J. Nonlinear Funct. Anal. 2023 (2023) 30 https://doi.org/10.23952/jnfa.2023.30

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

# STRONG CONVERGENCE ANALYSIS FOR SOLVING QUASI-MONOTONE VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS IN REFLEXIVE BANACH SPACES

A.O.E. OWOLABI[1], O.T. MEWOMO[1], J.C. YAO[2,3,*], X. QIN[4]

[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Durban, South Africa
[2]Center for General Education, China Medical University, Taichung, Taiwan
[3]Academy of Romanian Scientists, 50044 Bucharest, Romania
[4]Center for Advanced Information Technology, Kyung Hee University, Seoul 02447, Korea

**Abstract.** In this paper, we propose a modified inertial hybrid Tseng's extragradient algorithm with self-adaptive step sizes for finding a common solution of variational inequalities with quasimonotone operators and the fixed point problems of a finite family of Bregman quasi-nonexpansive mappings. By using the Bregman-distance approach, we prove a strong convergence result under some appropriate conditions on the control parameters in real reflexive Banach spaces. Our algorithm is based on a self-adaptive step size which generates a non-monotonic sequence. Unlike the existing results in the literature, our algorithm does not require any linesearch technique which uses inner loops and might consume additional computational time for determining the step size. Finally, we present some numerical examples to illustrate the efficiency of our algorithm in comparison with related methods in the literature.

**Keywords.** Bregman distance; Efficiency of algorithms; Numerical experiments; Quasi-monotone variational inequalities; Self-adaptive step sizes.

## 1. Introduction

Let $C$ be a convex and closed subset of a real Banach space $X$ with induced norm $\|\cdot\|$, and let $X^*$ be the dual of space $X$. Let $B : X \to X^*$ be a single-valued mapping. The variational inequality problem (VIP) is to find $\bar{x} \in C$ such that

$$\langle B\bar{x}, x-\bar{x}\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

We denote the solution set of (1.1) by $VI(C,B)$. The dual variational inequality problem (DVIP) also called Minty variational inequality problem is defined as follows: Find a point $\bar{x} \in C$ such that

$$\langle Bx, x-\bar{x}\rangle \geq 0, \ \forall x \in C. \tag{1.2}$$

*Corresponding author.
E-mail address: yaojc@mail.cmu.edu.tw (J.C. Yao).
Received June 14, 2023; Accepted September 2, 2023.

We denote the solution set of DVIP (1.2) by $VI(C,B)^m$. It is known that if $C$ is convex and $B$ is continuous, then $VI(C,B)^m$ is a subset of $VI(C,B)$. If $B$ is a pseudomonotone and continuous mapping, then $VI(C,B) = VI(C,B)^m$ (see [8]). However, the inclusion $VI(C,B) \subset VI(C,B)^m$ is false if $B$ is quasimonotone and continuous mapping (see [34]).

Generally, there are two main approaches for approximating the solutions of the VIP under suitable conditions. They are the projection-based methods and the regularized methods. In this article, we study the projection method. For the projection-based methods, projected-gradient method is simple and efficient, which is presented as follows: $x_1 \in C$ and $x_{n+1} = P_C(x_n - \lambda Bx_n)$ for all $n \geq 1$, where $P_C$ is the projection onto the subset and closed set $C$ of $H$ and $\lambda$ is a positive regularized constant. Based on the projected-gradient method, Korpelevich [14] introduced the following extragradient method (EGM) in order to overcome the drawback (the restriction on operator $B$) of the projected-gradient method for solving the VIP in a finite dimensional Euclidean space $\mathbb{R}^m$:

$$\begin{cases} x_0 \in C, y_n = P_C(x_n - \lambda Bx_n), \\ x_{n+1} = P_C(x_n - \lambda By_n), \quad n \geq 1, \end{cases} \tag{1.3}$$

where $C \subset \mathbb{R}^m$ is a convex and closed set, $B : C \to \mathbb{R}^m$ is monotone and $L$-Lipschitz continuous, and $\lambda \in \left(0, \frac{1}{L}\right)$. He demonstrated that the sequence generated by (1.3) converges weakly to a solution of the VIP in a finite dimensional space. It is well-known that the EGM requires computation of two projections onto set $C$ in every iteration. This is difficult to calculate when C is a general closed-convex set and the efficiency of the method is seriously affected accordingly. In order to overcome this drawback, Censor et al. [6] (see also [7]) proposed the following Subgradient Extragradient Method (SEGM) which presents a modification via a half-space. The SEGM is given as follows:

$$\begin{cases} x_0 \in H, y_n = P_C(x_n - \lambda Bx_n), \\ H_n = \{z \in H : \langle x_n - \lambda Bx_n - y_n, z - y_n \rangle \leq 0\}, \\ x_{n+1} = P_{H_n}(x_n - \lambda By_n). \end{cases}$$

They proved that, provided the solution set $VI(C,B) \neq \emptyset$, the sequence $\{x_n\}$ generated by the SEGM converges weakly to an element $p \in VI(C,B)$, where $p = \lim_{n\to\infty} P_{VI(C,B)}x_n$.

In [30], Tseng proposed the following iterative scheme, known as the Tseng's extragradient method (TEGM) in order to overcome the drawback in EGM :

$$\begin{cases} x_0 \in H, y_n = P_C(x_n - \lambda Bx_n), \\ x_{n+1} = y_n - \lambda(By_n - Bx_n), \end{cases}$$

where $B$ is a monotone and Lipschitz continuous operator and $\lambda \in \left(0, \frac{1}{L}\right)$. It is clear that the TEGM requires one projection to be calculated per iteration and hence has an advantage in computing projection over the EGM.

The concept of inertial technique introduced by Polyak [20] plays a vital role in speeding up the convergence rate of various iterative algorithms (see [3, 9, 18, 19, 31]). This technique originates from an implicit discretization method of the second-order dynamical systems in solving the smooth convex minimization problem. Alvarez and Attouch [1] employed the idea of the heavy-ball method for finding a zero point of a maximal monotone operator :

$$\begin{cases} x_0, x_1 \in H,\ y_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J^B_{\lambda_n} y_n, \end{cases} \tag{1.4}$$

where $J^B_{\lambda_n}$ is the resolvent operator of $B$, $\lambda_n > 0$, and $\theta_n(x_n - x_{n-1})$ is called the inertial extrapolation with $\theta_n \in [0,1)$. They proved that if $\{\lambda_n\}$ is increasing and $\theta_n \in [0,1)$ is selected so that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to a zero point of $B$.

In [29], Thong et al. employed the inertial technique for solving the monotone VIP in real Hilbert spaces. Their hybrid projection method is presented as follows:

$$\begin{cases} x_0, x_1 \in C,\ u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(u_n - \lambda B u_n), \\ z_n = \alpha_n u_n + (1-\alpha_n)(y_n - \lambda(By_n - Bu_n)), \\ E_n = \{w \in H : \|z_n - w\| \le \|u_n - w\|\}, \\ H_n = \{w \in H : \langle w - x_n, x_1 - x_n \rangle \le 0\}, \\ x_{n+1} = P_{E_n \cap H_n} x_0, \end{cases} \tag{1.5}$$

where $\{\alpha_n\} \subset [0,1)$ with $0 \le \alpha_n < \alpha < 1$, and $\{\theta_n\}$ is a bounded real sequence. It was demonstrated that the sequences $\{x_n\}$ generated by (1.5) converges to an element in $VI(C,B)$ in norm provided that $\lambda \in \left(0, \frac{1}{L}\right)$.

It is observed that the cost operator, in most of current literatures, is monotone and Lipschitz continuous. Recently, several authors (see, e.g., [10, 35]) proposed and studied pseudo-monotone VIPs. Furthermore, it would be highly interesting to extend or broaden the study of a class of monotone and pseudo-monotone VIPs to a more general class of quasi-monotone VIPs. In this article, we focus on the class of quasi-monotone VIPs.

A double projection algorithm for solving quasi-monotone variational inequalities in the finite dimensional Euclidean space $\mathbb{R}^m$ was introduced by Ye and He [34]. Salahuddin [26] improved on the result of EGM to solve a VIP with quasi-monotone and Lipschitz continuous operators. The Algorithm is presented as follows:

**Algorithm 1.1.**

**Data:** $x_0 \in C$ and $\{\lambda_n\} \in [a,b]$, where $0 < a \le b < \frac{1}{\xi}$.

**Step 0:** set $n = 0$.

**Step 1:** If $x_n = P_C(x_n - \lambda_n B x_n)$, then stop.

**Step 2:** Otherwise, set

$$x_{-n} = P_C(x_n - \lambda_n B x_n)) \text{ and } x_{n+1} = P_C(x_n - \lambda_n B x_{-n})).$$

Set $n := n+1$ and go to **Step 1**.

where $C$ is a convex and closed subset of a real Hilbert space $H$ and $B$ is quasi-monotone, Lipschitz continuous, and sequentially weakly continuous. It was proved that if the solution set, $\Gamma(C,B)$, is nonempty, then the sequence generated by Algorithm (1.1) converges weakly to a solution of $VI(C,B)$.

Recently, Liu and Yang [15] proposed the following iterative algorithm for solving a quasi-monotone variational inequality problem in infinite dimensional Hilbert spaces:

**Algorithm 1.2.**

**Step 0:** Set $\lambda_1 > 0$, $x_1 \in H$, and $0 < \sigma < 1$. Select a non-negative real sequence $\{\rho_n\}$ such that $\sum_{n=1}^{\infty} \rho_n < +\infty$.
**Step 1:** Given the current iterate $x_n$, compute $y_n = P_C(x_n - \lambda_n Bx_n)$. If $x_n = y_n$, (or $By_n = 0$), then stop and $y_n$ is a solution to the VIP. Otherwise, go to **Step 2.**
**Step 2:**Compute $x_{n+1} = y_n - \lambda_n(By_n - Bx_n)$, where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n - y_n\|}{\|Bw_n - By_n\|},\ \lambda_n + \rho_n\}, & Bw_n - By_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go to **Step 1**.

Under some appropriate conditions, the authors proved that the sequence generated by the Algorithm 1.2 converges weakly.

Inspired by Tseng's extragradient method, Wairojjana et al. [32] proposed an iterative algorithm based on the TEGM method for solving a quasi-monotone VIP. Their proposed method does not require prior knowledge of the Lipschitz constant of the operator, and it is presented as follows:

**Algorithm 1.3.**

**Step 0:** Select $x_0 \in C$ and $0 < \lambda < \frac{1}{L}$.
**Step 1:** Compute $y_n = P_C(x_n - \lambda_n Bx_n)$. If $x_n = y_n$, then stop and $y_n$ is a solution to the VIP. Otherwise, go to **Step 2.**
**Step 2:** Compute $x_{n+1} = y_n + \lambda_n[Bx_n - By_n]$,
Set $n := n+1$ and go to **Step 1**.

where $B$ is quasi-monotone, Lipschitz continuous and sequentially weakly continuous. A weak convergent result was obtained under suitable conditions.

The following iterative algorithm for solving quasi-monotone VIP in the frame work of Hilbert spaces was introduced by Chinedu et al. [12] to improve the result of Liu and Yang [15]:

**Algorithm 1.4.**

**Step 0:** let $\lambda_0, \lambda_1 > 0$, $x_1 \in H$, and $\sigma \in \left(\delta, \frac{1-2\delta}{2}\right)$, where $\delta \in \left(0, \frac{1}{4}\right)$. Select a non-negative real sequence $\{\rho_n\}$ such that $\sum_{n=1}^{\infty} \rho_n < +\infty$. Let $x_0, x_1 \in C$ be given starting points. Set $n := 1$.
**Step 1:** Compute $x_{n+1} = P_C(x_n - ((\lambda_n + \lambda_{n-1})Bx_n - \lambda_{n-1}Bx_{n-1})), n \geq 1$, where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|x_n - x_{n+1}\|}{\|Bx_n - Bx_{n+1}\|},\ \lambda_n + \rho_n\}, & \text{if } Bx_n \neq Bx_{n+1}, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

where $B$ is quasi-monotone and Lipschitz continuous. Under some certain conditions, the authors proved the weak convergence of the proposed algorithm.

Recently, an inertial Tseng's extragradient method with the viscosity technique was introduced by Alakoya et al. [2] for solving a quasi-monotone VIP in the frame work of Hilbert spaces. Their algorithm is presented as follows:

**Algorithm 1.5.**

**Step 1:** Select initial point $x_0, x_1 \in H_1$. Given the iterates $x_{n-1}$ and $x_n$ for each $n \geq 1$, choose $\theta_n$ such that $0 < \theta_n < \tilde{\theta}_n$, where

$$\tilde{\theta}_n = \begin{cases} \min\{\theta, \frac{\xi_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

**Step 2:** Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$
**Step 3:** Compute $y_n = P_C(w_n - \lambda_n B w_n)$. If $w_n = y_n$(or $By_n = 0$), then stop: $w_n$ is a solution of the VIP. Otherwise, go to **Step 4.**
**Step 4:** $z_n = y_n + \lambda_n(Bw_n - By_n)$.
**Step 5:** Compute $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)z_n$,
where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n - y_n\|}{\|Bw_n - By_n\|},\ \lambda_n + \rho_n\}, & \text{if } Bw_n - By_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to **Step 1.**

Furthermore, Mewomo et al. [16] extended the work of Alakoya et al. [2] from Hilbert spaces to 2-uniformly convex and uniformly smooth Banach spaces by proposing the following iterative algorithm:

**Algorithm 1.6.**

**Step 1:** Let $x_0, x_1 \in H_1$ be two arbitrary initial points and set $n = 1$. Given $(n-1)$ and $nth$ iterates, choose $\theta_n$ such that $0 < \theta_n < \tilde{\theta}_n$, where

$$\tilde{\theta}_n = \begin{cases} \min\{\theta, \frac{\xi_n}{\|x_n - x_{n-1}\|}\}, & \text{if } x_n \neq x_{n-1}, \\ \theta, & \text{otherwise.} \end{cases}$$

**Step 2:** Compute $w_n = J^{-1}(Jx_n + \theta_n(Jx_{n-1} - Jx_n))$
**Step 3:** Compute $y_n = \Pi_C J^{-1}(Jw_n - \lambda_n B w_n)$. If $w_n = y_n$(or $By_n = 0$), then stop: $w_n$ is a solution of the VIP. Otherwise, go to **Step 4.**
**Step 4:** $z_n = J^{-1}(Jy_n + \lambda_n(By_n - Bw_n))$.
**Step 5:** Compute $x_{n+1} = J^{-1}(\alpha_n Jq_n + (1-\alpha_n)Jz_n)$, where

$$\lambda_{n+1} = \begin{cases} \min\{\frac{(\sigma+\sigma_n)\|w_n - y_n\|}{\|Bw_n - By_n\|},\ \lambda_n + \rho_n\}, & \text{if } Bw_n - By_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

Set $n := n+1$ and go back to **Step 1.**

Here $\{\sigma_n\}$ and $\{\rho_n\}$ are non-negative sequences such that $\lim_{n\to\infty} \sigma_n = 0$ and $\sum_{n=1}^{\infty} \rho_n < \infty$. It is observed that the cost operator is non-Lipschitz quasi-monotone and uniformly continuous. Also, a strong convergent result was obtained under some appropriate conditions.

Recently, several new iterative algorithms with the Bregman distance were proposed to solve the VIP and other related problems. In 2023, Wang et al. [33] introduced three inertial like-algorithms with the Bregman distance based on Tseng's extragradient method, the extragradient method, and the subgradient extragradient method, for solving a quasi-monotone VIP in real Hilbert spaces. They obtained the weak convergence of these methods accordingly.

It is observed that several of the existing works on quasi-monotone VIPs in the literature are the ones whose cost operator is non-Lipschitz. In addition, most of the existing works on quasi-monotone VIPs are restricted to the framework of Hilbert spaces. As a result of this, we prove our results in the framework of Banach spaces. To the best of our knowledge, there are no existing works on solving common solutions of quasi-monotone VIPs (where the cost operator is non-Lipschitz) and fixed point problems in reflexive Banach spaces. Hence, it is pertinent to ask the following research-based question:

Is it possible to find a common solution of non-Lipschitz quasi-monotone variational inequalities with fixed-point restrictions in reflexive Banach spaces? In this paper, we propose and study an inertial algorithm which combines the Tseng's extragradient method with the hybrid projection technique for approximating a solution of the VIP with non-Lipschitz quasimonotone operators and fixed point problems of a finite family of Bregman quasi-nonexpansive mappings We establish a strong convergence theorem in reflexive Banach spaces. Finally, we present some numerical examples to illustrate the efficacy of our algorithm as well as compare it with some of the existing works in the literature. In Section 2, we recall some basic definitions and lemmas that are relevant in establishing our main result. In Section 3, we present our proposed method and highlight some of its important features, while in Section 4, we present some lemmas that are useful in proving the strong convergence theorem and then prove our strong convergence theorem. In Section 5, we present some numerical examples to illustrate the performance of our method and compare it with some related methods in the literature. Finally, in Section 6, we give a concluding remark.

## 2. Preliminaries

In this section, we recall some useful lemmas and definitions required to establish our result. We denote the strong convergence of a sequence $\{x_n\}$ to $x$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$. Let $C$ be a convex and closed subset of a real Banach space $X$. Let $X^*$ and $\langle \cdot, \cdot \rangle$ denote the dual space of $X$ and the duality pairing between elements of $X$ and $X^*$, respectively. Let $S_X$ be the unit sphere of $X$. Let $g : X \to (-\infty, \infty]$ be a proper, lower semi-continuous and convex function and let its domain be defined by dom $g := \{x \in X : g(x) < \infty\} \neq \emptyset$. Let $x \in \text{int}(dom\ g)$. For any $y \in X$, the directional derivative of $g$ at $x$ denoted by $g^0(x, y)$ is defined by

$$g^0(x,y) := \lim_{t \to 0^+} \frac{g(x+ty) - g(x)}{t}. \tag{2.1}$$

If the limit at $t \to 0^+$ in (2.1) exists for each $y$, then n $g$ is said to be Gâteaux differentiable at $x$. In this case $g^0(x,y) = \langle \nabla g(x), y \rangle$ (or $g'(x)$), where $\nabla g(x)$ is the value of the gradient of $g$ at $x$. Moreover, when the limit in (2.1) holds uniformly for any $y \in S_X$ and $x \in$ int(dom g), one says that $g$ is Fréchet differentiable.

The Fenchel conjugate of $g$ is the function $g^* : X^* \to \mathbb{R}$ defined by

$$g^*(x^*) = \sup\{\langle x, x^* \rangle - g(x) : x \in X\},\ \forall x^* \in X^*.$$

A function $g$ is called a Legendre function if and only if the following conditions are satisfied:

(i) $\text{int}(\text{dom}\ g) \neq \emptyset, \text{dom}\nabla g = \text{int}(\text{dom}\ g)$ and $g$ is G$\hat{a}$teaux differentiable;

(ii) $\text{int}(\text{dom}\ g^*) \neq \emptyset, \text{dom}\nabla g^* = \text{int}(\text{dom} g^*)$ and $g^*$ is G$\hat{a}$teaux differentiable.

Let $g : X \to \mathbb{R}$ be a Legendre function. Let $V_g : X \times X^* \to [0, \infty)$ associated with $g$ be defined by

$$V_g(x,x^*) = g(x) - \langle x, x^* \rangle + g^*(x^*), \forall x \in X, x^* \in X^*. \tag{2.2}$$

It is clear from (2.2) that $V_g$ is non-negative and $V_g(x,x^*) = D_g(x, \nabla g^*(x^*))$. Moreover, $V_g$ satisfies the following inequality (see [27]):

$$V_g(x,x^*) + \langle y^*, \nabla g^*(x^*) - x \rangle \leq V_g(x, x^* + y^*), \forall x \in X, x^*, y^* \in X^*.$$

**Definition 2.1.** Let $g : X \to \mathbb{R} \cup \{+\infty\}$ be a G$\hat{a}$teaux differentiable function. The Bregman distance $D_g : dom\ g \times \text{int}(dom\ g) \to \mathbb{R}$ is defined by

$$D_g(x,y) = g(x) - g(y) - \langle x - y, \nabla g(y)\rangle, \forall x \in dom\ g, y \in \text{int}(dom\ g). \tag{2.3}$$

It is known that $D_g$ is not a real metric. In addition,

$$D_g(x,y) + D_g(y,x) = \langle x - y, \nabla g(x) - \nabla g(y)\rangle, \forall x, y \in \text{int}(\text{dom } g), \tag{2.4}$$

$$D_g(x,y) + D_g(y,z) - D_g(x,z) = \langle x - y, \nabla g(z) - \nabla g(y)\rangle, \forall x \in \text{dom } g, y, z \in \text{int}(\text{dom } g), \tag{2.5}$$

and

$$D_g(x,y) - D_g(x,z) - D_g(w,y) + D_g(w,z) = \langle x - w, \nabla g(z) - \nabla g(y)\rangle \tag{2.6}$$

for all $x, w \in \text{dom } g$ and $y, z \in \text{int}(\text{dom } g)$.

Let $v_g : [0, +\infty) \to [0, +\infty)$ be the modulus of total convexity of $g$ at $x \in$ int(dom g), defined by $v_g(x,t) := \inf\{D_g(y,x) : y \in \text{dom } g, \|y - x\|\}$. $g$ is totally convex at $x$ if $v_g(x,t) > 0$ whenever $t > 0$. Moreover, let $v_g : \text{int(dom g)} \times [0, \infty) \to [0, \infty]$ be the modulus of total convexity of the function $g$ on the set $A$ defined by $v_g(A,t) := \inf\{v_g(x,t) : x \in A \cap \text{dom } g\}$. $g$ is said to be totally convex on bounded sets of $X$ if $v_g(A,t) > 0$ for any nonempty bounded subset of $A$ of $X$ and $t > 0$. A Gâteaux differentiable function $g$ is said to be $\sigma$-strongly convex if there exists a constant $\tau > 0$ such that

$$g(x) \geq g(y) + \langle x - y, \nabla g(y)\rangle + \frac{\tau}{2}\|x - y\|^2, \forall x \in dom\ g, y \in \text{int}(dom\ g).$$

It is clear that if $g$ is a $\sigma$-strongly convex function, then it is a uniformly convex function. In view of the Bregman distance, one sees that

$$D_g(x,y) \geq \frac{\tau}{2}\|x - y\|^2, \forall x \in dom\ g, y \in \text{int}(dom\ g). \tag{2.7}$$

**Definition 2.2.** [13] Let $T : C \to C$ be a single-valued mapping. A point $x$ in set $C$ is called a fixed point of mapping $T$ if and only if $Tx = x$. The set of fixed points of $T$ is denoted by $F(T)$. Recall that a point $x$ in set $C$, is called an asymptotic fixed point of $T$ if and only if there exists a sequence $x_n$ in set $C$ such that $x_n \rightharpoonup x$ and $\lim_{n\to\infty} \|x_n – Tx_n\| = 0$.

Let the set of asymptotic fixed points of $T$ be denoted by $\hat{F}(T)$ in this paper.

**Definition 2.3.** [13, 24] Let $T : C \to C$ be a mapping. Then $T$ is called

(i) Bregman nonexpansive if $D_g(Tx, Ty) \leq D_g(x,y)$ for all $x, y \in C$,
(ii) Bregman relatively nonexpansive if $F(T) \neq \emptyset$ and $D_g(q, Tx) \leq D_g(q, x)$ for all $x \in C$, $q \in F(T)$, and $\hat{F}(T) = F(T)$,
(iii) Bregman firmly nonexpansive (BFNE) if

$$\langle \nabla g(Tx) - \nabla g(Ty), Tx - Ty\rangle \leq \langle \nabla g(x) - \nabla g(y), Tx - Ty\rangle, \forall x, y \in C;$$

(iv) Bregman strongly nonexpansive (BSNE) with respect to $\hat{F}(T)$ if $D_g(q, Tx) \leq D_g(q, x)$ for all $q \in F(T)$ and $x \in C$ and if whenever $\{x_n\} \subset C$ is bounded and $\lim_{n\to\infty}(D_g(q, x_n) - D_g(q, Tx_n)) = 0$, then $\lim_{n\to\infty} D_g(x_n, Tx_n) = 0$.
(v) Bregman quasi-nonexpansive (BQNE) if $F(T) \neq \emptyset$ and $D_g(q, Tx) \leq D_g(q, x)$ for all $q \in F(T)$ and $x \in C$.

If $F(T)=\hat{F}(T)$, then BFNE $\Rightarrow$ BSNE $\Rightarrow$ BQNE. It is well-known that Bregman relatively nonexpansive mapping is a Bregman quasi-nonexpansive mapping. Furthermore, Bregman relatively nonexpansive mapping is a Bregman weak relatively nonexpansive mapping, but the converse in not true in general (see [17])

**Definition 2.4.** Let $B : C \to X^*$ be an operator. Then $B$ is said to be

(i) monotone, if $\langle Bx - By, x-y\rangle \geq 0$ for all $x,y \in C$;
(ii) pseudo-monotone if $\langle Bx, y-x\rangle \geq 0 \Rightarrow \langle By, y-x\rangle \geq 0$ for all $x,y\in C$,
(iii) quasi-monotone if $\langle Bx, y-x\rangle > 0 \Rightarrow \langle By, y-x\rangle \geq 0$ for all $x,y\in C$,
(iv) $L$-Lipschitz continuous if there exists a constant $L>0$ such that $\|Bx-By\| \leq L\|x-y\|$ for all $x,y\in C$.
(v) uniformly continuous if, for every $\varepsilon>0$, there exists $\delta=\delta(\varepsilon)>0$ such that $\|Bx-By\| < \varepsilon$ whenever $\|x-y\|<\delta$ for all $x,y,\in C$.

From the definition above, the monotonicity $\Rightarrow$ pseudo-monotonicity $\Rightarrow$ quasi-monotonicity . However, the converse is not always true. It is known that the uniform continuity is a weaker notion than the Lipschitz continuity. Also, it is well known that if $D$ is a convex subset of $X$, then $B : D \to \text{range}(B)$ is uniformly continuous if and only if, for every $\varepsilon>0$, there exists a constant $Q<+\infty$ such that

$$\|Bx-By\| \leq Q\|x-y\| + \varepsilon, \quad \forall x,y\in D. \tag{2.8}$$

**Example 2.5.** [26] Let $B : \mathbb{R} \to \mathbb{R}$ be quasi-monotone defined by $B(x)=x^2$. $B$ is quasi-monotone but pseudo-monotone on $\mathbb{R}$ since $B$ is not pseudo-monotone at $x=0$. Also, if $B(x)=-x$, then $B$ is neither pseudo-monotone nor quasi-monotone.

**Example 2.6.** [26] Let $B : \mathbb{R}^+ \to \mathbb{R}$ be defined by $B(x)=[x,2x]$ for all $x\in\mathbb{R}^+$. Let $v=2x\in B(x)$ and $u=y\in B(y)$. If $x,y\in\mathbb{R}$ such that $x<y<2x$, then

$$\langle u-v, y-x\rangle = \langle y-2x, y-x\rangle = (y-2x)(y-x) < 0,$$

which implies that $B$ is not monotone. Suppose that $x,y\in\mathbb{R}^+$. Obviously, if $\sup_{v\in B(x)}\langle v, y-x\rangle = 2x(y-x)>0$, then $y>x$. Hence $\inf_{u\in B(y)}\langle u, y-x\rangle = y(y-x)>0$. Thus, $B$ is quasi-monotone.

**Lemma 2.7.** [4, 23] *Suppose that $g : X\to\mathbb{R}$ is Gâteaux differentiable and $C\subset int(dom\ g)$ is a nonempty closed and convex set. Then the Bregman projection $\Pi_C^g : X\to C$ satisfies the following properties:*

(i) *$w=\Pi_C^g(x)$ if and only if $\langle \nabla g(x)-\nabla g(w), y-w\rangle \leq 0$, for all $y\in C$;*

(ii) *$D_g(y,\Pi_C^g(x)) + D_g(\Pi_C^g(x),x) \leq D_g(y,x)$ for all $y\in C$ and $x\in X$.*

**Lemma 2.8.** [23] *Let $g : X\to\mathbb{R}$ be a Gâteaux differentiable and totally convex function. Suppose that $x\in X$. If $\{D_g(x_n,x)\}$ is bounded, then $\{x_n\}$ is bounded.*

**Lemma 2.9.** [24] *Suppose that $g : X\to\mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of $X$. Then $\nabla g$ is norm-to-norm uniformly continuous on bounded subsets of $X$ and thus, both $g$ and $\nabla g$ are bounded on bounded subsets of $X$.*

**Lemma 2.10.** [21] *If $g : X \to (-\infty, +\infty]$ is a proper, lower semi-continuous, and convex function, $g^* : X^* \to (-\infty, +\infty]$ is a weak$^*$ lower semi-continuous and convex function, then, for all $w \in X$,*

$$D_g\left(w, \nabla g^*\left(\sum_{i=1}^{N} \delta_i \nabla g(x_i)\right)\right) \le \sum_{i=1}^{N} \delta_i D_g(w, x_i),$$

*where $\{x_i\} \subset X$ and $\{\delta_i\} \subseteq (0,1)$ satisfying $\sum_{i=1}^{N} \delta_i = 1$.*

**Lemma 2.11.** [5] *The function $g : X \to (-\infty, \infty]$ is said to be sequentially consistent if and only if it is totally convex on bounded subsets of $X$.*

**Remark 2.12.** From Lemma 2.11, one supposes that $g$ is a Legendre function, which is uniformly Fréchet differentiable and bounded on bounded subsets of $X$. Then, for any sequence $\{x_n\} \subset \text{dom } g$ and $\{y_n\} \subset \text{int}(\text{dom } g)$ such that the first one is bounded,

$$\lim_{n\to\infty} D_g(x_n, y_n) = 0 \Rightarrow \lim_{n\to\infty} \|x_n - y_n\| = 0 \Rightarrow \lim_{n\to\infty} \|\nabla g(x_n) - \nabla g(y_n)\| = 0.$$

**Lemma 2.13.** [25] *If $x \in int(dom\ g)$, then the following statements are equivalent:*

(i) *The function $g$ is totally convex at $x$;*
(ii) *For any sequence $\{y_n\} \subset dom\ g$, $\lim_{n\to\infty} D_g(y_n, x) = 0 \Rightarrow \lim_{n\to\infty} \|y_n - x\| = 0$.*

Recall that the function $g$ is called sequentially consistent [4] if any two sequences $\{x_n\} \subset \text{dom } g$ and $\{y_n\} \subset \text{int}(\text{dom } g)$ such that the first one is bounded and $\lim_{n\to\infty} D_g(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

**Lemma 2.14.** [17] *Let $g : X \to R$ be a continuous uniformly and convex function on bounded subsets of $X$ and $r > 0$ be a constant. Then*

$$g\left(\sum_{k=0}^{n} \tau_k x_k\right) \le \sum_{k=0}^{n} \tau_k g(x_k) - \tau_i \tau_j \omega_r(\|x_i - x_j\|), \tag{2.9}$$

*for $x_k \in B_r, i, j \in \mathbb{N} \cup \{0\}, \tau_k \in (0,1)$ and $k \in \mathbb{N} \cup \{0\}$ with $\sum_{k=0}^{n} \tau_k = 1$, where $\omega_r$ is the gauge of uniform convexity of $g$.*

**Lemma 2.15.** [23] *Let $g : X \to R$ be a Gâteaux differentiable and totally convex function, $x \in X$, and let $C$ be a convex and closed set in $X$. Suppose that $\{x_n\}$ is bounded and that any weak subsequential limit of $\{x_n\}$ belongs to $C$. If $D_g(x_n, x) \le D_g(\Pi_C^g(x), x)$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $\Pi_C^g(x)$.*

**Lemma 2.16.** [28] *Suppose $\{\lambda_n\}$ and $\{\theta_n\}$ are two non-negative real sequences such that $\lambda_{n+1} \le \lambda_n + \phi_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} \phi_n < +\infty$, then $\lim_{n\to\infty} \lambda_n$ exists.*

**Lemma 2.17.** [34] *If one of the following conditions hold:*

(i) *$B$ is pseudo-monotone on $C$ and $VI(C,B) \neq \emptyset$,*
(ii) *$B$ is the gradient of $G$, where $G$ is a differentiable quasi-convex function on an open set $K \supset C$ and attains its global minimum on $C$;*
(iii) *$B$ is quasi-monotone on $C$, $B \neq 0$ on $C$ and $C$ is bounded;*
(iv) *$B$ is quasi-monotone on $C, B \neq 0$ on $C$ and there exists a positive number $r$ such that, for every $x \in C$ with $\|x\| \ge r$, there exists $y \in C$ such that $\|y\| \le r$ and $\langle Bx, y - x\rangle \le 0$;*
(v) *$B$ is quasi-monotone on $C, intC$ is nonempty and there exists $x^* \in S$ such that $Bx^* \neq 0$,*

*then $VI(C,B)^m$ is nonempty.*

## 3. Proposed Method

**Assumption 3.1.**

(a) $B : X \to X^*$ satisfies the following property
whenever $\{x_n\} \subset C, x_n \rightharpoonup z$, one has $\|Bz\| \leq \liminf_{n\to\infty} \|Bx_n\|$;

(b) The mapping $B : X \to X^*$ is quasi-monotone and uniformly continuous on $X$;

(c) For $i = 1,2,\cdots,N, \{T_i\}$ is a finite family of Bregman quasi-nonexpansive mappings on $X$ such that $F(T) = \hat{F}(T), \forall i = 1,2,\cdots,N$;

(d) The solution set $\Gamma = VI(C,B)^m \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$;

(e) Let $\{\rho_n\}$ be a non-negative sequence such that $\sum_{n=1}^\infty \rho_n < +\infty$;

(f) $\{\delta_{n,i}\} \subset (0,1), \sum_{i=0}^N \delta_{n,i} = 1$, and $\liminf_{n\to\infty} \delta_{n,0}\delta_{n,i} > 0$, for all $i = 1,2,\cdots,N$ and $n \geq 1$;

(g) $\alpha_n + \beta_n + \gamma_n = 1$ such that $\alpha_n, \beta_n$ and $\gamma_n \in (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$.

**Assumption 3.2.** The function $g : X \to \mathbb{R}$ satisfies the following:

(a) $g$ is uniformly Fr*é*chet differentiable;

(b) $g$ is proper, convex and lower semi-continuous;

(c) $g$ is strongly coercive and Legendre;

(d) $g$ is strongly convex on $X$ with strong convexity constant $\tau > 0$.

Now, we present our proposed algorithm as follows.

---

**Algorithm 3.3.**

---

**Step 0:** Select $v, x_0 \in X, \lambda_1 > 0, \theta_n \in [-\theta, \theta]$ for some $\theta > 0$ and $\sigma \in (0,\tau)$ and set $n = 1$.
**Step 1:** Compute $w_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_n) - \nabla g(x_{n-1})))$.
**Step 2:** Compute $y_n = \Pi_C^g(\nabla g^*(\nabla g(w_n) - \lambda_n Bw_n))$. If $w_n - y_n = 0$ : set $w_n = z_n$ and go to Step 4. Else do Step 3.
**Step 3:** Compute $z_n = \nabla g^*(\nabla g(y_n) - \lambda_n(By_n - Bw_n))$.
**Step 4:** Compute $s_n = \nabla g^*(\delta_{n,0}\nabla g(z_n) + \sum_{i=1}^N \delta_{n,i}\nabla g(T_i z_n))$, and $t_n = \nabla g^*(\alpha_n \nabla g(v) + \beta_n \nabla g(s_n) + \gamma_n \nabla g(s_n))$.
**Step 5:** Compute $x_{n+1} = \Pi^g_{E_n\cap H_n} x_1$, and construct two half-spaces $E_n$ and $H_n$ as follows:

$$E_n = \{r \in X : D_g(r,t_n) \leq \alpha_n D_g(r,v) + (1-\alpha_n)\left[D_g(r,w_n) - \mu_n\right]\},$$
$$H_n = \{r \in X : \langle r - x_n, \nabla g(x_1) - \nabla g(x_n)\rangle \leq 0\},$$

$$\mu_n = \left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right) D_g(z_n,y_n) + \left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right) D_g(y_n,w_n),$$

and

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n-y_n\|}{\|Bw_n-By_n\|},\ \lambda_n+\rho_n\}, & Bw_n - By_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise}. \end{cases} \tag{3.1}$$

Set $n := n+1$ and go to **Step 1**.

---

**Remark 3.4.**

(i) The authors in [10, 31] considered a VIP with a pseudo-monotone operator. Our method solves the VIP whose cost operator is quasi-monotone, a more general class of mappings. Furthermore, our cost operator is uniformly continuous a weaker notion than Lipschitz continuity. Also, assumption 3.1(a) is weaker than the sequentially weakly continuity condition assumed in [16].

(ii) Our result is an extension and improvement of the results obtained in [10, 15, 26, 31] from Hilbert spaces and 2-uniformly convex to reflexive Banach spaces.
(iii) In [11], the authors employed a linesearch technique which is computationally time consuming to implement. A more efficient step size rule which generates a non-monotonic sequence of step sizes is employed. Moreover, it reduces the dependence of the algorithm on the initial step sizes $\lambda_1$.
(iv) Our method employs the inertial technique to accelerate the rate of convergence.
(v) We prove the strong convergence of our proposed method since the strong convergence is relatively more desirable than the weak convergence obtained in [15, 26, 32]
(vi) The fixed point of a relatively nonexpansive mapping was considered in [10] while we consider the fixed point of a Bregman quasi-nonexpansive mapping which is different from relatively nonexpansive mappings.
(vii) Unlike several of the existing works on quasi-monotone VIPs in the literature, in our study, the cost operator of the quasi-monotone VIP is non-Lipschitz.
(viii) The existing works in the literature are solutions of quasimonotone variational inequalities in Hilbert space and 2-uniformly convex and uniformly smooth Banach spaces (Mewomo et al. [16]). To the best of our knowledge, there is no existing work on common solutions of quasi-monotone variational inequality and fixed point problems in reflexive Banach space. Hence, we find common solutions of quasi-monotone variational inequality and fixed point problems in reflexive Banach spaces.

## 4. Convergence Analysis

In this section, we prove some lemmas that are required to establish our strong convergence theorem.

**Lemma 4.1.** *Let $\{\lambda_n\}$ be the sequence of step sizes generated by Algorithm 3.3. Then $\{\lambda_n\}$ is well-defined and $\lim_{n\to\infty}\lambda_n = \lambda \in [\min\{\frac{\sigma}{N},\lambda_1\},\lambda_1+\Psi]$, where $\Psi = \sum_{n=1}^{\infty}\rho_n$ and for some $N>0$.*

*Proof.* Observe that $B$ is uniformly continuous. By (2.8), we have that, for any given $\varepsilon>0$, there exists $Q<+\infty$ such that $\|Bw_n - By_n\| \le Q\|w_n - y_n\| + \varepsilon$. Thus, for the case that $Bw_n - By_n \neq 0$ for all $n\ge 1$,

$$\frac{\sigma\|w_n-y_n\|}{\|Bw_n-By_n\|} \ge \frac{\sigma\|w_n-y_n\|}{Q\|w_n-y_n\|+\varepsilon} = \frac{\sigma\|w_n-y_n\|}{(Q+\varepsilon_1)\|w_n-y_n\|} = \frac{\sigma}{L},$$

where $\varepsilon = \varepsilon_1\|w_n-y_n\|$ for some $\varepsilon_1\in(0,1)$ and $L = Q+\varepsilon_1$. Therefore, by the definition of $\lambda_{n+1}$, $\{\lambda_n\}$ has lower bound $\min\{\frac{\sigma}{L},\lambda_1\}$ and has upper bound $\lambda_1+\Psi$. By Lemma 2.16, $\lim_{n\to\infty}\lambda_n$ exists and denoted by $\lambda = \lim_{n\to\infty}\lambda_n$. Clearly, $\lambda\in\left[\min\{\frac{\sigma}{L},\lambda_1\},\lambda_1+\Psi\right]$. □

From (3.1), we have

$$\lambda_{n+1} = \min\left\{\frac{\sigma\|w_n-y_n\|}{\|Bw_n-By_n\|},\lambda_n+\rho_n\right\} \le \frac{\sigma\|w_n-y_n\|}{\|Bw_n-By_n\|},$$

which implies that

$$\|Bw_n-By_n\| \le \frac{\sigma}{\lambda_{n+1}}\|w_n-y_n\|, \quad \forall n\ge 1. \tag{4.1}$$

**Lemma 4.2.** *Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by Algorithm 3.3. Then*

$$D_g(q,z_n) \le D_g(q,w_n) - \left[\left(1-\frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(z_n,y_n) + \left(1-\frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(y_n,w_n)\right], \forall q\in\Gamma, n\ge 1.$$

*Proof.* Since $q \in \Gamma$, then

$$\begin{aligned} D_g(q,z_n) &= g(q) - \langle \nabla g(y_n) - \lambda_n(By_n - Bw_n), q - z_n\rangle - f(z_n) \\ &= g(q) - \langle \nabla g(y_n), q-y_n\rangle - f(y_n) + \langle \nabla g(y_n), q-y_n\rangle + f(y_n) + \langle \nabla g(y_n), z_n - q\rangle \\ &+ \langle \lambda_n(By_n - Bw_n), q - z_n\rangle - f(z_n) \\ &= D_g(q,y_n) - D_g(z_n,y_n) + \langle \lambda_n(By_n - Bw_n), q - z_n\rangle. \end{aligned} \tag{4.2}$$

From (2.5), we have

$$D_g(q,y_n) = D_g(q,w_n) - D_g(y_n,w_n) + \langle \nabla g(w_n) - \nabla g(y_n), q - y_n\rangle. \tag{4.3}$$

By substituting (4.3) into (4.2), we obtain

$$\begin{aligned} &D_g(q,z_n) \\ &= D_g(q,w_n) - D_g(y_n,w_n) - D_g(z_n,y_n) + \langle \nabla g(w_n) - \nabla g(y_n), q-y_n\rangle + \lambda_n\langle By_n - Bw_n, q - z_n\rangle. \end{aligned} \tag{4.4}$$

We know that $y_n = \Pi_C^g(\nabla g^*(\nabla g(w_n) - \lambda_n Bw_n))$. By using Lemma 2.7(i) together with the fact that $q \in VI(C,B)^m \subset VI(C,B)$, we have $\langle \nabla g(w_n) - \lambda_n Bw_n - \nabla g(y_n), q - y_n\rangle \le 0$. Hence, $\langle \nabla g(w_n) - \nabla g(y_n), q-y_n\rangle \le \lambda_n\langle Bw_n, q - y_n\rangle$, which together with (4.4) yields

$$\begin{aligned} D_g(q,z_n) &\le D_g(q,w_n) - D_g(y_n,w_n) - D_g(z_n,y_n) + \lambda_n\langle Bw_n, q-y_n\rangle \\ &\quad + \lambda_n\langle By_n, q - z_n\rangle - \lambda_n\langle Bw_n, q-z_n\rangle \\ &= D_g(q,w_n) - D_g(y_n,w_n) - D_g(z_n,y_n) + \lambda_n\langle Bw_n, z_n - y_n\rangle - \lambda_n\langle By_n, y_n - q\rangle \\ &\quad + \lambda_n\langle By_n, y_n - z_n\rangle \\ &= D_g(q,w_n) - D_g(y_n,w_n) - D_g(z_n,y_n) + \lambda_n\langle Bw_n - By_n, z_n - y_n\rangle - \lambda_n\langle By_n, y_n - q\rangle. \end{aligned} \tag{4.5}$$

Since $q \in VI(C,B)^m$ and $y_n \in C$, then it follows from the dual VIP that $\langle By_n, y_n - q\rangle \ge 0$. Hence, from (4.5), (4.1), and (2.7), one has

$$\begin{aligned} D_g(q,z_n) &\le D_g(q,w_n) - D_g(z_n,y_n) - D_g(y_n,w_n) + \frac{\lambda_n}{\lambda_{n+1}}\lambda_{n+1}\|Bw_n - By_n\|\|z_n - y_n\| \\ &\le D_g(q,w_n) - D_g(z_n,y_n) - D_g(y_n,w_n) + \frac{\sigma}{2}\cdot\frac{\lambda_n}{\lambda_{n+1}}\|z_n - y_n\|^2 + \frac{\sigma}{2}\cdot\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 \\ &\le D_g(q,w_n) - \left(1-\frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(z_n,y_n) - \left(1-\frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(y_n,w_n) \\ &= D_g(q,w_n) - \mu_n. \end{aligned} \tag{4.6}$$

□

**Lemma 4.3.** *Let $\{w_n\}$ and $\{y_n\}$ be sequences generated by Algorithm 3.3 such that Assumption 3.1 and 3.2 are satisfied and suppose that $\{x_n\}$ is bounded. Let $\{w_{n_j}\}$ be a subsequence of $\{w_n\}$ which converges weakly to some $\bar{x} \in X$ as $j \to \infty$ and $\lim_{j\to\infty}\|w_{n_j} - y_{n_j}\| = 0$. Then $\bar{x} \in VI(C,B)^m$ or $B\bar{x} = 0$.*

*Proof.* We divide the proof of this lemma into two.
**CASE A:** If $\limsup_{n\to\infty}\|By_{n_j}\|=0$, then $\lim_{n\to\infty}\|By_{n_j}\|=\liminf_{j\to\infty}\|By_{n_j}\|=0$. Since $B$ is uniformly continuous and $y_{n_j}\rightharpoonup\bar{x}$, then $0\le\|B\bar{x}\|\le\liminf_{j\to\infty}\|By_{n_j}\|=0$. Hence $B\bar{x}=0$.
**CASE B:** Suppose that $\limsup_{n\to\infty}\|By_{n_j}\|>0$, without loss of generality. Let $\lim_{j\to\infty}\|By_{n_j}\|=K>0$. Thus, there exists a constant $H\in\mathbb{N}$ such that $\|By_{n_j}\|>\frac{K}{2}$, for all $j\ge H$.

Since $y_{n_j}=\Pi_C^g(\nabla g^*(\nabla g(w_{n_j})-\lambda_{n_j}Bw_{n_j}))$, it follows from Lemma 2.7(i) that

$$\langle\nabla g(w_{n_j})-\lambda_{n_j}Bw_{n_j}-\nabla g(y_{n_j}),x-y_{n_j}\rangle\le 0,\forall x\in C,$$

from which we obtain $\langle\nabla g(w_{n_j})-\nabla g(y_{n_j}),x-y_{n_j}\rangle\le\lambda_{n_j}\langle Bw_{n_j},x-y_{n_j}\rangle$ for all $x\in C$. Hence,

$$\frac{1}{\lambda_{n_j}}\langle\nabla g(w_{n_j})-\nabla g(y_{n_j}),x-y_{n_j}\rangle+\langle Bw_{n_j},y_{n_j}-w_{n_j}\rangle\le\langle Bw_{n_j},x-w_{n_j}\rangle,\ \forall x\in C. \tag{4.7}$$

Since $\lim_{j\to\infty}\lambda_{n_j}=\lambda>0,\lim_{j\to\infty}\|w_{n_j}-y_{n_j}\|=0$, and $\nabla g$ is uniformly continuous, then it follows from (4.7) that

$$0\le\liminf_{j\to\infty}\langle Bw_{n_j},x-w_{n_j}\rangle\le\limsup_{j\to\infty}\langle Bw_{n_j},x-w_{n_j}\rangle<+\infty,\ \forall x\in C. \tag{4.8}$$

It is clear that

$$\langle By_{n_j},x-y_{n_j}\rangle=\langle By_{n_j}-Bw_{n_j},x-w_{n_j}\rangle+\langle Bw_{n_j},x-w_{n_j}\rangle+\langle By_{n_j},w_{n_j}-y_{n_j}\rangle. \tag{4.9}$$

From the hypothesis of the lemma, we have $\lim_{j\to\infty}\|Bw_{n_j}-By_{n_j}\|=0$. Thus, (4.8) and (4.9) yield

$$0\le\liminf_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle\le\limsup_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle<+\infty,\ \forall x\in C. \tag{4.10}$$

If $\limsup_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle>0$, then there exists a subsequence $\{y_{n_j}\}$ such that $\lim_{j\to\infty}\langle By_{n_{j_k}},x-y_{n_{j_k}}\rangle>0$. Thus $k_0\in\mathbb{N}$ such that $\langle By_{n_{j_k}},x-y_{n_{j_k}}\rangle>0$ for all $k\ge k_0$. By the quasimonotonicity of $B$, we obtain $\langle Bx,x-y_{n_{j_k}}\rangle\ge 0$ for all $k\ge k_0$. As $k\to\infty$, we obtain $\bar{x}\in VI(C,B)^m$. Suppose that $\limsup_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle=0$. Hence, from (4.10), one has

$$\lim_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle=\limsup_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle=\liminf_{j\to\infty}\langle By_{n_j},x-y_{n_j}\rangle=0. \tag{4.11}$$

Set $\phi_j:=|\langle By_{n_j},x-y_{n_j}\rangle|+\frac{1}{j+1}$. Therefore,

$$\langle By_{n_j},x-y_{n_j}\rangle+\phi_j>0,\forall x\in C. \tag{4.12}$$

For $m_{n_j}\in X$ with $\lim_{j\to\infty}m_{n_j}=c$, let $m_{n_j}=\frac{By_{n_j}}{\|By_{n_j}\|^2}$. Thus, we obtain $\langle By_{n_j},m_{n_j}\rangle=1$. From (4.12), we have $\langle By_{n_j},x-y_{n_j}\rangle+\phi_j\langle By_{n_j},m_{n_j}\rangle>0$, which implies that $\langle By_{n_j},x+\phi_jm_{n_j}-y_{n_j}>0$. The quasimonotonicity of $B$ implies that $\langle B(x+\phi_jm_{n_j}),x+\phi_jm_{n_j}-y_{n_j}\rangle\ge 0,\forall j\ge H$. By (2.8), we have

$$\begin{aligned}\langle Bx,x+\phi_jm_{n_j}-y_{n_j}\rangle&\ge\langle Bx-B(x+\phi_jm_{n_j}),x+\phi_jm_{n_j}-y_{n_j}\rangle\\&\ge-\phi_j(Q^*+\varepsilon_1^*)\|m_{n_j}\|\|x+\phi_jm_{n_j}-y_{n_j}\|,\end{aligned} \tag{4.13}$$

where $Q^*$ is a constant and $\varepsilon^*=\varepsilon_1^*\|\phi_jm_{n_j}\|$. Taking the limit as $j\to\infty$ in (4.13), by the boundedness of $\{\|x+\phi_jm_{n_j}-y_{n_j}\|\}$ and $\{m_{n_j}\}$ with the fact that $\lim_{j\to\infty}\phi_j=0$, we obtain $\langle Bx,x-\bar{x}\rangle\ge 0,\forall x\in C$. Hence, $\bar{x}\in VI(C,B)^m$. □

**Theorem 4.4.** *Let $C$ be a convex and closed subset of a reflexive Banach space $X$ with the dual of $X$, denoted by $X^*$. Let $B : X \to X^*$ be quasi-monotone and uniformly continuous, and let $T_i : X \to X^*$ be a finite family of Bregman quasi-nonexpansive mappings. Let $\Gamma = VI(C,B)^m \bigcap_{i=1}^N F(T_i) \neq \emptyset$. If $g$ satisfies Assumption 3.2 and $\{x_n\}$ is a sequence generated by Algorithm 3.3, then $\{x_n\}$ converges strongly to a point $\bar{x}$, where $\bar{x} = \Pi_\Gamma^g(x_1)$.*

*Proof.* We divide the proof into the following lemmas.

**Lemma 4.5.** *Let $E_n$ and $H_n$ be defined as in Algorithm 3.3. Then $E_n \cap H_n$ is closed and convex for all $n \in \mathbb{N}$.*

It is clear from the definition of $H_n$ that $H_n$ is closed and convex for all $n \in \mathbb{N}$. It is known that $E_n$ is defined as $D_g(r,t_n) \leq \alpha_n D_g(r,v) + (1-\alpha_n)\big[D_g(r,w_n) - \mu_n\big]$. From the definition of $\mu_n$ together with the fact that $\lim_{n\to\infty} \lambda_n$ exists and $\sigma \in (0,\tau)$, we have

$$\lim_{n\to\infty}\left(1 - \frac{\sigma}{\tau}\frac{\lambda_n}{\lambda_{n+1}}\right) = \left(1 - \frac{\sigma}{\tau}\right) > 0.$$

This implies that there exists $n_0 \in \mathbb{N}$ such that $1 - \frac{\sigma}{\tau}\frac{\lambda_n}{\lambda_{n+1}} > 0$ for all $n \geq n_0$. Therefore $\mu_n \geq 0$. By Assumption 3.1(g), we have

$$\begin{aligned} D_g(r,t_n) &\leq \alpha_n D_g(r,v) + (1-\alpha_n) D_g(r,w_n) - \mu_n(1-\alpha_n) \\ &\leq \alpha_n D_g(r,v) + (1-\alpha_n) D_g(r,w_n). \end{aligned}$$

By applying (2.3), we obtain

$$\begin{aligned} g(r) - g(t_n) - \langle \nabla r - t_n, g(t_n)\rangle \leq \alpha_n &\big[g(r) - g(v) - \langle r - v, \nabla g(v)\rangle\big] \\ &+ (1-\alpha_n)[g(r) - g(w_n) - \langle r - w_n, \nabla g(w_n)\rangle], \end{aligned}$$

from which we have

$$\begin{aligned} &\alpha_n\big[\langle r - v, g(v)\rangle - \langle r - t_n, g(t_n)\rangle\big] + (1-\alpha_n)\big[\langle r - w_n, g(w_n)\rangle - \langle r - t_n, g(t_n)\rangle\big] \\ &\quad \leq \alpha_n(g(t_n) - g(v)) + (1-\alpha_n)(g(t_n) - g(w_n)). \end{aligned}$$

Thus $E_n$ is closed and convex for all $n \geq 1$. Therefore, for all $n \in \mathbb{N}$, $E_n \cap H_n$ is closed and convex.

**Lemma 4.6.** *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3 Then the following assertions hold:*

(i) *$\{x_n\}$ is well-defined.*
(ii) *$\{x_n\}$ is bounded*

(i) Let $q \in \Gamma$. By applying Lemma 2.10 and the definition of $s_n$ together with the fact that $T_i$ is a Bregman quasi-nonexpansive mapping, we have

$$D_g(q,s_n) \leq \delta_{n,0} D_g(q,z_n) + \sum_{i=1}^N \delta_{n,i} D_g(q,T_i z_n) \leq D_g(q,z_n). \tag{4.14}$$

From (4.6), we have

$$D_g(q,z_n) \leq D_g(q,w_n) - \mu_n. \tag{4.15}$$

From the definition of $t_n$, we have $D_g(q,t_n) \leq \alpha_n D_g(q,v) + (1-\alpha_n) D_g(q,s_n)$. By (4.15) and (4.14), we have $D_g(q,s_n) \leq D_g(q,w_n) - \mu_n$. Hence, $D_g(q,t_n) = \alpha_n D_g(q,v) + (1-\alpha_n)\big[D_g(q,w_n) -$

$\mu_n]$, which implies that $\Gamma \subset E_n$ for all $n \in \mathbb{N}$. If $n = 1$, we have $H_1 = X$, thus $\Gamma \subset E_1 \cap H_1$. Suppose that $\Gamma \subset E_k \cap H_k$, for some $k \geq 1$. Thus, $x_{k+1} = \Pi^g_{E_k \cap H_k} x_1$ is well defined. By Lemma 2.7, we have $\langle r - x_{k+1}, \nabla g(x_1) - \nabla g(x_{k+1}) \rangle \leq 0$ for all $r \in E_k \cap H_k$. Since $\Gamma \subset E_k \cap H_k$, we obtain $\langle y - x_{k+1}, \nabla g(x_1) - \nabla g(x_{k+1}) \rangle \leq 0$ for all $y \in \Gamma$, which implies that $\Gamma \subset E_{k+1}$. Thus $\Gamma \subset E_{k+1} \cap H_{k+1}$. By induction, we have $\Gamma \subset E_n \cap H_n$, for all $n \geq 1$. Thus $\{x_n\}$ is well-defined.

(ii) By the definition of $H_n$, we have

$$\langle y - x_n, \nabla g(x_1) - \nabla g(x_n) \rangle \leq 0, \forall y \in H_n. \tag{4.16}$$

By Lemma 2.7, we see that $x_n = \Pi^g_{H_n}(x_1)$. It follows that

$$D_g(x_n, x_1) \leq D_g(y, x_1) - D_g(y, x_n) \leq D_g(y, x_1), \forall y \in H_n. \tag{4.17}$$

Since $\Gamma \subset H_n$, then we $D_g(x_n, x_1) \leq D_g(q, x_1)$ for all $q \in \Gamma$. Hence $\{D_g(x_n, x_1)\}$ is bounded. By Lemma 2.8, we obtain that $\{x_n\}$ is bounded.

**Lemma 4.7.** *Let $C$ be a convex and closed subset of $X$, and let $B$ be quasimonotone and Lipschitz continuous, and let $T_i : X \to X^*$ be a Bregman-quasi nonexpansive mapping. If $\Gamma = VI(C,B)^m \cap \bigcap_{i=1}^N F(T_i) \neq \emptyset$, then $\bar{x} \in \Gamma$.*

First, we show that $\bar{x} \in VI(C,B)^m$. By (4.17) and the fact that $x_{n+1} \in H_n$, it follows that $D_g(x_n, x_1) \leq D_g(x_{n+1}, x_1)$. Hence $D_g(x_n, x_1)$ is increasing and thus $\lim_{n\to\infty} D_g(x_n, x_1)$ exists. Recall that $x_{n+1} \in H_n$. From (2.5) and (4.16), we have $D_g(x_{n+1}, x_n) \leq D_g(x_{n+1}, x_1) - D_g(x_n, x_1) \to 0$ as $n \to \infty$, which yields $D_g(x_{n+1}, x_n) \to 0$ as $n \to \infty$. In view of Remark 2.12, we have that $\lim_{n\to\infty} \|\nabla g(x_{n+1}) - \nabla g(x_n)\| = 0$. Since $w_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_n) - \nabla g(x_{n-1})))$, we have $\lim_{n\to\infty} \|\nabla g(w_n) - \nabla g(x_n)\| = 0$. In view of these, we obtain $\lim_{n\to\infty} \|\nabla g(x_{n+1}) - \nabla g(w_n)\| = 0$. Since $g$ is uniformly Fréchet differentiable, and $\nabla g^*$ is uniformly continuous on bounded subsets of $X^*$, we have $\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0$ and then $\lim_{n\to\infty} \|\nabla g(w_n) - \nabla g(x_n)\| = 0$, which implies that $\lim_{n\to\infty} \|w_n - x_n\| = 0$. From (2.4), we have

$$\begin{aligned} D_g(x_{n+1}, w_n) &\leq \langle x_{n+1} - w_n, \nabla g(x_{n+1}) - \nabla g(w_n) \rangle \\ &\leq L\|\nabla g(x_{n+1}) - \nabla g(w_n)\| \to 0 \text{ as } n \to \infty, \end{aligned} \tag{4.18}$$

where $L > 0$. Since $x_{n+1} \in E_n$, then $D_g(x_{n+1}, t_n) \leq \alpha_n D_g(x_{n+1}, v) + (1 - \alpha_n) D_g(x_{n+1}, w_n)$. It follows from (4.18) that $\lim_{n\to\infty} D_g(x_{n+1}, t_n) = 0$, which implies that $\lim_{n\to\infty} \|x_{n+1} - t_n\| = 0$. Since $t_n = \nabla g^*(\alpha_n \nabla g(v) + \beta_n \nabla g(s_n) + \gamma_n \nabla g(s_n))$, we obtain by the condition on $\alpha_n$ that $\|\nabla g(t_n) - \nabla g(s_n)\| \leq \alpha_n \|\nabla g(v) - \nabla g(s_n)\| \to 0$ as $n \to \infty$. Furthermore, by our assumption that $g$ is uniformly Fréchet differentiable, $\nabla g^*$ is uniformly continuous on bounded subsets of $X^*$. Hence, $\lim_{n\to\infty} \|t_n - s_n\| = 0$. We hence assert that $\|t_n - w_n\| \leq \|t_n - x_{n+1}\| + \|x_{n+1} - w_n\| \to 0$ as $n \to \infty$, which implies that $\lim_{n\to\infty} \|t_n - w_n\| = 0$. Also, $\|s_n - w_n\| \leq \|s_n - t_n\| + \|t_n - w_n\|$. Hence $\lim_{n\to\infty} \|s_n - w_n\| = 0$. By Lemma 2.9, we have $\lim_{n\to\infty} \|\nabla g(s_n) - \nabla g(w_n)\| = 0$. From (4.14), we have

$$\begin{aligned} &\left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right) D_g(z_n, y_n) + \left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right) D_g(y_n, w_n) \\ &\leq D_g(q, w_n) - D_g(q, s_n) \\ &\leq \langle q - w_n, \nabla g(s_n) - \nabla g(w_n) \rangle \\ &\leq P\|\nabla g(s_n) - \nabla g(w_n)\| \to 0, \end{aligned}$$

where $P>0$. Hence,

$$\lim_{n\to\infty}\left[\left(1-\frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(z_n,y_n)+\left(1-\frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(y_n,w_n)\right]=0,$$

which implies that

$$\lim_{n\to\infty}\left[\left(1-\frac{\sigma}{\tau}\right)D_g(z_n,y_n)+\left(1-\frac{\sigma}{\tau}\right)D_g(y_n,w_n)\right]=0.$$

Since $\sigma\in(0,\tau)$, we have $\lim_{n\to\infty}(D_g(z_n,y_n)+D_g(y_n,w_n))=0$. Thus $\lim_{n\to\infty}D_g(z_n,y_n)=\lim_{n\to\infty}D_g(y_n,w_n)=0$. By Remark 2.12, we have $\lim_{n\to\infty}\|z_n-y_n\|=\lim_{n\to\infty}\|y_n-w_n\|=0$, which implies that

$$\lim_{n\to\infty}\|\nabla g(z_n)-\nabla g(y_n)\|=\lim_{n\to\infty}\|\nabla g(y_n)-\nabla g(w_n)\|=0.$$

which yields $\lim_{n\to\infty}\|\nabla g(z_n)-\nabla g(w_n)\|=0$. Thus $\|\nabla g(s_n)-\nabla g(z_n)\|=0$ as $n\to\infty$. Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{n_j}\}\subset\{x_n\}$ such that $x_{n_j}\rightharpoonup\bar{x}\in X$. Note that $w_{n_j}\rightharpoonup\bar{x}$. Since $\lim_{n\to\infty}\|y_n-w_n\|=0$, we get $\lim_{n\to\infty}\|w_{n_j}-y_{n_j}\|=0$. It then follows from Lemma 4.3 that $\bar{x}\in VI(C,B)^m$.

Next, we show that $\bar{x}\in\bigcap_{i=1}^N F(T_i)$. Let $q\in\Gamma$. By applying (2.2) and Lemma 2.14, we have

$$\begin{aligned}
D_g(q,s_n)&=V_g\left(q,\delta_{n,0}\nabla g(z_n)+\sum_{i=1}^N\delta_{n,i}\nabla g(T_iz_n)\right)\\
&=g(q)-\delta_{n,0}\langle q,\nabla g(z_n)\rangle+\sum_{i=1}^N\delta_{n,i}\langle q,\nabla g(T_iz_n)\rangle+\delta_{n,0}g^*(\nabla g(z_n))\\
&+\sum_{i=1}^N\delta_{n,i}g^*(\nabla g(T_iz_n))-\delta_{n,0}\delta_{n,i}\omega_r(\|\nabla g(z_n)-\nabla g(T_iz_n)\|)\\
&=\delta_{n,0}D_g(q,z_n)+\sum_{i=1}^N\delta_{n,i}D_g(q,T_iz_n)-\delta_{n,0}\delta_{n,i}\omega_r(\|\nabla g(z_n)-\nabla g(T_iz_n)\|)\\
&\le\delta_{n,0}D_g(q,z_n)+\sum_{i=1}^N\delta_{n,i}D_g(q,z_n)-\delta_{n,0}\delta_{n,i}\omega_r(\|\nabla g(z_n)-\nabla g(T_iz_n)\|)\\
&\le D_g(q,z_n)-\delta_{n,0}\delta_{n,i}\omega_r(\|\nabla g(z_n)-\nabla g(T_iz_n)\|).
\end{aligned}\tag{4.19}$$

Indeed, from (2.5), we have

$$D_g(q,z_n)-D_g(q,s_n)\le\langle q-z_n,\nabla g(s_n)-\nabla g(z_n)\rangle\le N\|\nabla g(s_n)-\nabla g(z_n)\|,$$

where $N>0$. Thus $\lim_{n\to\infty}(D_g(q,z_n)-D_g(q,s_n))=0$. Obviously, we obtain from (4.19) that

$$\delta_{n,0}\delta_{n,i}\omega_r(\|\nabla g(z_n)-\nabla g(T_iz_n)\|)\le D_g(q,z_n)-D_g(q,s_n).$$

From the fact that $\liminf_{n\to\infty}\delta_{n,0}\delta_{n,i}>0$ with the property of $\omega_r$, we obtain $\lim_{n\to\infty}\|\nabla g(z_n)-\nabla g(T_iz_n)\|=0$. Also, since $g$ is uniform Fréchet differentiable, then $\nabla g^*$ is uniformly continuous on bounded subsets of $X^*$. Hence, we have $\lim_{n\to\infty}\|z_n-T_iz_n\|=\lim_{n\to\infty}\|z_n-x_n\|=0$. It is not hard to see that $\bar{x}\in\hat{F}(T_i)=F(T_i),\forall i=1,2,\cdots,N$. Thus $\bar{x}\in\bigcap_{i=1}^N F(T_i)$. Therefore $\bar{x}\in\Gamma$.

**Lemma 4.8.** *Let $\{x_n\}$ be a sequence generated by Algorithm 3.3. Then $x_n\to\Pi_\Gamma^g(x_1)$, where $g$ satisfies Assumption 3.2.*

Let $\bar{x} = \Pi^g_\Gamma(x_1)$. In view of $x_{n+1} = \Pi^g_{E_n\cap H_n}(x_1)$ and $\Gamma \subset E_n \cap H_n$, one has $D_g(x_{n+1},x_1) \leq D_g(\bar{x},x_1)$. Then, by Lemma 2.15, $x_n$ converges strongly to $\bar{x} = \Pi^g_\Gamma(x_1)$. □

**Corollary 4.9.** *Let $C$ be a convex and closed subset of a 2-uniformly convex and uniformly smooth Banach space $X$, and let $B : X \to X^*$ be a quasi-monotone and uniformly continuous mapping. Let $T_i$ be a relatively nonexpansive mapping. Suppose that $\alpha_n+\beta_n+\gamma_n = 1$ such that $\alpha_n,\beta_n$, and $\gamma_n \in (0,1)$ with $\lim_{n\to\infty}\alpha_n = 0$. Let $\{x_n\}$ be the sequence generated as follows.*

---

**Algorithm 4.10.**

---

**Step 0:** Select $x_0, v \in X, \lambda_1 > 0, \theta_n \in [-\theta,\theta], \theta > 0$, and $\sigma \in (0,d)$. Set $n = 1$.
**Step 1:** Compute $w_n = J^{-1}(Jx_n + \theta_n(Jx_n - Jx_{n-1}))$.
**Step 2:** Compute $y_n = \Pi_C(J^{-1}(Jw_n - \lambda_n Bw_n))$. If $w_n - y_n = 0$, set $w_n = z_n$ and go to Step 4. Else do Step 3.
**Step 3:** Compute $z_n = J^{-1}(Jy_n - \lambda_n(By_n - Bw_n))$.
**Step 4:** Compute $s_n = J^{-1}(\delta_{n,0}Jz_n + \sum_{i=1}^N \delta_{n,i}JT_iz_n)$ and $t_n = J^{-1}(\alpha_nJv + \beta_nJs_n + \gamma_nJs_n)$.
**Step 5:** Compute $x_{n+1} = \Pi^g_{E_n\cap H_n}x_1$, and construct two half-spaces $E_n$ and $H_n$ as follows:

$$E_n = \{r \in X : \phi(r,t_n) \leq \alpha_n\phi(r,v) + (1-\alpha_n)\big[\phi(r,w_n) - \mu_n\big]\},$$
$$H_n = \{r \in X : \langle r - x_n, Jx_1 - Jx_n\rangle \leq 0\},$$

where $\mu_n = \left(1 - \frac{\sigma\lambda_n}{d\lambda_{n+1}}\right)\phi(z_n,y_n) + \left(1 - \frac{\sigma\lambda_n}{d\lambda_{n+1}}\right)\phi(y_n,w_n)$, and

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n-y_n\|}{\|Bw_n-By_n\|},\ \lambda_n+\rho_n\}, & Bw_n - By_n \neq 0,\\ \lambda_n+\rho_n, & \text{otherwise.}\end{cases}$$

Set $n := n+1$ and go to **Step 1**.

---

*Suppose $\Gamma = VI(C,B)^m\bigcap_{i=1}^N F(T_i) \neq \emptyset$, $\{\delta_{n,i}\} \subset (0,1)$, $\sum_{i=0}^N \delta_{n,i} = 1$, and $\liminf_{n\to\infty}\delta_{n,0}\delta_{n,i} > 0$ for all $i = 1,2,\cdots,N$ and $n \geq 1$. Then sequence $\{x_n\}$ generated by Algorithm 4.10 converges strongly to $\bar{x} = \Pi_\Gamma x_1$.*

Let $s_n = 0$ in Algorithm 3.3. Then we have the following result.

**Corollary 4.11.** *Let $C$ be a convex and closed set in a reflexive Banach space $X$, and let $B : X \to X^*$ be a quasimonotone monotone and uniformly continuous mapping. Let $T_i$ be a Bregman relatively nonexpansive mapping. Let $\{x_n\}$ be the sequence generated as follows.*

---

**Algorithm 4.12.**

---

**Step 0:** Select $x_0, v \in X, \lambda_1 > 0, \theta_n \in [-\theta,\theta]$ for some $\theta > 0$, and $\sigma \in (0,\tau)$. Set $n = 1$.
**Step 1:** Compute $w_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_n) - \nabla g(x_{n-1})))$.
**Step 2:** Compute $y_n = \Pi^g_C(\nabla g^*(\nabla g(w_n) - \lambda_nBw_n))$. If $w_n - y_n = 0$, set $w_n = z_n$ and go to Step 4. Else do Step 3.
**Step 3:** Compute $z_n = \nabla g^*(\nabla g(y_n) - \lambda_n(By_n - Bw_n))$.
**Step 4:** Compute $t_n = \nabla g^*(\alpha_n\nabla g(v) + \beta_n\nabla g(s_n) + \gamma_n\nabla g(s_n))$.
**Step 5:** Compute $x_{n+1} = \Pi^g_{E_n\cap H_n}x_1$, and construct two half-spaces $E_n$ and $H_n$ as follows:

$$E_n = \{r \in X : D_g(r,t_n) \leq \alpha_nD_g(r,v) + (1-\alpha_n)\big[D_g(r,w_n) - \mu_n\big]\},$$
$$H_n = \{r \in X : \langle r - x_n, \nabla g(x_1) - \nabla g(x_n)\rangle \leq 0\},$$

where the adaptive step-size is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n - y_n\|}{\|Bw_n - By_n\|},\ \lambda_n + \rho_n\}, & Bw_n - By_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

and

$$\mu_n = \left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right) D_g(z_n, y_n) + \left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right) D_g(y_n, w_n),$$

Set $n := n+1$ and go to **Step 1**.

*Let Assumption 3.1 and Assumption 3.2 be satisfied. Then the sequence $\{x_n\}$ generated by Algorithm 4.12 converges strongly to $\bar{x} = \Pi_{\Gamma}^{g} x_1$.*

## 5. Numerical Examples

In this section, we give some numerical examples in finite and infinite dimensional spaces to illustrate our proposed method as well as compare it with some related methods in the literature. We compare our Algorithm 3.3 with Algorithm 1.5 (Alakoya *et al.* Alg.), Algorithm 1.6 (Mewomo *et al.* Alg.), Appendix 6.1 (Reich *et al.* Alg.), and Appendix 6.2 (Wang *et al.* Alg.). We use Matlab version R2022(b) for all the codes and numerical computations and the numerical results with control Parameters are demonstrated in Tables 1 - 3 and Figures 1 - 8.

**Example 5.1.** [15] Let $X = \mathbb{R}, C := [-1,1]$ and

$$B(x) = \begin{cases} 2x - 1, & x > 1, \\ x^2, & x \in [-1,1], \\ -2x - 1, & x < -1, \end{cases}$$

where $B$ is a quasi-monotone and Lipschitz continuous mapping, $VI(C,B)^m = \{-1\}$, and $VI(C,B) = \{-1,0\}$. Let $|x_{n+1} - x_n| < 10^{-4}$ be our stopping criterion, and $q_n = \frac{n}{3n+2} x_0$. Let $x_0 \neq 0$ be any element in $X$, for $i = 1,2,\cdots,7$, and let $T_i = T : X \to X$ be defined by

$$T(x) = \begin{cases} \left(\frac{1}{2} + \frac{1}{2^{n+1}}\right) x_0, & \text{if } x = \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0; \\ -x, & \text{if } x \neq \left(\frac{1}{2} + \frac{1}{2^n}\right) x_0, \end{cases}$$

for all $n \geq 0$. It is clear that $T$ is Bregman quasi-nonexpansive (see [17]).

We use $|x_{n+1} - x_n| < 10^{-4}$ as the stopping criterion and choose the starting points as follows:

Case 1. $x_0 = 0.5003, x_1 = 0.2000$;

Case 2. $x_0 = \frac{47}{99}, x_1 = \frac{9}{50}$;

Case 3. $x_0 = 0.4950, x_1 = 0.1879$;

Case 4. $x_0 = \frac{1}{2}, x_1 = \frac{1}{5}$.

TABLE 1. Control Parameters for Examples 5.1- 5.2

| | | | | | |
|---|---|---|---|---|---|
| Alakoya *et al. Alg.* | $\theta = 0.50$ | $\lambda_1 = 0.68$ | $\xi = \frac{1}{(4n+5)^2}$ | $\alpha_n = \frac{1}{4n+5}$ | $\rho_n = \frac{10}{n^2}$ $\sigma = 0.50$ |
| Mewomo *et al. Alg.* | $\theta = 0.50$ | $\lambda_1 = 0.68$ | $\xi = \frac{1}{(4n+5)^2}$ | $\alpha_n = \frac{1}{4n+5}$ | $\rho_n = \frac{10}{n^2}$ $\sigma = 0.50$ |
| Reich *et al. Alg.* | $\theta_n = \frac{1}{3n+1}$ | $\lambda_1 = 0.68$ | $\rho_n = \frac{10}{n^2}$ | $\sigma = 0.50$ | |
| Wang *et al. Alg.* | $\theta = 0.50$ | $\lambda_1 = 0.68$ | $\xi = \frac{1}{(4n+5)^2}$ | $\rho_n = \frac{10}{n^2}$ | $\rho = 0.69$ |
| Proposed Alg. 3.3 | $\theta_n = \frac{1}{3n+1}$ <br> $\rho_n = \frac{10}{n^2}$ | $\lambda_1 = 0.68$ <br> $\sigma = 0.50$ | $\alpha_n = \frac{1}{4n+5}$ <br> $\tau = 0.38$ | $\beta_n = \frac{n}{2n+1}$ <br> $N = 7$ | $\gamma_n = 1 - \alpha_n - \beta_n$ <br> $\delta_{n,i} = \frac{1}{N}$ |

TABLE 2. Numerical Results for Example 5.1

| | Case 1 | | Case 2 | | Case 3 | | Case 4 | |
|---|---|---|---|---|---|---|---|---|
| | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time |
| Alakoya *et al. Alg.* | 101 | 0.0146 | 99 | 0.0141 | 100 | 0.0143 | 101 | 0.0147 |
| Mewomo *et al. Alg.* | 98 | 0.0040 | 96 | 0.0039 | 98 | 0.0045 | 98 | 0.0042 |
| Reich *et al. Alg.* | 121 | 0.0044 | 121 | 0.0040 | 121 | 0.0040 | 121 | 0.0040 |
| Wang *et al. Alg.* | 137 | 0.0046 | 135 | 0.0044 | 137 | 0.0046 | 137 | 0.0043 |
| Proposed Alg. 3.3 | 7 | 0.0053 | 7 | 0.0055 | 7 | 0.0059 | 7 | 0.0054 |

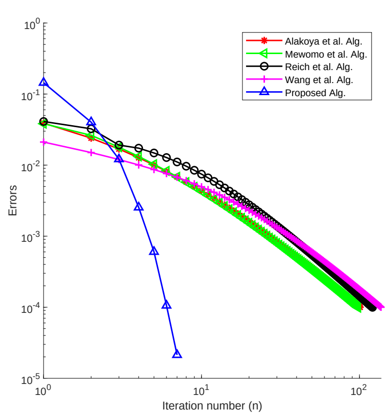


FIGURE 1. Example 5.1 Case 1

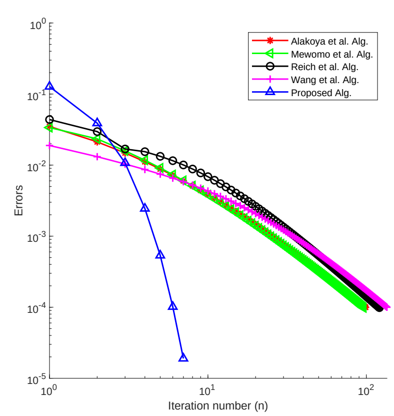


FIGURE 2. Example 5.1 Case 2

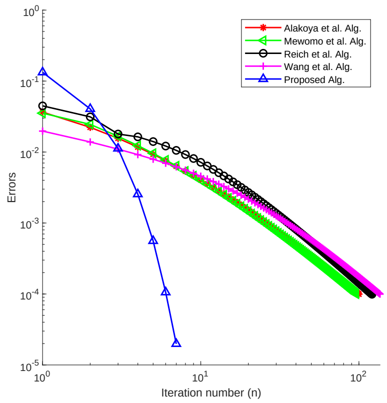


FIGURE 3. Example 5.1 Case 3

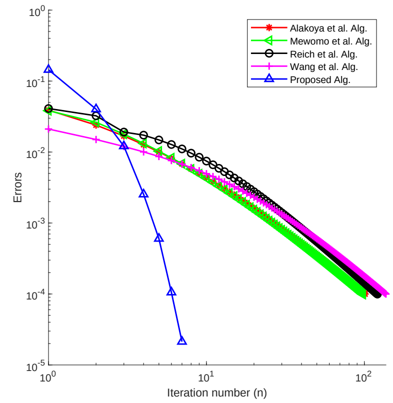


FIGURE 4. Example 5.1 Case 4

**Example 5.2.** Let $X = \ell_2 := \{x = (x_1, x_2, \cdots, x_i, \cdots) : \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$. Let $t, v \in \mathbb{R}$ such that $t > v > \frac{t}{2}$. Let $C_v = \{x \in X : \|x\| \le v\}$ and $B_t(x) = (t - \|x\|)x$. Then $B$ is quasi-monotone and Lipschitz continuous. Let $v = 2, t = 3$, and $q_n = \frac{2n}{3n+1} x_1$. For $i = 1, 2, \cdots, 7$, let $T_i = T : X \to X$ be defined by $T(x) = \begin{cases} \frac{n}{n+1} x & \text{if } x = x_n; \\ -x & \text{if } x \ne x_n, \end{cases}$ where $\{x_n\} \subset X$ is a sequence defined as follows: $x_0 = (1, 0, 0, 0, \cdots)$, $x_1 = (1, 1, 0, 0, 0, \cdots)$, $x_2 = (1, 0, 1, 0, 0, 0, \cdots)$, $\cdots$, $x_n = (\tau_{n,1}, \tau_{n,2}, \cdots, \tau_{n,j}, \cdots)$ and

$$\tau_{n,j} = \begin{cases} 1, & \text{if } j = 1, n+1, \\ 0, & \text{if } j \ne 1, j \ne n+1 \end{cases}$$

for all $n \in \mathbb{N}$. It is clear that $F(T) = \{0\}$ and $T$ is Bregman quasi-nonexpansive (see [17]). we use $\|x_{n+1} - x_n\| < 10^{-9}$ as the stopping criterion and the starting points as follows:

Case 1. $x_0 = (-3, 1, -\frac{1}{3}, \cdots), x_1 = (0.2, 0.02, 0.002, \cdots)$.
Case 2. $x_0 = (-0.3, 0.03, -0.003, \cdots), x_1 = (-0.1, 0.01, -0.001, \cdots)$.
Case 3. $x_0 = (2, 1, \frac{1}{2}, \cdots), x_1 = (-0.15, 0.015, -0.0015, \cdots)$.
Case 4. $x_0 = (1, -0.1, 0.01, \cdots), x_1 = (0.2, -0.02, 0.002, \cdots)$.

TABLE 3. Numerical Results for Example 5.1

| | Case 1 | | Case 2 | | Case 3 | | Case 4 | |
|---|---|---|---|---|---|---|---|---|
| | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time | Iter. | CPU Time |
| Alakoya *et al. Alg.* | 41 | 0.0158 | 41 | 0.0159 | 40 | 0.0157 | 41 | 0.0157 |
| Mewomo *et al. Alg.* | 37 | 0.0080 | 41 | 0.0083 | 36 | 0.0083 | 41 | 0.0083 |
| Reich *et al. Alg.* | 76 | 0.0046 | 72 | 0.0045 | 75 | 0.0049 | 75 | 0.0046 |
| Wang *et al. Alg.* | 77 | 0.0078 | 72 | 0.0060 | 75 | 0.0063 | 77 | 0.0059 |
| Proposed Alg. 3.3 | 13 | 0.0087 | 13 | 0.0080 | 13 | 0.0081 | 13 | 0.0081 |

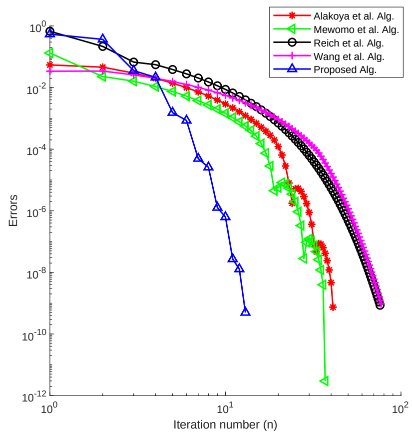


FIGURE 5. Example 5.1 Case 1

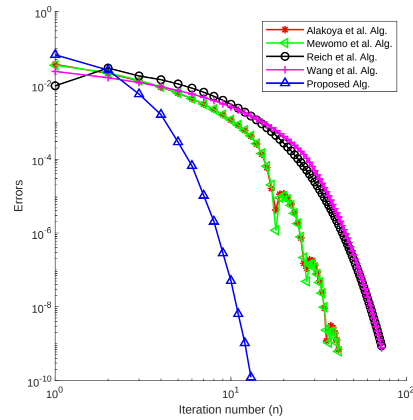


FIGURE 6. Example 5.1 Case 2

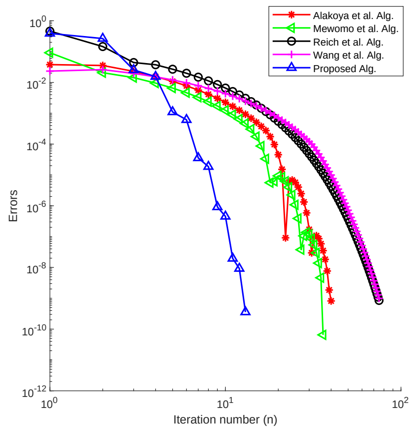


FIGURE 7. Example 5.1 Case 3

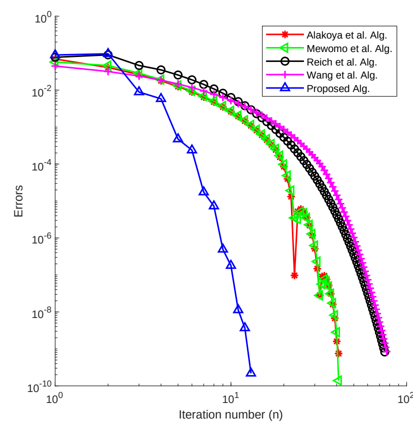


FIGURE 8. Example 5.1 Case 4

## 6. CONCLUSION

In this paper, we introduced and studied a modified inertial hybrid tseng's extragradient method for approximating a common solution of a non-Lipschitz quasi-monotone variational inequality and fixed points of a finite family of Bregman quasi-nonexpansive mappings in the framework of reflexive Banach spaces. Our method does not involve the linesearch technique but employes a self-adaptive step size which generates a non-monotonic sequence of step sizes. We established that the sequence generated by our method converged strongly. We gave some numerical examples to illustrate the efficacy of our method as well as compare it with related methods in the literature.

**Funding**

The first author acknowledges with thanks the International Mathematical Union Breakout Graduate Fellowship (IMU-BGF) Award for his doctoral study. The second author is supported by the National Research Foundation (NRF) of South Africa Incentive Funding for Rated Researchers (Grant Number 119903) and DSI-NRF Centre of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), South Africa (Grant Number 2022-087-OPA). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS and NRF.

## REFERENCES

[1] F. Alvarez, H. Attouch, An inertial proximal method for monotone operators via discretization of a nonlinear oscillator with damping, Set-Valued Var. Anal. 9 (2001) 3–11.

[2] T.O. Alakoya, O.T. Mewomo, Y. Shehu, Strong convergence results for quasimonotone variational inequalities, Math. Meth. Oper. Res. 95 (2022) 249-279.

[3] T.O. Alakoya, V.A. Uzor, O.T. Mewomo, J.-C. Yao, On a system of monotone variational inclusion problems with fixed-point constraint, J. Inequ. Appl. 2022 (2022) 47.

[4] D. Butnariu, E. Resmerita, Bregman distances, totally convex functions and a method for solving operator equations in Banach spaces, Abstr. Appl. Anal. 2006 (2006) 84919.

[5] D. Butnariu, A.N. Iusem, Totally Convex Functions for Fixed Points Computation and Infinite Dimensional Optimization, Kluwer Academic Publishers, Dordrecht, 2000.

[6] Y. Censor, A. Gibali, S. Reich, Extensions of Korpelevich's extragradient methods for the variational inequality problem in Euclidean space, Optimization 61 (2012) 119-1132.

[7] Y. Censor, A. Gibali, S. Reich, Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space, Optim. Meth. Softw. 26 (2011) 827-845.

[8] R.W. Cottle, J.C. Yao, Pseudo-monotone complementarity problems in Hilbert space, J. Optim. Theory Appl. 75 (1992) 281–295.

[9] E.C. Godwin, T.O. Alakoya, O.T. Mewomo, J.-C. Yao, Relaxed inertial Tseng extragradient method for variational inequality and fixed point problems, Appl. Anal. (2022), DOI: 10.1080/00036811.2022.2107913.

[10] S. Hu, Y. Wang, Q-L. Dong, Self-adaptive extragradient methods for variational inequalities and fixed point problems in 2-uniformly convex and uniformly smooth Banach spaces, Numer. Funct. Anal. Optim. 43 (2022) 1401-1422.

[11] S. Hu, Y. Wang, X. Li, Q-L. Dong, A new modified extragradient method with line-search process for solving pseudomonotone variational inequality in Hilbert spaces, Optim. (2022) DOI: 10.1080/02331934.2022.2098126.

[12] C. Izuchukwu, Y. Shehu, J.-C. Yao, A simple projection method for solving quasimonotone variational inequality problems, Optim. Eng. (2022), DOI: 10.1017/s11081-022-09713-8.

[13] G. Kassay, S. Reich, S. Sabach, Iterative methods for solving system of variational inequalities in reflexive Banach spaces, SIAM J. Optim. 21 (2011) 1319–1344.

[14] G.M. Korpelevich, The extragradient method for finding saddle points and other problems, Ekonomika Matematicheskie Melody, 12 (1976) 747-756.

[15] H. Liu, J. Yang, Weak convergence of iterative methods for solving quasimonotone variational inequalities, Comput. Optim. Appl. 77 (2020), 491-508.

[16] O.T. Mewomo, T.O. Alakoya, J-C Yao, L. Akinyemi, Strong convergent inertial Tseng's extragradient method for solving non-Lipschitz quasimonotone variational inequalities in Banach spaces, J. Nonlinear Var. Anal. 7 (2023) 147-152.

[17] E. Naraghirad, J-C. Yao, Bregman weak relatively nonexpansive mappings in Banach spaces, Fixed Point Theory Appl. 2013 (2013) 141.

[18] A.O.E. Owolabi, T.O. Alakoya, O.T. Mewomo, A self-adaptive inertial extragradient method for a class of split pseudomonotone variational inequality problems, Open Maths. 21 (2023) DOI: 10.1515/math-2022-0571.

[19] G.N. Ogwo, T.O. Alakoya, O.T. Mewomo, An inertial forward-backward method with self-adaptive step sizes for finding minimum-norm solutions of inclusion and split equilibrium problems, Appl. Set-Valued Anal. Optim. 4 (2022) 185-206.

[20] B.T. Polyak, Some methods of speeding up the convergence of iteration methods, USSR Comput. Math. Math. Phys. 4 (1964) 1–17.
[21] R.R. Phelps, Convex Functions, Monotone Operators and Differentiablity, 2nd edn. Lecture Notes in Mathematics, vol. 1364, Spinger, Berlin, 1993.
[22] S. Reich, T.M. Tuyen, P. Sunthrayuth, P. Cholamjiak, Two new inertial algorithms for solving variational inequalities in reflexive Banach spaces, Numer. Funct. Anal. Optim. 42 (2021) 1954-1984.
[23] S. Reich, S. Sabach, Two strong convergence theorem for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim. 31 (2010) 22–44.
[24] S. Reich, S. Sabach, A strong convergence theorem for proximal-type algorithm in reflexive Banach spaces, J. Nonlinear Convex Anal. 10 (2009) 471–485.
[25] E. Resmerita, On total convexity, Bregman projections and stability in Banach spaces, J. Convex Anal. 11 (2004) 1–16.
[26] Salahuddin, The extragradient method for quasi-monotone variational inequalities, Optimization 71 (2020) 2519–2528.
[27] P. Senakka, P. Cholamjiak, Approximation method for solving fixed point problem of Bregman strongly nonexpansive mappings in reflexive Banach spaces, Ric. Mat. 65 (2016) 209–220.
[28] K.K. Tan, H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993) 301-308.
[29] D.V. Thong, N.T. Vinh, D.V. Hieu, Accelerated hybrid and shrinking projection methods for variational inequality problems, Optimization 68 (2019) 981–998.
[30] P. Tseng, A modified forward-backward splitting method for maximal method for maximal monotone mappings, SIAM J. Control Optim. 38 (2000) 431-446.
[31] V.A. Uzor, T.O. Alakoya, O.T. Mewomo, Strong convergence of a self-adaptive inertial Tseng's extragradient method for pseudomonotone variational inequalities and fixed point problems, Open Math. 20 (2022) 234–257.
[32] N. Wairojjana, N. Pakkaranang, W. Jirakitpuwapa, N. Pholasa, The Tseng's extragradient method for quasimonotone variational inequalities, Thai J. Math. 19 (2021) 913-923.
[33] Z-B. Wang, P. Sunthrayuth, A. Adamu, P. Cholamjiak, Modified accelerated Bregman projection methods for solving quasi-monotone variational inequalities, Optim. (2023) DOI: 10.1080/02331934.2023.2187663.
[34] M. Ye, Y. He, A double projection method for solving variational inequalities without monotonicity, Comput. Optim. Appl. 60 (2015) 141–150.
[35] T.Y. Zhao, et al., Quasi-inertial Tseng's extragradient algorithms for pseudomonotone variational inequalities and fixed point problems of quasinonexpansive operators, Numer. Funct. Anal. Optim. 42 (2020) 69–90.

**Appendix 6.1.** Algorithm 1 of [22]
**Step 0:** Select $x_0 \in X, \lambda_1 > 0, \theta_n \in [-\theta, \theta]$, and $\sigma \in (0, \tau)$. Set $n = 1$.
**Step 1:** Compute $w_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_n) - \nabla g(x_{n-1})))$.
**Step 2:** Compute $y_n = \Pi_C^g(\nabla g^*(\nabla g(w_n) - \lambda_n B w_n))$. If $w_n - y_n = 0$, set $w_n = z_n$ and go to Step 4. Else do Step 3.
**Step 3:** Compute $z_n = \nabla g^*(\nabla g(y_n) - \lambda_n(By_n - Bw_n))$.
**Step 4:** Compute $x_{n+1} = \Pi_{E_n \cap H_n}^g x_1$, and construct two half-spaces $E_n$ and $H_n$ as follows:

$$E_n = \{r \in X : D_g(r, z_n) \leq D_g(r, t_n) - \mu_n\},$$
$$H_n = \{r \in X : \langle r - x_n, \nabla g(x_1) - \nabla g(x_n)\rangle \leq 0\},$$

where the adaptive step-size is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n - y_n\|}{\|Bw_n - By_n\|},\ \lambda_n\}, & Bw_n - By_n \neq 0, \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{6.1}$$

and

$$\mu_n = \left[\left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(z_n, y_n) + \left(1 - \frac{\sigma\lambda_n}{\tau\lambda_{n+1}}\right)D_g(y_n, w_n)\right]$$

Set $n := n + 1$ and go to **Step 1**.

Here $g : X \to \mathbb{R}$ is strongly convex, Legendre, bounded, and uniformly Fréchet differentiable on bounded subsets of $X$, and $B : X \to X^*$ is monotone and Lipschitz continuous with constant $L > 0$.

**Appendix 6.2.** Algorithm 1 of [33]
**Initialization:** Select $\beta \in [0, 1), \delta \in (0, 1], \lambda_1 > 0$, and $\sigma \in (0, \tau)$. Let sequences $\{\xi_n\} \subset [0, \infty)$ and $\{\rho_n\} \subset [0, \infty)$ be sequences such that $\sum_{n=1}^{\infty} \xi_n < \infty$ and $\sum_{n=1}^{\infty} \rho_n < \infty$.
**Iterative steps:** Given the iterates $x_{n-1}$ and $x_n$ for each $n \geq 1$, calculate $x_{n+1}$ as follows:
**Step 1.** Select $\theta_n$ such that $\theta_n \in [0, \bar{\theta}_n]$, where

$$\bar{\theta}_n = \begin{cases} \min\{\frac{\xi_n}{\|\nabla g(x_{n-1}) - \nabla g(x_n)\|},\ \beta\}, & \nabla g(x_{n-1}) \neq \nabla g(x_n), \\ \beta, & \text{otherwise.} \end{cases}$$

**Step 2:** Compute

$$\begin{cases} w_n = \nabla g^*(\nabla g(x_n) + \theta_n(\nabla g(x_{n-1}) - \nabla g(x_n))), \\ y_n = \Pi_C^g(\nabla g^*(\nabla g(w_n) - \lambda_n B w_n)), \\ z_n = \nabla g^*(\nabla g(y_n) - \lambda_n(By_n - Bw_n)), \\ x_{n+1} = \nabla g^*((1 - \delta)\nabla g(x_n) + \delta \nabla g(z_n)),\ \forall n \geq 1, \end{cases}$$

where the adaptive step-size is given by

$$\lambda_{n+1} = \begin{cases} \min\{\frac{\sigma\|w_n - y_n\|}{\|Bw_n - By_n\|},\ \lambda_n + \rho_n\}, & Bw_n - By_n \neq 0, \\ \lambda_n + \rho_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and go to **Step 1**.

Here $g$ is strongly convex with constant $\kappa > 0$, Legendre which is uniformly Fréchet differentiable, and $B : X \to X^*$ is quasi-monotone and Lipschitz continuous.