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SUPERCLOSE ANALYSIS OF H^1-GALERKIN MIXED FINITE ELEMENT METHODS COMBINED WITH TWO-GRID SCHEME FOR SEMILINEAR PARABOLIC EQUATIONS

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Abstract. In this paper, we investigate a two-grid scheme for semilinear parabolic equations discretized by H^1-Galerkin mixed finite element method combined with Crank-Nicolson scheme. Based on the interpolation and duality argument technique, we discuss superclose properties for two-grid method and H^1-Galerkin mixed finite element method. The interpolation theory plays an important role in convergence analysis. Theoretical results demonstrate that the two methods have the same convergence order by choosing $h = H^2$. Finally, a numerical example is given to verify the theoretical results.

Keywords. Crank-Nicolson scheme; Mixed finite element; Semilinear parabolic equations; Superclose; Two-grid.

1. Introduction

In this paper, we consider the following semilinear parabolic equations

$$y_t - \Delta y = f(y), \quad (x,t) \in \Omega \times J, \tag{1.1}$$

$$y|_{\partial\Omega} = 0, \quad (x,t) \in \partial\Omega \times J, \tag{1.2}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^2$ is a rectangle with the boundary $\partial\Omega$, $J = (0,T]$, and $f(y) = f(y,x,t)$ is a given real-valued function on Ω. There exists a positive constant M such that

$$|f'(y)| + |f''(y)| \leq M, \; y \in \mathbf{R}.$$

The two-grid method was introduced by Xu [18, 19, 20] and proven to be an effective discretization method for solving nonsymmetric, indefinite and nonlinear partial differential equations. In the recent thirty years, numerous scholars applied various numerical methods combined with two-grid method to solve nonlinear partial differential equations. Bi and Ginting [2]

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designed two-grid finite volume element schemes for the two-dimensional second-order nonselfadjoint and indefinite linear elliptic problems and the two-dimensional second-order nonlinear elliptic problems. Next, they presented a two-grid scheme of symmetric interior penalty discontinuous Galerkin method for a class of quasi-linear elliptic problems in [3]. Shi et al. [17] derived superconvergence analysis of a two-grid finite element method for semilinear parabolic equations. Zhong et al. [21] developed several two-grid methods for the Nédélec edge finite element approximation of the time-harmonic Maxwell equations. Dawson et al. [7] presented a two-level finite difference scheme for the approximation of nonlinear parabolic equations. Rui and Liu [16] proposed a two-grid block-centered finite difference method for solving the two-dimensional Darcy-Forchheimer model describing non-Darcy flow in porous media. Chen et al. [6] considered expanded mixed finite element approximations of semilinear reaction-diffusion equations and designed a two-step two-grid algorithm. Chen and Chen [5] applied two-grid method and mixed finite element method to solve nonlinear reaction diffusion equations. However, we find that there exist a few works on convergence analysis of two-grid method combined with Crank-Nicolson scheme for nonlinear parabolic problems in the literature. Recently, Chen et al. [4] presented a second-order accurate Crank-Nicolson scheme for the two-grid finite element methods of the nonlinear Sobolev equations. Hou et al. [11] investigated a two-grid scheme for semilinear parabolic equations by P_0^2-P_1 mixed finite element methods combined with Crank-Nicolson scheme.

In [15], Pani first presented H^1-Galerkin mixed finite element method to discuss a priori error estimates for parabolic equations. Compared with standard mixed finite element method, the method does not need the inf-sup condition. Moreover, the approximating finite element spaces for the method are allowed to be of different polynomial degree. This paper considers two-grid method of H^1-Galerkin mixed finite element method combined with Crank-Nicolson scheme for the problem (1.1)-(1.3) and gives superclose analysis.

In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum\limits_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum\limits_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p=2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$. We define the space $L^s(J;W^{m,p}(\Omega))$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \left(\int_0^T \|v\|_{W^{m,p}(\Omega)}^s dt \right)^{\frac{1}{s}}$ for $s \in [1,\infty)$, and the standard modification for $s=\infty$. In this paper, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. In addition C denotes a general positive constant.

The plan of this paper is as follows. In Section 2, we construct the fully discretized scheme for problem (1.1)-(1.3). In Section 3, we derive the superclose estimates of order $\mathscr{O}(h^2 + (\Delta t)^2)$. In Section 4, we present the two-grid algorithm and obtain the superclose estimates of order $\mathscr{O}(h^2 + H^4 + (\Delta t)^2)$. Finally, a numerical example is demonstrated to verify the theoretical results in Section 5.

2. Fully Discretized Scheme

Let

$$W = H_0^1(\Omega), \;\; \boldsymbol{V} = H(\mathrm{div}, \Omega) = \left\{ \boldsymbol{v} \in \left(L^2(\Omega)\right)^2 | \mathrm{div}\boldsymbol{v} \in L^2(\Omega) \right\}.$$

Set $\boldsymbol{p} = \nabla y$, as in [15], a mixed weak form of (1.1)-(1.3) can be given by

$$(\boldsymbol{p}_t, \boldsymbol{v}) + (\mathrm{div}\boldsymbol{p}, \mathrm{div}\boldsymbol{v}) = -(f(y), \mathrm{div}\boldsymbol{v}),\ \forall\, \boldsymbol{v} \in \boldsymbol{V}, \tag{2.1}$$

$$(\nabla y, \nabla w) - (\boldsymbol{p}, \nabla w) = 0,\ \forall\, w \in W. \tag{2.2}$$

Let $\boldsymbol{T}_h$ denote a uniform rectangular partition of Ω, h_τ denote the diameter of the element τ, and $h = \max_{\tau \in \boldsymbol{T}_h} h_\tau$. Let $W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$ be the following finite element spaces

$$W_h = \left\{ w_h \in W \cap C^0(\overline{\Omega}) : w_h|_\tau \in Q_{1,1}(\tau),\ \forall\, \tau \in \mathscr{T}_h \right\},$$

$$\boldsymbol{V}_h = \left\{ \boldsymbol{v}_h \in \boldsymbol{V} : \boldsymbol{v}_h|_\tau \in Q_{1,0}(\tau) \times Q_{0,1}(\tau),\ \forall\, \tau \in \mathscr{T}_h \right\},$$

where $Q_{m,n}(\tau)$ indicates the space of polynomials of degree no more than m and n in x_1 and x_2 on element τ, respectively.

Now, we define the Lagrange interpolation operator (see [13]) $R_h : W \to W_h$, which has the approximation property

$$\|\phi - R_h\phi\|_s \le Ch^{2-s}\|\phi\|_2,\ s = 0, 1,\ \forall\, \phi \in H^2(\Omega). \tag{2.3}$$

Furthermore, the following inequality holds (see [12, Corollary 2.1])

$$(\nabla(\phi - R_h\phi), \nabla w_h) \le Ch^2\|\phi\|_3\|\nabla w_h\|,\ \forall\, w_h \in W_h,\ \phi \in H^3(\Omega). \tag{2.4}$$

Next, we recall the Fortin projection (see [8, 9]) $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies: for any $\boldsymbol{q} \in \boldsymbol{V}$

$$(\mathrm{div}(\Pi_h\boldsymbol{q} - \boldsymbol{q}), \mathrm{div}\boldsymbol{v}_h) = 0,\ \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.5}$$

$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\| \le Ch\|\boldsymbol{q}\|_1,\ \forall\, \boldsymbol{q} \in (H^1(\Omega))^2,$$

$$\|\mathrm{div}(\boldsymbol{q} - \Pi_h\boldsymbol{q})\|_{-s} \le Ch^{1+s}\|\mathrm{div}\boldsymbol{q}\|_1,\ s = 0, 1,\ \forall\, \mathrm{div}\boldsymbol{q} \in H^1(\Omega). \tag{2.6}$$

Moreover, the following super-approximation result holds (see [9, Lemma 2.2])

$$(\boldsymbol{q} - \Pi_h\boldsymbol{q}, \boldsymbol{v}_h) \le Ch^2\|\boldsymbol{q}\|_2\|\boldsymbol{v}_h\|,\ \forall\, \boldsymbol{q} \in (H^2(\Omega))^2,\ \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.7}$$

Let $\Delta t > 0$, $N = T/\Delta t \in \mathbb{Z}$, $t_n = n\Delta t$, $\psi^n = \psi(x, t_n)$, and $dt\psi^n = (\psi^n - \psi^{n-1})/\Delta t$. Then the fully discrete scheme is to find $(\boldsymbol{p}_h^n, y_h^n) \in \boldsymbol{V}_h \times W_h, n = 1, 2, \cdots, N$, such that

$$(dt\boldsymbol{p}_h^n, \boldsymbol{v}_h) + \frac{1}{2}(\mathrm{div}(\boldsymbol{p}_h^n + \boldsymbol{p}_h^{n-1}), \mathrm{div}\boldsymbol{v}_h) = -\frac{1}{2}(f(y_h^n) + f(y_h^{n-1}), \mathrm{div}\boldsymbol{v}_h), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.8}$$

$$(\nabla y_h^n, \nabla w_h) - (\boldsymbol{p}_h^n, \nabla w_h) = 0,\ \forall\, w_h \in W_h, \tag{2.9}$$

$$\boldsymbol{p}_h^0 = \Pi_h\boldsymbol{p}_0. \tag{2.10}$$

For the proof of existence and uniqueness of the solution for nonlinear algebraic system (2.8)-(2.10), we refer to [1, 14].

3. Superclose Analysis

Before presenting our main results, we need the following discrete Gronwall's inequality.

Lemma 3.1. [10] *Assume that $\{k_n\}$ and $\{p_n\}$ are nonnegative sequences, and the sequence ϕ_n satisfies*

$$\phi_0 \le g_0,\quad \phi_n \le g_0 + \sum_{l=0}^{n-1} p_l + \sum_{l=0}^{n-1} k_l\phi_l,\quad n \ge 1,$$

where $g_0 \geq 0$. Then $\{\phi_n\}$ satisfies

$$\phi_n \leq \left(g_0 + \sum_{l=0}^{n-1} p_l \right) \exp \left(\sum_{l=0}^{n-1} k_l \right), \quad n \geq 1.$$

Theorem 3.2. *Let $(y, \boldsymbol{p})$ and $(y_h^n, \boldsymbol{p}_h^n)$ be the solutions to (2.1)-(2.2) and (2.8)-(2.10), respectively. Then, for $1 \leq n \leq N$,*

$$\begin{aligned} &\|\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n\| + \|\nabla(R_h y^n - y_h^n)\| \\ &\leq Ch^2(\|\boldsymbol{p}\|_{L^\infty(H^2)} + \|\boldsymbol{p}_t\|_{L^2(H^2)} + \|y\|_{L^\infty(H^3)}) + C(\Delta t)^2 \|\boldsymbol{p}_{ttt}\|_{L^2(L^2)}. \end{aligned} \tag{3.1}$$

Proof. For convenience, let

$$\boldsymbol{\xi}^n = \Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n, \quad \boldsymbol{\eta}^n = \boldsymbol{p}^n - \Pi_h \boldsymbol{p}^n, \rho^n = R_h y^n - y_h^n, \quad \theta^n = y^n - R_h y^n.$$

Using (2.1)-(2.2), (2.8)-(2.9), and (2.5), for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and $w_h \in W_h$, we obtain the following error equations

$$\begin{aligned} &(dt\boldsymbol{\xi}^n, \boldsymbol{v}_h) + \frac{1}{2}(\mathrm{div}(\boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1}), \mathrm{div}\boldsymbol{v}_h) \\ &= (\boldsymbol{\sigma}^n, \boldsymbol{v}_h) - (dt\boldsymbol{\eta}^n, \boldsymbol{v}_h) - \frac{1}{2}(f(y^n) - f(y_h^n) + f(y^{n-1}) - f(y_h^{n-1}), \mathrm{div}\boldsymbol{v}_h), \end{aligned} \tag{3.2}$$

$$(\nabla \rho^n, \nabla w_h) = (\boldsymbol{\xi}^n + \boldsymbol{\eta}^n, \nabla w_h) - (\nabla \theta^n, \nabla w_h), \tag{3.3}$$

where $\boldsymbol{\sigma}^n = dt\boldsymbol{p}^n - (\boldsymbol{p}_t^n + \boldsymbol{p}_t^{n-1})/2$. Choosing $\boldsymbol{v}_h = \boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1}$ in (3.2), multiplying (3.2) by $2\Delta t$, and summing over n from 1 to l, we derive

$$\begin{aligned} &2\sum_{n=1}^{l}(dt\boldsymbol{\xi}^n, \boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1})\Delta t + \sum_{n=1}^{l} \|\mathrm{div}(\boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1})\|^2 \Delta t \\ &= 2\sum_{n=1}^{l}(\boldsymbol{\sigma}^n, \boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1})\Delta t - 2\sum_{n=1}^{l}(dt\boldsymbol{\eta}^n, \boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1})\Delta t \\ &- \sum_{n=1}^{l}(f(y^n) - f(y_h^n), \mathrm{div}(\boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1}))\Delta t \\ &- \sum_{n=1}^{l}(f(y^{n-1}) - f(y_h^{n-1}), \mathrm{div}(\boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1}))\Delta t := \sum_{i=1}^{4} I_i. \end{aligned} \tag{3.4}$$

Since $\boldsymbol{\xi}^0 = 0$, it is easy to see that

$$2\sum_{n=1}^{l}(dt\boldsymbol{\xi}^n, \boldsymbol{\xi}^n + \boldsymbol{\xi}^{n-1})\Delta t = 2\sum_{n=1}^{l}(\|\boldsymbol{\xi}^n\|^2 - \|\boldsymbol{\xi}^{n-1}\|^2) = 2\|\boldsymbol{\xi}^l\|^2. \tag{3.5}$$

Now, we estimate I_1-I_4, respectively. For I_1, from the results given in [4], we have

$$I_1 \leq C(\Delta t)^4 \|\boldsymbol{p}_{ttt}\|_{L^2(L^2)}^2 + C\sum_{n=1}^{l} \|\boldsymbol{\xi}^n\|^2 \Delta t.$$

For I_2, using Cauchy mean value theorem, (2.7), and Cauchy-Schwarz inequality, for some ζ^n, we obtain

$$I_2 = -2\sum_{n=1}^{l}(\boldsymbol{\eta}_t(\zeta^n), \boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1})\Delta t \le Ch^2\sum_{n=1}^{l}\|\boldsymbol{p}_t(\zeta^n)\|_2\|\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1}\|\Delta t$$
$$\le Ch^4\|\boldsymbol{p}_t\|^2_{L^2(H^2)} + C\sum_{n=1}^{l}\|\boldsymbol{\xi}^n\|^2\Delta t. \tag{3.6}$$

For I_3, using Cauchy mean value theorem, Cauchy-Schwarz inequality, the assumption on f, and (2.4), for some $\widetilde{y}^n$, we have

$$I_3 = -\sum_{n=1}^{l}(f'(\widetilde{y}^n)(\rho^n+\theta^n), \mathrm{div}(\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1}))\Delta t$$
$$\le C\sum_{n=1}^{l}\|f'(\widetilde{y}^n)\|_{0,\infty}(\|\rho^n\|+\|\theta^n\|)\cdot\|\mathrm{div}(\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1})\|\Delta t$$
$$\le Ch^4\|y\|^2_{L^2(H^2)} + C\sum_{n=1}^{l}\|\rho^n\|^2\Delta t + \frac{1}{3}\sum_{n=1}^{l}\|\mathrm{div}(\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1})\|^2\Delta t. \tag{3.7}$$

Choosing $w_h = \rho^n$ in (3.3) and using Green' formula, we derive

$$(\nabla\rho^n, \nabla\rho^n) = (\boldsymbol{\xi}^n, \nabla\rho^n) - (\mathrm{div}\boldsymbol{\eta}^n, \rho^n) - (\nabla\theta^n, \nabla\rho^n). \tag{3.8}$$

From (2.6), we see that

$$(\mathrm{div}\boldsymbol{\eta}^n, \rho^n) \le C|\mathrm{div}\boldsymbol{\eta}^n|_{-1}|\rho^n|_1 \le Ch^2\|\mathrm{div}\boldsymbol{p}^n\|_1\|\nabla\rho^n\|. \tag{3.9}$$

Using (3.8)-(3.9), (2.3), Cauchy-Schwarz inequality, and Poincare's inequality, we conclude that

$$\|\rho^n\| \le C\|\nabla\rho^n\| \le Ch^2(\|\mathrm{div}\boldsymbol{p}^n\|_1 + \|y^n\|_3) + C\|\boldsymbol{\xi}^n\|. \tag{3.10}$$

Now, substituting (3.10) into (3.7), we have

$$I_3 \le Ch^4(\|y\|^2_{L^2(H^3)} + \|\boldsymbol{p}\|^2_{L^2(H^2)}) + C\sum_{n=1}^{l}\|\boldsymbol{\xi}^n\|^2\Delta t + \frac{1}{3}\sum_{n=1}^{l}\|\mathrm{div}(\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1})\|^2\Delta t. \tag{3.11}$$

Similar to I_3, we can estimate I_4 as

$$I_4 \le Ch^4(\|y\|^2_{L^2(H^3)} + \|\boldsymbol{p}\|^2_{L^2(H^2)}) + C\sum_{n=1}^{l-1}\|\boldsymbol{\xi}^n\|^2\Delta t + \frac{1}{3}\sum_{n=1}^{l}\|\mathrm{div}(\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1})\|^2\Delta t. \tag{3.12}$$

Using the estimates of I_1-I_4, (3.4)-(3.5) and (3.10), we obtain

$$\|\nabla\rho^l\|^2 + \|\boldsymbol{\xi}^l\|^2 \le Ch^4(\|\boldsymbol{p}\|^2_{L^\infty(H^2)} + \|\boldsymbol{p}_t\|^2_{L^2(H^2)} + \|y\|^2_{L^\infty(H^3)})$$
$$+ C(\Delta t)^4\|\boldsymbol{p}_{ttt}\|^2_{L^2(L^2)} + C\sum_{n=1}^{l}\|\boldsymbol{\xi}^n\|^2\Delta t. \tag{3.13}$$

Finally, for sufficiently small Δt, applying discrete Gronwall's inequality (see Lemma 3.1) to (3.13), we derive (3.1). Thus, the proof of the theorem is ended. □

Theorem 3.3. *Let $(y,\boldsymbol{p})$ and $(y_h^n,\boldsymbol{p}_h^n)$ be the solutions of* (2.1)-(2.2) *and* (2.8)-(2.10)*, respectively. If $h=\Delta t$, then we have*

$$\begin{aligned}&\|\mathrm{div}(\Pi_h\boldsymbol{p}^n-\boldsymbol{p}_h^n)\|\le Ch^2(\|\boldsymbol{p}\|_{L^\infty(H^2)}+\|\boldsymbol{p}_t\|_{L^2(H^2)}+\|y_t\|_{L^2(H^3)}\\&+\|y\|_{L^\infty(H^3)})+C(\Delta t)^2\|\boldsymbol{p}_{ttt}\|_{L^2(L^2)},\ 1\le n\le N.\end{aligned}\tag{3.14}$$

Proof. Letting $\boldsymbol{v}_h=dt\boldsymbol{\xi}^n$ in (3.2), multiplying the resulting equation by Δt and summing over n from 1 to l, we see that

$$\begin{aligned}&\sum_{n=1}^{l}\|dt\boldsymbol{\xi}^n\|^2\Delta t+\frac{1}{2}\sum_{n=1}^{l}(\mathrm{div}(\boldsymbol{\xi}^n+\boldsymbol{\xi}^{n-1}),\mathrm{div}dt\boldsymbol{\xi}^n)\Delta t\\&=\sum_{n=1}^{l}(\boldsymbol{\sigma}^n,dt\boldsymbol{\xi}^n)\Delta t-\sum_{n=1}^{l}(dt\boldsymbol{\eta}^n,dt\boldsymbol{\xi}^n)\Delta t-\frac{1}{2}\sum_{n=1}^{l}(f(y^n)-f(y_h^n),\mathrm{div}dt\boldsymbol{\xi}^n)\Delta t\\&-\frac{1}{2}\sum_{n=1}^{l}(f(y^{n-1})-f(y_h^{n-1}),\mathrm{div}dt\boldsymbol{\xi}^n)\Delta t:=\sum_{i=1}^{4}J_i.\end{aligned}\tag{3.15}$$

Now, we estimate J_1-J_4, respectively. Similar to I_1-I_2, we can estimate J_1 and J_2 as

$$J_1+J_2\le C(\Delta t)^4\|\boldsymbol{p}_{ttt}\|^2_{L^2(L^2)}+Ch^4\|\boldsymbol{p}_t\|^2_{L^2(H^2)}+\frac{1}{2}\sum_{n=1}^{l}\|dt\boldsymbol{\xi}^n\|^2\Delta t.\tag{3.16}$$

For J_3, using Cauchy mean value theorem, we find that

$$\begin{aligned}J_3=&\frac{1}{2}\sum_{n=2}^{l}\left(\frac{f(y^n)-f(y_h^n)-[f(y^{n-1})-f(y_h^{n-1})]}{\Delta t},\mathrm{div}\boldsymbol{\xi}^{n-1}\right)\Delta t-\frac{1}{2}(f(y^l)-f(y_h^l),\mathrm{div}\boldsymbol{\xi}^l)\\=&\frac{1}{2}\sum_{n=2}^{l}\left(\frac{(f'(\widetilde{y}^n)-f'(\widetilde{y}^{n-1}))(y^n-y_h^n)+f'(\widetilde{y}^{n-1})(y^n-y^{n-1}+y_h^{n-1}-y_h^n)}{\Delta t},\mathrm{div}\boldsymbol{\xi}^{n-1}\right)\Delta t\\&-\frac{1}{2}(f(y^l)-f(y_h^l),\mathrm{div}\boldsymbol{\xi}^l)\\=&\frac{1}{2}\sum_{n=2}^{l}\left(\frac{f''(\bar{y}^n)(\widetilde{y}^n-\widetilde{y}^{n-1})(y^n-y_h^n)}{\Delta t},\mathrm{div}\boldsymbol{\xi}^{n-1}\right)\Delta t+\frac{1}{2}\sum_{n=2}^{l}(f'(\widetilde{y}^{n-1})dt\rho^n,\mathrm{div}\boldsymbol{\xi}^{n-1})\Delta t\\&+\frac{1}{2}\sum_{n=2}^{l}(f'(\widetilde{y}^{n-1})dt\theta^n,\mathrm{div}\boldsymbol{\xi}^{n-1})\Delta t-\frac{1}{2}(f(y^l)-f(y_h^l),\mathrm{div}\boldsymbol{\xi}^l):=\sum_{i=1}^{4}M_i,\end{aligned}\tag{3.17}$$

where $\bar{y}^n$ is located between $\widetilde{y}^n$ and $\widetilde{y}^{n-1}$. Notice that

$$\begin{aligned}&|\widetilde{y}^n-\widetilde{y}^{n-1}|\le|y^n-y_h^{n-1}|+|y^n-y^{n-1}|+|y_h^n-y^{n-1}|+|y_h^n-y_h^{n-1}|\\&\le 4|y^n-y^{n-1}|+2|y^n-y_h^n|+2|y^{n-1}-y_h^{n-1}|.\end{aligned}\tag{3.18}$$

Using (3.18), Cauchy mean value inequality, Cauchy-Schwarz inequality, the assumption on f, (3.1) and $h = \Delta t$, we derive

$$\begin{aligned}
M_1 \leq & \frac{1}{2}\sum_{n=2}^{l}\left(\frac{|f''(\bar{y}^n)|\cdot|\widetilde{y}^n-\widetilde{y}^{n-1}|\cdot|y^n-y_h^n|}{\Delta t}, |\mathrm{div}\boldsymbol{\xi}^{n-1}|\right)\Delta t \\
\leq & \sum_{n=2}^{l}\left(\frac{|f''(\bar{y}^n)|(2|y^n-y^{n-1}|+|y^n-y_h^n|+|y^{n-1}-y_h^{n-1}|)|y^n-y_h^n|}{\Delta t}, |\mathrm{div}\boldsymbol{\xi}^{n-1}|\right)\Delta t \\
\leq & C\sum_{n=2}^{l}\|f\|_{2,\infty}\left(\|y_t\|_{0,\infty}\|y^n-y_h^n\|+\frac{\|y^n-y_h^n\|_{0,4}+\|y^{n-1}-y_h^{n-1}\|_{0,4}}{\Delta t}\|y^n-y_h^n\|_{0,4}\right) \\
& \|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t \\
\leq & C\sum_{n=2}^{l}\left(\|y^n-y_h^n\|+\frac{\|\theta^n\|_{0,4}+\|\rho^n\|_{0,4}+\|\theta^{n-1}\|_{0,4}+\|\rho^{n-1}\|_{0,4}}{\Delta t}(\|\theta^n\|_{0,4}+\|\rho^n\|_{0,4})\right) \\
& \|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t \\
\leq & C\sum_{n=2}^{l}\|y^n-y_h^n\|\cdot\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t + C\sum_{n=2}^{l}\frac{h^2+(\Delta t)^2}{\Delta t}(h^2\|y^n\|_{2,4}+\|\rho^n\|_{0,4})\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t \\
\leq & Ch^4\|y\|^2_{L^2(W^{2,4})}+C\sum_{n=2}^{l}\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|^2\Delta t + C\sum_{n=2}^{l}(\|y^n-y_h^n\|^2+\|\nabla\rho^n\|^2)\Delta t, \qquad (3.19)
\end{aligned}$$

where we also used the embedding $\|v\|_{0,4} \leq C\|v\|_1$, interpolation theory and Poincare' inequality.

Taking the difference in time of (3.3) and choosing $w_h = dt\rho^n$ to obtain

$$(\nabla dt\rho^n, \nabla dt\rho^n) = (dt\boldsymbol{\xi}^n, \nabla dt\rho^n) - (\mathrm{div}dt\boldsymbol{\eta}^n, dt\rho^n) - (\nabla dt\theta^n, \nabla dt\rho^n). \qquad (3.20)$$

Similar to (3.10), we derive

$$\|dt\rho^n\| \leq C\|\nabla dt\rho^n\| \leq C\|dt\boldsymbol{\xi}^n\| + Ch^2(\|\mathrm{div}dt\boldsymbol{p}^n\|_1 + \|dty^n\|_3). \qquad (3.21)$$

By use of Cauchy-Schwarz inequality, the assumption on f and (3.21), we conclude that

$$\begin{aligned}
M_2 \leq & \sum_{n=2}^{l}\|f\|_{1,\infty}\|dt\rho^n\|\cdot\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t \\
\leq & C\sum_{n=2}^{l}(\|dt\boldsymbol{\xi}^n\|+h^2\|\mathrm{div}dt\boldsymbol{p}^n\|_1+h^2\|dty^n\|_3)\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t \\
\leq & \frac{1}{4}\sum_{n=2}^{l}\|dt\boldsymbol{\xi}^n\|^2\Delta t + Ch^4\left(\|\mathrm{div}\boldsymbol{p}_t\|^2_{L^2(H^1)}+\|y_t\|^2_{L^2(H^3)}\right)+C\sum_{n=2}^{l}\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|^2\Delta t, \qquad (3.22)
\end{aligned}$$

$$M_3 \leq \sum_{n=2}^{l}\|f\|_{1,\infty}\|dt\theta^n\|\cdot\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|\Delta t \leq Ch^4\|y_t\|^2_{L^2(H^2)} + C\sum_{n=2}^{l}\|\mathrm{div}\boldsymbol{\xi}^{n-1}\|^2\Delta t, \qquad (3.23)$$

$$M_4 = -\frac{1}{2}(f'(\widetilde{y}^l)(y^l-y_h^l), \mathrm{div}\boldsymbol{\xi}^l) \leq C\|y^l-y_h^l\|^2 + \frac{1}{6}\|\mathrm{div}\boldsymbol{\xi}^l\|^2. \qquad (3.24)$$

Now, combining (3.17), (3.19) with (3.22)-(3.24), we get

$$\begin{aligned} J_3 \leq & C\sum_{n=2}^{l} \left\|\mathrm{div}\boldsymbol{\xi}^{n-1}\right\|^2\Delta t + C\sum_{n=2}^{l}(\|y^n-y_h^n\|^2+\|\nabla\rho^n\|^2)\Delta t + \frac{1}{4}\sum_{n=2}^{l}\|dt\boldsymbol{\xi}^n\|^2\Delta t \\ & + Ch^4(\|\mathrm{div}\boldsymbol{p}_t\|^2_{L^2(H^1)}+\|y_t\|^2_{L^2(H^3)}+\|y\|^2_{L^2(H^3)}) + C\|y^l-y_h^l\|^2+\frac{1}{6}\|\mathrm{div}\boldsymbol{\xi}^l\|^2. \end{aligned} \tag{3.25}$$

Similar to J_3, we can estimate J_4 as

$$\begin{aligned} J_4 \leq & Ch^4(\|\mathrm{div}\boldsymbol{p}_t\|^2_{L^2(H^1)}+\|y_t\|^2_{L^2(H^3)}+\|y\|^2_{L^2(H^3)}) + C\|y^{l-1}-y_h^{l-1}\|^2+\frac{1}{6}\|\mathrm{div}\boldsymbol{\xi}^l\|^2 \\ & + C\sum_{n=1}^{l-1}\|\mathrm{div}\boldsymbol{\xi}^n\|^2\Delta t + C\sum_{n=1}^{l-1}(\|y^n-y_h^n\|^2+\|\nabla\rho^n\|^2)\Delta t + \frac{1}{4}\sum_{n=1}^{l-1}\|dt\boldsymbol{\xi}^n\|^2\Delta t. \end{aligned} \tag{3.26}$$

Substituting the estimates of J_1-J_4 into (3.15), using (3.13), (2.4), Poincare's inequality and discrete Gronwall's inequality, we complete the proof of theorem. □

4. Two-Grid Algorithm and Superclose Analysis

In this section, we present the following two-grid algorithm and discuss its convergence.

Step1: On the coarse grid $\mathscr{T}_H$, compute $(\boldsymbol{p}_H^n, y_H^n)$ to satisfy the following original nonlinear system:

$$(dt\boldsymbol{p}_H^n,\boldsymbol{v}_H)+\frac{1}{2}(\mathrm{div}(\boldsymbol{p}_H^n+\boldsymbol{p}_H^{n-1}),\mathrm{div}\boldsymbol{v}_H) = -\frac{1}{2}(f(y_H^n)+f(y_H^{n-1}),\mathrm{div}\boldsymbol{v}_H), \forall\, \boldsymbol{v}_H\in\boldsymbol{V}_H, \tag{4.1}$$

$$(\nabla y_H^n,\nabla w_H)-(\boldsymbol{p}_H^n,\nabla w_H)=0,\ \forall\, w_H\in W_H, \tag{4.2}$$

$$\boldsymbol{p}_H^0=\Pi_H\boldsymbol{p}_0. \tag{4.3}$$

Step2: On the fine grid $\mathscr{T}_h$, compute $(\bar{\boldsymbol{p}}_h^n, \bar{y}_h^n)$ to satisfy the following linear system:

$$(\nabla\bar{y}_h^n,\nabla w_h)-(\bar{\boldsymbol{p}}_h^n,\nabla w_h)=0,\ \forall\, w_h\in W_h, \tag{4.4}$$

$$\begin{aligned} &(dt\bar{\boldsymbol{p}}_h^n,\boldsymbol{v}_h)+\frac{1}{2}(\mathrm{div}(\bar{\boldsymbol{p}}_h^n+\bar{\boldsymbol{p}}_h^{n-1}),\mathrm{div}\boldsymbol{v}_h) = -\frac{1}{2}(f(y_H^n)+f'(y_H^n)(\bar{y}_h^n-y_H^n),\mathrm{div}\boldsymbol{v}_h) \\ &-\frac{1}{2}(f(y_H^{n-1})+f'(y_H^{n-1})(\bar{y}_h^{n-1}-y_H^{n-1}),\mathrm{div}\boldsymbol{v}_h),\ \forall\, \boldsymbol{v}_h\in\boldsymbol{V}_h, \end{aligned} \tag{4.5}$$

$$\bar{\boldsymbol{p}}_h^0=\Pi_h\boldsymbol{p}_0. \tag{4.6}$$

Theorem 4.1. *Let $(y,\boldsymbol{p})$ and $(\bar{y}_h^n,\bar{\boldsymbol{p}}_h^n)$ be the solutions to (2.1)-(2.2) and (4.1)-(4.6), respectively. Assume that $y\in L^\infty(H^3)$, $\boldsymbol{p}\in L^\infty(H^2)$, $\boldsymbol{p}_t\in L^2(H^2)$, and $\boldsymbol{p}_{ttt}\in L^2(L^2)$. Then, for Δt small enough and $1\leq n\leq N$,*

$$\|\Pi_h\boldsymbol{p}^n-\bar{\boldsymbol{p}}_h^n\|+\|\nabla(R_hy^n-\bar{y}_h^n)\|\leq C(H^4+h^2+(\Delta t)^2). \tag{4.7}$$

Proof. Let $\alpha^n = \Pi_h \boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n$ and $\gamma^n = R_h y^n - \bar{y}_h^n$. Similar to (3.4)-(3.5), we conclude that

$$
\begin{aligned}
&2\|\alpha^l\|^2 + \sum_{n=1}^{l} \|\mathrm{div}(\alpha^n + \alpha^{n-1})\|^2 \Delta t \\
&\le - \sum_{n=1}^{l} (f(y^{n-1}) - f(y_H^{n-1}) - f'(y_H^{n-1})(\bar{y}_h^{n-1} - y_H^{n-1}), \mathrm{div}(\alpha^n + \alpha^{n-1}))\Delta t \\
&+ 2\sum_{n=1}^{l} (\boldsymbol{\sigma}^n - dt\boldsymbol{\eta}^n, \alpha^n + \alpha^{n-1})\Delta t \\
&- \sum_{n=1}^{l} (f(y^n) - f(y_H^n) - f'(y_H^n)(\bar{y}_h^n - y_H^n), \mathrm{div}(\alpha^n + \alpha^{n-1}))\Delta t := \sum_{i=1}^{3} K_i. \qquad (4.8)
\end{aligned}
$$

From [6, Lemma 4.2], we know that for some function $\hat{y}^n$

$$
f(y^n) - f(y_H^n) - f'(y_H^n)(\bar{y}_h^n - y_H^n) = f'(y_H^n)(\theta^n + \gamma^n) + f''(\hat{y}^n)(y^n - y_H^n)^2/2. \qquad (4.9)
$$

Similar to (3.8)-(3.10), we have

$$
\|\gamma^n\| \le C\|\nabla\gamma^n\| \le Ch^2(\|\mathrm{div}\boldsymbol{p}^n\|_1 + \|y^n\|_3) + C\|\alpha^n\|. \qquad (4.10)
$$

Thus, due to (4.9)-(4.10), the assumption on f, and (2.4), we find that

$$
\begin{aligned}
K_1 + K_3 &\le Ch^4(\|y\|_{L^2(H^3)}^2 + \|\boldsymbol{p}\|_{L^2(H^2)}^2) + C\sum_{n=1}^{l} \|\alpha^n\|^2 \Delta t \\
&+ C\sum_{n=0}^{l} \|(y^n - y_H^n)^2\|^2 \Delta t + \sum_{n=1}^{l} \|\mathrm{div}(\alpha^n + \alpha^{n-1})\|^2 \Delta t. \qquad (4.11)
\end{aligned}
$$

As in (3.19), using the embedding $\|v\|_{0,4} \le C\|v\|_1$ and interpolation theory, we see that

$$
\begin{aligned}
\|(y^n - y_H^n)^2\|^2 &= \|y^n - y_H^n\|_{0,4}^4 \\
&\le (\|y^n - R_H y^n\|_{0,4} + \|R_H y^n - y_H^n\|_{0,4})^4 \\
&\le C(H^2\|y^n\|_{2,4} + \|R_H y^n - y_H^n\|_1)^4. \qquad (4.12)
\end{aligned}
$$

Now, combining (4.8), (4.11)-(4.12), Poincare's inequality with the estimates of I_1-I_2, we obtain

$$
\|\alpha^l\|^2 \le C(h^4 + H^8 + (\Delta t)^4) + C\sum_{n=0}^{l} \|\nabla(R_H y^n - y_H^n)\|^4 \Delta t + C\sum_{n=1}^{l} \|\alpha^n\|^2 \Delta t. \qquad (4.13)
$$

For sufficiently small Δt, applying discrete Gronwall's inequality to (4.13), using (3.1) and (4.10), we complete the proof of theorem. □

5. Numerical Experiments

In this section, we verify the theoretical results by a numerical example. We consider the following semilinear parabolic equation

$$
y_t - \Delta y = f(y), \quad (x,t) \in \Omega \times J, \qquad (5.1)
$$

$$
y|_{\partial\Omega} = 0, \quad (x,t) \in \partial\Omega \times J, \qquad (5.2)
$$

$$
y(x,0) = y_0(x), \quad x \in \Omega, \qquad (5.3)
$$

with $\Omega = (0,1)^2$ and $J = (0,1]$. We choose the right function

$$f(y) = (1+8\pi^2)e^t \sin(2\pi x_1)\sin(2\pi x_2) - e^{e^t \sin(2\pi x_1)\sin(2\pi x_2)} + e^y$$

and the exact solution

$$y(x,t) = e^t \sin(2\pi x_1)\sin(2\pi x_2).$$

In Table 1, we show the numerical errors of $\|\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n\|$, $\|\mathrm{div}(\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n)\|$ and $\|\nabla(R_h y^n - y_h^n)\|$ solved by mixed finite element method (MFEM) with $h = \Delta t$. Next, we consider the two-grid algorithm with $h = \Delta t = H^2$, and we show the numerical errors of $\|\Pi_h \boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n\|$ and $\|\nabla(R_h y^n - \bar{y}_h^n)\|$ in Table 2. The convergence order for errors are also displayed in the two tables. We can easily see from Table 1 and Table 2 that the two-grid method and the mixed finite element method have the same convergence order. These numerical results are coincide with the theoretical analysis.

$h=\Delta t$	$\|\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n\|$	Rate	$\|\mathrm{div}(\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n)\|$	Rate	$\|\nabla(R_h y^n - y_h^n)\|$	Rate
1/16	7.8051e-3	-	6.1360e-2		9.8605e-2	-
1/36	1.6126e-3	1.94	1.2861e-2	1.92	1.9635e-2	1.99
1/64	5.1400e-4	1.99	4.1047e-3	1.98	6.2211e-3	2.00
1/100	2.1096e-4	2.00	1.6852e-3	1.99	2.5491e-3	2.00

Table 1. Errors of $\|\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n\|$, $\|\mathrm{div}(\Pi_h \boldsymbol{p}^n - \boldsymbol{p}_h^n)\|$ and $\|\nabla(R_h y^n - y_h^n)\|$ by MFEM with $n = N/2$.

H	$h=\Delta t$	$\|\Pi_h \boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n\|$	Rate	$\|\nabla(R_h y^n - \bar{y}_h^n)\|$	Rate
1/4	1/16	2.2606e-2	-	1.0841e-1	-
1/6	1/36	8.2922e-3	1.24	2.4285e-2	1.84
1/8	1/64	3.1529e-3	1.68	8.0980e-3	1.91
1/10	1/100	1.4219e-3	1.78	3.4166e-3	1.93

Table 2. Errors of $\|\Pi_h \boldsymbol{p}^n - \bar{\boldsymbol{p}}_h^n\|$ and $\|\nabla(R_h y^n - \bar{y}_h^n)\|$ by two-grid algorithm with $n = N/2$.

In Tables 3-4, we compare the computing time of two methods at the same time level, where the computing time of two-grid method at every time level is equal to the sum of the computing time of (4.1)-(4.2) and the computing time of (4.4)-(4.5). We can see that the computing time for two-grid method is significantly less than that for MFEM.

Time level	Computing Time(MFEM)	Computing Time(Two-grid)
4	15s	5s
8	12s	5s
12	15s	2s
16	8s	5s

Table 3. The computing time of MFEM ($h = \Delta t = 1/16$) and two-grid algorithm ($H = 1/4, h = \Delta t = 1/16$).

Time level	Computing Time(MFEM)	Computing Time(Two-grid)
8	1046s	417s
16	1063s	397s
24	1078s	409s
32	1034s	381s
40	1061s	498s
48	1073s	460s
56	1214s	502s
64	1131s	523s

Table 4. The computing time of MFEM ($h = \Delta t = 1/64$) and two-grid algorithm ($H = 1/8, h = \Delta t = 1/64$).

At last, in Figures 1-3, we plot the profiles of the exact solution y, the mixed finite element solution of y and the two-grid solution of y on 36×36 triangle mesh at $t = 0.5$ respectively.

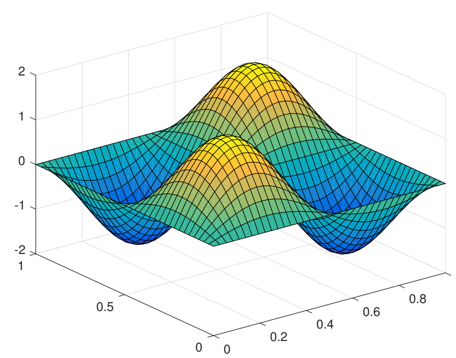

Figure 1. The profile of the exact solution y on 36×36 triangle mesh at $t = 0.5$.

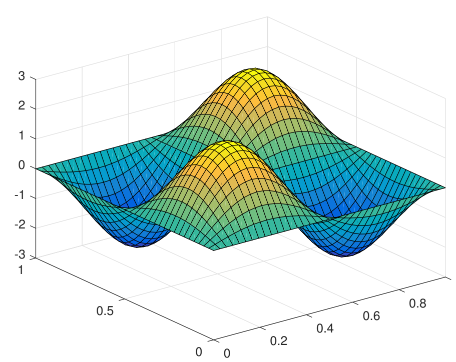

Figure 2. The profile of the mixed finite element solution of y on 36×36 triangle mesh at $t = 0.5$.

All numerical experiments performance on a PC with Intel(R) Core(TM) i7-8550U CPU 2.00 GHz processor, 8GB main memory and Window operating system.

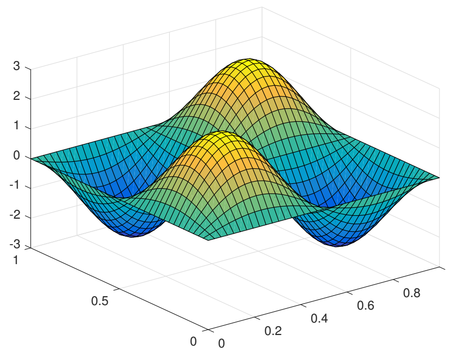

Figure 3. The profile of the two-grid solution of y on 36×36 triangle mesh at $t = 0.5$.

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