ &=\max h_1(t)=\frac{(b^\rho-a^\rho)^{(\delta+\gamma-2)}(\delta-1)^{\delta-1}(\gamma-1)^{\gamma-1}}{(\delta+\gamma-2)^{(\delta+\gamma-2)}}.\end{aligned}$$

It follows that

$$\begin{aligned}|G(t,s)|&\le\frac{(\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}}{(\delta+\gamma-2)^{\delta+\gamma-2}}\cdot\frac{\rho^{1-\delta}s^{\rho-1}(b^\rho-a^\rho)^{\delta+\gamma-2}}{\Gamma(\alpha)(b^\rho-a^\rho)^{\gamma-1}}\\ &=\frac{(\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}(b^\rho-a^\rho)^{\delta-1}\rho^{1-\delta}s^{\rho-1}}{\Gamma(\alpha)(\delta+\gamma-2)^{\delta+\gamma-2}}.\end{aligned}\tag{4.7}$$

Finally, we indicate that (iii) holds. In fact, for any $(t,s)\in[a,b]\times[a,b]$, it can easily be proved that $0\le y_2(t,s)\le y_2(s,s)=y_1(s,s)$. Differentiating the function $y_1(t,s)$ with respect to t, we obtain

$$\frac{\partial y_1(t,s)}{\partial t}=\rho t^{\rho-1}(\gamma-1)(\delta-1)\bigg(-\Big(\frac{b^\rho-s^\rho}{t^\rho-s^\rho}\Big)^{2-\delta}+\Big(\frac{b^\rho-a^\rho}{t^\rho-a^\rho}\Big)^{2-\gamma}\bigg)\le 0,$$

which indicate that $y_1(t,s)$ is increasing with respect to $t \in [s,b]$. It follows that $y_1(b,s) \leq y_1(t,s) \leq y_1(s,s) = y_2(s,s)$, and hence

$$|y_1(t,s)| \leq \max\{\max_{s\in[a,b]} |y_1(b,s)|, \max_{s\in[a,b]} |y_1(s,s)|\}. \tag{4.8}$$

For any $s \in [a,b]$, it is easy to verify that $y_1(b,s)$ is an increasing function. Thus

$$y_1(b,a) \leq y_1(b,s) \leq y_1(b,b), \tag{4.9}$$

$y_1(b,a) = (\delta-\gamma)(b^\rho - s^\rho) \leq 0$, and $y_1(b,b) = (\delta-1)(b^\rho - a^\rho) > 0$. Observe that $|y_1(b,s)| \leq \max\{y_1(b,b), -y_1(b,a)\} = (b^\rho - a^\rho)\max\{\gamma-\delta, \delta-1\}$, which together with (4.8) and (4.9) yields that $|y_1(t,s)| = (b^\rho - a^\rho)\max\{\gamma-\delta, \delta-1\}$. Thus

$$|Y(t,s)| \leq \frac{(b^\rho - s^\rho)^{\delta-2}\rho^{1-\delta}s^{\rho-1}}{\Gamma(\alpha)(\gamma-1)}(b^\rho - a^\rho)\max\{\gamma-\delta, \delta-1\}.$$

This completes the proof. □

4.2. **Lyapunov-type Inequalities for BVPs (1.3) and (1.4).**

Theorem 4.5. *Let* $(\mathbb{A})$ *hold. If* $x(t)$ *is the a nontrivial continuous solution of BVPs* (1.3)*,* $q(t) \in C([a,b],\mathbb{R})$*, then*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)\rho^{\delta-1}(\delta+\gamma-2)^{\delta+\gamma-2}}{\Lambda_1[1+M(b)\sum_{i=1}^{m-2}\omega_i]\max\{a^{\rho-1}, b^{\rho-1}\}}, \tag{4.10}$$

where

$$\Lambda_1 = (\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}(b^\rho - a^\rho)^{\delta-1}.$$

Proof. $x(t)$ is a solution to BVPs (1.3) if and only if it satisfies the integral equation

$$x(t) = \int_a^b G(t,s)q(s)x(s)ds + M(t)\sum_{i=1}^{m-2}\omega_i\int_a^b G(\varphi_i,s)q(s)x(s)ds.$$

Observe that

$$\begin{aligned}
|x(t)| &\leq \int_a^b |G(t,s)||q(s)||x(s)|ds + |M(t)|\sum_{i=1}^{m-2}\omega_i\int_a^b |G(\varphi_i,s)||q(s)||x(s)|ds \\
&\leq \frac{\Lambda_1\rho^{1-\delta}s^{\rho-1}\|x\|_\infty}{\Gamma(\alpha)(\delta+\gamma-2)^{\delta+\gamma-2}}(1+M(b)\textstyle\sum_{i=1}^{m-2}\omega_i)\displaystyle\int_a^b |q(s)|ds \\
&\leq \frac{\Lambda_1\rho^{1-\delta}\max\{a^{\rho-1}, b^{\rho-1}\}\|x\|_\infty}{\Gamma(\alpha)(\delta+\gamma-2)^{\delta+\gamma-2}}(1+M(b)\textstyle\sum_{i=1}^{m-2}\omega_i)\displaystyle\int_a^b |q(s)|ds,
\end{aligned}$$

which implies that (4.10) holds. Otherwise, BVPs (1.3) have a uniqueness solution $x(t) \equiv 0$. This completes the proof. □

The following corollary 4.6 coincides with [15, Theorem 4.1].

Corollary 4.6. *Consider the following Hilfer-Katugampola fractional m-point BVPs*

$$\begin{cases} {}^\rho D_{a^+}^{\alpha,\mu}x(t) + q(t)x(t) = 0, \ 0 < a < t < b, \ 1 < \alpha < 2, \ \rho > 0, \\ x(a) = 0, \ x(b) = \sum\limits_{i=1}^{m-2}\omega_i x(\varphi_i), \end{cases} \tag{4.11}$$

where $q(t) \in C([a,b],\mathbb{R})$ *and* ${}^{\rho}D_{a^+}^{\alpha,\mu}$ *denotes the Hilfer-Katugampola fractional derivative of order* α *and type* μ $(0 \le \mu \le 1)$*. If* (4.11) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds \ge \frac{[2(\alpha-1)+\mu(2-\alpha)]^{2(\alpha-1)+\mu(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Delta_1[1+Q(b)\sum_{i=1}^{m-2}\omega_i]\max\{a^{\rho-1},b^{\rho-1}\}}. \tag{4.12}$$

Proof. Applying Theorem 4.5 with $\alpha=\beta$, $\gamma=\alpha+\mu(2-\alpha)$, and $\delta=\alpha$, one has

$$\Lambda_1 = (\alpha-1)^{\alpha-1}[\alpha-1+\mu(2-\alpha)]^{\alpha-1+\mu(2-\alpha)}(b^\rho-a^\rho)^{\alpha-1} = \Delta_1$$

$$M(b) = \frac{(b^\rho-a^\rho)^{1-(2-\alpha)(1-\mu)}}{(b^\rho-a^\rho)^{1-(2-\alpha)(1-\mu)}-\sum_{i=1}^{m-2}\omega_i(\varphi_i^\rho-a^\rho)^{1-(2-\alpha)(1-\mu)}} = Q(b)$$

Then (4.10) reduce to (4.12). □

The following corollary coincides with [12, Theorem 3.1].

Corollary 4.7. *Consider the following Hilfer fractional m-point BVPs*

$$\begin{cases} D_{a^+}^{\alpha,\mu}x(t)+q(t)x(t)=0,\ t\in(a,b),\ 1<\alpha\le 2,\ 0\le\beta\le 1, \\ x(a)=0,\ \ x(b)=\sum_{i=1}^{m-2}\omega_i x(\varphi_i), \end{cases} \tag{4.13}$$

where $q(t) \in C([a,b],\mathbb{R})$ *and* $D_{a^+}^{\alpha,\mu}$ *denotes the Hilfer fractional derivative of order* α *and type* μ*. If* (4.13) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)}{\tilde{\Delta}_1}\cdot\frac{1}{1+\sum_{i=1}^{m-2}\omega_i T(b)},$$

where

$$\begin{aligned}\tilde{\Delta}_1 &= \lim_{\rho\to1}\frac{\Lambda_1}{\rho^{\delta-1}(\delta+\gamma-2)^{\delta+\gamma-2}} = \frac{(\gamma-1)^{\gamma-1}(\delta-1)^{\delta-1}(b-a)^{\delta-1}}{(\delta+\gamma-2)^{\delta+\gamma-2}} \\ &= \frac{(\alpha-1)^{\alpha-1}(\alpha-1+2\mu-\alpha\mu)^{\alpha-1+2\mu-\alpha\mu}(b-a)^{\alpha-1}}{(2\alpha-2+2\mu-\alpha\mu)^{2\alpha-2+2\mu-\alpha\mu}}, \\ T(b) &= \lim_{\rho\to1}\frac{(b^\rho-a^\rho)^{\gamma-1}}{(b^\rho-a^\rho)^{\gamma-1}-\sum_{i=1}^{m-2}\omega_i(\varphi_i^\rho-a^\rho)^{\gamma-1}} \\ &= \frac{(b-a)^{1-(2-\alpha)(1-\mu)}}{(b-a)^{1-(2-\alpha)(1-\mu)}-\sum_{i=1}^{m-2}\omega_i(\varphi_i-a)^{1-(2-\alpha)(1-\mu)}}.\end{aligned}$$

Proof. Applying Theorem 4.5 with$\alpha=\beta$ and $\rho\to1$, one has

$$\lim_{\rho\to1}\frac{\Gamma(\alpha)\rho^{\delta-1}(\delta+\gamma-2)^{\delta+\gamma-2}}{\Lambda_1[1+M(b)\sum_{i=1}^{m-2}\omega_i]\max\{a^{\rho-1},b^{\rho-1}\}} = \frac{\Gamma(\alpha)}{\tilde{\Delta}_1}\cdot\frac{1}{1+\sum_{i=1}^{m-2}\omega_i T(b)}.$$

□

The following corollary is consistent with in [23, Theorem 5].

Corollary 4.8. *Consider the following Katugampola fractional Dirichlet problem*

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha}x(t)+q(t)x(t)=0,\ 0<a<t<b,\ 1<\alpha<2, \\ x(a)=0,\ \ x(b)=0, \end{cases} \tag{4.14}$$

where $q(t)\in C([a,b],\mathbb{R})$ *and* ${}^{\rho}D_{a^+}^{\alpha}$ *denotes the Katugampola fractional derivative of order* α. *If* (4.14) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|\,ds \geq \frac{\Gamma(\alpha)}{\max\{a^{\rho-1},b^{\rho-1}\}}\Big(\frac{4\rho}{b^{\rho}-a^{\rho}}\Big)^{\alpha-1}. \tag{4.15}$$

Proof. Applying Theorem 4.5 with $\alpha=\beta$, $\mu=0$, and $\omega_i=0$, we obtain

$$\begin{aligned}\int_a^b |q(s)|\,ds \geq & \frac{\Gamma(\alpha)\rho^{\delta-1}(\delta+\gamma-2)^{\delta+\gamma-2}}{\Lambda_1[1+M(b)\sum_{i=1}^{m-2}\omega_i]\max\{a^{\rho-1},b^{\rho-1}\}} \\ = & \frac{\Gamma(\alpha)\rho^{\alpha-1}(2\alpha-2)^{(2\alpha-2)}}{(\alpha-1)^{(2\alpha-2)}(b^{\rho}-a^{\rho})^{\alpha-1}\max\{a^{\rho-1},b^{\rho-1}\}} \\ = & \frac{\Gamma(\alpha)}{\max\{a^{\rho-1},b^{\rho-1}\}}\Big(\frac{4\rho}{b^{\rho}-a^{\rho}}\Big)^{\alpha-1}.\end{aligned}$$

One sees that (4.10) is reduced to (4.15) immediately. □

The following corollary coincides with [18, Theorem 3.7].

Corollary 4.9. *Consider the following Caputo-Hadamard fractional m-point BVPs*

$$\begin{cases} {}_{H}^{C}D_{a^+}^{\alpha}x(t)+q(t)x(t)=0,\ 0<a<t<b,\ 1<\alpha<2, \\ x(a)=0,\ \ x(b)=\sum\limits_{i=1}^{m-2}\omega_i x(\varphi_i), \end{cases} \tag{4.16}$$

where $q(t)\in C([a,b],\mathbb{R})$ *and* ${}_{H}^{C}D_{a^+}^{\alpha}$ *is the Caputo-Hadamard fractional derivative of order* α. *If* (4.16) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|ds \geq \frac{a\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\ln b-\ln a)]^{\alpha-1}}\cdot\frac{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\omega_i\ln\frac{\varphi_i}{a}}{\ln\frac{b}{a}+\sum_{i=1}^{m-2}\omega_i\ln\frac{b}{\varphi_i}}.$$

Proof. Applying Theorem 4.5 with $\alpha=\beta$, $\mu=1$, and $\rho\to 0^+$, we have

$$\begin{aligned}&\lim_{\rho\to 0^+}\frac{[2(\alpha-1)+\mu(2-\alpha)]^{2(\alpha-1)+\mu(2-\alpha)}\Gamma(\alpha)\rho^{\alpha-1}}{\Lambda_1[1+M(b)\sum_{i=1}^{m-2}\varphi_i]\max\{a^{\rho-1},b^{\rho-1}\}} \\ &=\frac{a\alpha^{\alpha}\Gamma(\alpha)}{(\alpha-1)^{\alpha-1}}\lim_{\rho\to 0^+}\frac{\rho^{\alpha-1}}{(b^{\rho}-a^{\rho})^{\alpha-1}}\cdot\lim_{\rho\to 0^+}\frac{(b^{\rho}-a^{\rho})^{\alpha-1}-\sum_{i=1}^{m-2}\omega_i({\varphi_i}^{\rho}-a^{\rho})}{(b^{\rho}-a^{\rho})(1+\sum_{i=1}^{m-2}\varphi_i)-\sum_{i=1}^{m-2}\omega_i({\varphi_i}^{\rho}-a^{\rho})} \\ &\underline{\underline{\text{L'Hospital's rule}}}\frac{a\alpha^{\alpha}\Gamma(\alpha)}{[(\alpha-1)(\ln b-\ln a)]^{\alpha-1}}\cdot\frac{\ln\frac{b}{a}-\sum_{i=1}^{m-2}\omega_i\ln\frac{\varphi_i}{a}}{\ln\frac{b}{a}+\sum_{i=1}^{m-2}\omega_i\ln\frac{b}{\varphi_i}}.\end{aligned}$$

□

The following corollary coincides with [9, Theorem 2].

Corollary 4.10. *Consider the following Hadamard fractional Dirichlet problem*

$$\begin{cases} {}^{H}D^{\alpha}_{a^+}x(t)+q(t)x(t)=0,\ 0<a<t<b,\ 1<\alpha<2, \\ x(a)=0,\ x(b)=0 \end{cases} \tag{4.17}$$

where $q(t)\in C([a,b],\mathbb{R})$ *and* ${}^{H}D^{\alpha}_{a^+}$ *is the Hadamard fractional derivative of order* α. *If* (4.17) *has a nontrivial continuous solution, then*

$$\int_a^b |q(s)|\,ds \geq 4^{(\alpha-1)}a\Gamma(\alpha)\big(\ln\frac{b}{a}\big)^{1-\alpha}.$$

Proof. Let $\alpha=\beta$, $\mu=0$, $\omega_i=0$, and $\rho\to0^+$ in the right-side (4.10). It follows that

$$\begin{aligned}
&\lim_{\rho\to0^+}\frac{\Gamma(\alpha)\rho^{\delta-1}(\delta+\gamma-2)^{\delta+\gamma-2}}{\Lambda_1[1+M(b)\sum_{i=1}^{m-2}\omega_i]\max\{a^{\rho-1},b^{\rho-1}\}}\\
&=4^{(\alpha-1)}a\Gamma(\alpha)\lim_{\rho\to0^+}\frac{\rho^{\alpha-1}}{(b^\rho-a^\rho)^{\alpha-1}}\\
&\underline{\underline{\text{L'Hospital's rule}}}(\ln\frac{b}{a})^{1-\alpha}4^{(\alpha-1)}a\Gamma(\alpha).
\end{aligned}$$

□

Theorem 4.11. *Let* $(\mathbb{B})$ *hold. If* $x(t)$ *is the a nontrivial continuous solution to BVPs* (1.4) *and* $q(t)\in C([a,b],\mathbb{R})$, *then*

$$\int_a^b (b^\rho-s^\rho)^{\delta-2}|q(s)|ds \geq \frac{(\gamma-1)\rho^{\delta-1}\Gamma(\alpha)}{\Lambda_2\left(1+L(b)\sum_{i=1}^{m-2}\lambda_i\right)}, \tag{4.18}$$

where

$$\Lambda_2=(b^\rho-a^\rho)\max\{\gamma-\delta,\delta-1\}\max\{a^{\rho-1},b^{\rho-1}\}.$$

Proof. According to Lemma 4.2, $x(t)$ is a solution to BVPs (1.4) if and only if satisfies the integral equation

$$x(t)=\int_a^b Y(t,s)q(s)x(s)ds+L(t)\sum_{i=1}^{m-2}\lambda_i\int_a^b Y(\eta_i,s)q(s)x(s)ds,\ t\in[a,b].$$

Thus

$$|x(t)|\leq\int_a^b|Y(t,s)||q(s)||x(s)|ds+|L(t)|\sum_{i=1}^{m-2}\lambda_i\int_a^b|Y(\eta_i,s)||q(s)||x(s)|ds.$$

Combining with Lemma 4.4 (iii), one finds

$$\begin{aligned}
\|x\|_\infty \leq&\frac{\rho^{1-\delta}s^{\rho-1}(b^\rho-a^\rho)\max\{\gamma-\delta,\delta-1\}\|x\|_\infty}{\Gamma(\alpha)(\gamma-1)}(1+L(b)\textstyle\sum_{i=1}^{m-2}\lambda_i)\\
&\times\int_a^b(b^\rho-a^\rho)^{\delta-2}|q(s)|ds\\
\leq&\frac{\Lambda_2\rho^{1-\delta}\|x\|_\infty}{\Gamma(\alpha)(\gamma-1)}(1+L(b)\textstyle\sum_{i=1}^{m-2}\lambda_i)\displaystyle\int_a^b(b^\rho-a^\rho)^{\beta-2+\mu(\alpha-\beta)}|q(s)|\,ds,
\end{aligned}$$

which implies that (4.18) holds. Otherwise, BVPs (1.4) have a uniqueness solution $x(t)\equiv0$. Hence, the proof is complete. □

The following corollary is consistent with [15, Theorem 4.2].

Corollary 4.12. *Consider the following Hilfer-Katugampola fractional m-point BVPs*

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha,\mu}x(t)+q(t)x(t)=0,\ t\in(a,b),\ 1<\alpha<2,\ \rho>0, \\ x(a)=0,\ t^{1-\rho}\frac{d}{dt}x(t)|_{t=b}=\sum\limits_{i=1}^{m-2}\lambda_i x(\eta_i), \end{cases} \tag{4.19}$$

where $q(t)\in C([a,b],\mathbb{R})$ *and* ${}^{\rho}D_{a^+}^{\alpha,\mu}$ *denotes the Hilfer-Katugampola fractional derivative of order* α *and type* μ $(0\le\mu\le1)$. *If* (4.19) *has a nontrivial continuous solution, then*

$$\int_a^b (b^\rho-s^\rho)^{\alpha-2}|q(s)|ds \ge \frac{[1-(2-\alpha)(1-\mu)]\rho^{\alpha-1}\Gamma(\alpha)}{\Delta_2[1+R(b)\sum_{i=1}^{m-2}\lambda_i]},$$

where

$$\Delta_2=(b^\rho-a^\rho)\max\{\mu(2-\alpha),\alpha-1\}\max\{a^{\rho-1},b^{\rho-1}\},$$

$$R(b)=\frac{(b^\rho-a^\rho)^{1-(2-\alpha)(1-\mu)}}{[1-(2-\alpha)(1-\mu)]\rho(b^\rho-a^\rho)^{-(2-\alpha)(1-\mu)}-\sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho-a^\rho)^{[1-(2-\alpha)(1-\mu)]}}.$$

Proof. Letting $\alpha=\beta$ in the right-side inequality (4.18), one sees that

$$\begin{aligned}\int_a^b (b^\rho-s^\rho)^{\delta-2}|q(s)|ds &\ge \frac{(\gamma-1)\rho^{\delta-1}\Gamma(\alpha)}{\Lambda_2\left(1+L(b)\sum_{i=1}^{m-2}\lambda_i\right)} \\ &= \frac{[1-(2-\alpha)(1-\mu)]\rho^{\alpha-1}\Gamma(\alpha)}{\Delta_2\left(1+R(b)\sum_{i=1}^{m-2}\lambda_i\right)}.\end{aligned}$$

□

The following result coincides with [15, Corollary 4.5].

Corollary 4.13. *Consider the following Katugampola fractional m-point BVPs*

$$\begin{cases} {}^{\rho}D_{a^+}^{\alpha}x(t)+q(t)x(t)=0,\ t\in(a,b),\ 1<\alpha<2,\ \rho>0, \\ x(a)=0,\ t^{1-\rho}\frac{d}{dt}x(t)|_{t=b}=\sum\limits_{i=1}^{m-2}\lambda_i x(\eta_i), \end{cases} \tag{4.20}$$

where $q(t)\in C([a,b],\mathbb{R})$ *and* ${}^{\rho}D_{a^+}^{\alpha}$ *denotes the Katugampola fractional derivative of order* α. *If* (4.20) *has a nontrivial continuous solution, then*

$$\int_a^b (b^\rho-s^\rho)^{\alpha-2}|q(s)|ds \ge \frac{\rho^{\alpha-1}\Gamma(\alpha)}{\tilde{\Delta}_2(1+H(b)\sum_{i=1}^{m-2}\lambda_i)},$$

where $\tilde{\Delta}_2=(b^\rho-a^\rho)\max\{a^{\rho-1},b^{\rho-1}\}$ *and*

$$H(b)=\frac{(b^\rho-a^\rho)^{\alpha-1}}{(\alpha-1)\rho(b^\rho-a^\rho)^{\alpha-2}-\sum_{i=1}^{m-2}\lambda_i(\eta_i^\rho-a^\rho)^{\alpha-1}}.$$

Proof. Let $\alpha=\beta$ and $\mu=0$ in the right-side inequality (4.18). This completes the proof. □

5. CONCLUSIONS

In this paper, we provided a new definition of the bi-order Hilfer-Katugampola fractional derivative and proved its property, which plays a vital role in the study of BVPs (1.3) and (1.4). Based on this type of fractional calculus, some fractional Lyapunov-type inequalities for m-point BVPs were studied. Comparing with previous work, we obtained new Lyapunov type inequalities that utilize general fractional derivatives. Clearly, there are two special cases of BVPs (1.3) and (1.4). If $\alpha=\beta,\rho\to 0$, then our results are an extension of that recently established in [12]. If $\alpha=\beta$, and then BVPs (1.3) and (1.4) degenerate to the results presented in [15]. On the one hand, if $\alpha=\beta,\rho\to 1$, then BVPs (1.3) is reduced to the result presented in [18]. In addition, if $\alpha=\beta,\mu=0$, then BVPs (1.3) can be viewed as an extension of the results obtained recently in [23]. Finally, as a possible extension directions for fractional Lyapunov-type inequalities for BVPs in the future. The following problems are interesting: the Lyapunov-type inequalities for bi-order Hilfer-Katugampola fractional differential equation under Sturm-Liouville boundary conditions, the Lyapunov-type inequalities for fractional Langevin equations, and the Lyapunov-type inequality for the fractional boundary value problems associated with anti-periodic boundary conditions, and so on.

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