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ON NEW FIXED POINT RESULTS VIA A GENERALIZE EXTENDED SIMULATION FUNCTION IN b-METRIC SPACES AND AN APPLICATION TO HOMOTOPY

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Abstract. In this paper, we investigate the existence of fixed points of nonlinear mappings by using the concept of an e_s-simulation function in b-metric spaces. We also provide an example to illustrate our main results. As an application, we investigate the existence of a unique solution to homotopy theory.

Keywords. Admissibility; b-metric spaces; Fixed point; Homotopy theory; Simulation function.

1. Introduction and Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$, and $\mathbb{R}$ the sets of positive integers, non-negative real numbers, and real numbers, respectively.

The Banach contraction principle, proved by Banach [1], was the starting point of exhaustive research in the fixed point theory. This principle emphasizes on complete metric spaces. For metric spaces, the one of extension of a metric space was considered in the concept of a b-metric space by Czerwik [5] who proved some fixed point theorems of single-valued and multi-valued mapping in b-metric space.

Definition 1.1 ([5])**.** Let X be a nonempty set and $s \geq 1$ be a fixed real number. Suppose that the mapping $d : X \times X \to \mathbb{R}_+$ satisfies the following condition for all $x, y, z, \in X$

(1) $d(x,y) = d(y,x)$;
(2) $d(x,y) \leq s[d(x,z) + d(z,y)]$;
(3) $x = y$ if and only if $d(x,y) = 0$.

Then d is called a b-metric and (X,d) is called a b-metric space with coefficient s.

Every metric is b-metric for $s = 1$, but the converse does not hold in general. Furthermore, many authors studied variational principle for single-valued and multi-valued operators in b-metric spaces recently; see, e.g., [2, 3, 6, 7].

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The following examples are some known b-metric spaces.

Example 1.2. Let $X=\mathbb{R}$, $p\geq 1$ and the mapping $d:X\times X\to\mathbb{R}_+$ be defined by $d(x,y)=|x-y|^p$ for all $x,y\in X$. Then (X,d) is a b-metric space with coefficient $s=2^{p-1}$.

Example 1.3. Let $X=\{0,1,2,3,4\}$ and $d:X\times X\to\mathbb{R}_+$ be defined by

$$d(x,y)=\begin{cases}(x+y)^2 & \text{if } x\neq y;\\ 0 & \text{if } x=y.\end{cases}$$

It is easy to see that (X,d) is a b-metric space with coefficient $s=\frac{49}{25}$.

Definition 1.4 ([2])**.** Let (X,d) be a b-metric space and $\{x_n\}$ be a sequence in X.

- (i)**:** $\{x_n\}$ is b-convergent if there exists $x\in X$ such that $d(x_n,x)\to 0$ as $n\to\infty$; In this case, we write $\lim\limits_{n\to\infty}x_n=x$.
- (ii)**:** $\{x_n\}$ is called a b-Cauchy sequence if $d(x_n,x_m)\to 0$ as $n,m\to\infty$.
- (iii)**:** (X,d) is called b-complete if every b-Cauchy sequence in X b-converges.

Proposition 1.5 ([2])**.** *In a b-metric space (X,d), the following assertions hold.*

- (p_1)**:** *A b-convergent sequence has a unique limit.*
- (p_2)**:** *Each b-convergent sequence is a b-Cauchy sequence.*
- (p_3)**:** *In general, a b-metric is not continuous.*

In 2015, Khojasteh et al. [8] introduced the concept of a simulation function and a $\mathscr{Z}$-contraction mapping. They also proved the existence and uniqueness theorems of fixed points of $\mathscr{Z}$-contraction mapping in metric spaces. Here, we review some basic knowledge related to our investigation from [8].

Definition 1.6 ([8])**.** A function $\zeta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called a simulation function if it satisfies the following conditions:

- $(\zeta 1)$ $\zeta(t,s)<s-t$ for all $t,s>0$;
- $(\zeta 2)$ $\zeta(0,0)=0$;
- $(\zeta 3)$ if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ with $\lim\limits_{n\to\infty}t_n=\lim\limits_{n\to\infty}s_n>0$, then $\limsup\limits_{n\to\infty}\zeta(t_n,s_n)<0$.

We denote by $\mathscr{Z}$ the class of all simulation functions.

Example 1.7 ([8])**.** Let $\zeta_1,\zeta_2,\zeta_3:[0,\infty)\times[0,\infty)\to\mathbb{R}$ be defined by

- (a) $\zeta_1(t,s)=\psi(s)-\phi(t)$ for all $t,s\in[0,\infty)$, where $\psi,\phi:[0,\infty)\to[0,\infty)$ are two continuous functions such that $\psi(t)=\phi(t)=0$ if and only if $t=0$ and $\psi(t)<t\leq\phi(t)$ for all $t>0$;
- (b) $\zeta_2(t,s)=s-\varphi(s)-t$ for all $t,s\in[0,\infty)$, where $\varphi:[0,\infty)\to[0,\infty)$ is a continuous function such that $\varphi(t)=0$ if and only if $t=0$;
- (c) $\zeta_3(t,s)=s-\frac{f(t,s)}{g(t,s)}t$ for all $t,s\in[0,\infty)$, where $f,g:[0,\infty)\to[0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s)>g(t,s)$ for all $t,s>0$.

Then ζ_1,ζ_2, and ζ_3 are simulation functions.

Definition 1.8 ([8])**.** Let (X,d) be a metric space and $\zeta\in\mathscr{Z}$. A mapping $T:X\to X$ is called a $\mathscr{Z}$-contraction with respect to ζ if $\zeta(d(Tx,Ty),d(x,y))\geq 0$ for all $x,y\in X$.

Definition 1.9 ([4])**.** Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be asymptotically regular at point $x\in X$ if $\lim_{n\to\infty} d(T^n x, T^{n+1}x)=0$.

Recently, Hierro and Samet [9] introduced the class of extended simulation functions, which is more large than the class of simulation functions. In addition, They obtained a φ-admissibility result involving extended simulation functions for a new class of mappings with respect to a lower semi-continuous function.

Definition 1.10. Recall that $\theta:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called an extended simulation function (for short, an e-simulation function) if it satisfies the following conditions:

(θ_1)**:** for any sequences $\{\alpha_n\}$, $\{\beta_n\}\subset(0,\infty)$,

$$\lim_{n\to\infty}\alpha_n, =\lim_{n\to\infty}\beta_n = l\in(0,\infty),\ \ \beta_n>l \ \ \Rightarrow\ \ \limsup_{n\to\infty}\theta(\alpha_n,\beta_n)<0;$$

(θ_2)**:** $\theta(\alpha,\beta)<\beta-\alpha$ for every $\alpha,\beta>0$;

(θ_3)**:** for any sequence $\{\alpha_n\}\subset(0,\infty)$,

$$\lim_{n\to\infty}\alpha_n = l\in[0,\infty), \theta(\alpha_n,l)\geq 0,\ \ \Rightarrow\ \ l=0.$$

In this paper, we extend and generalize the notion in the result of Hierro and Samet, which is e_s-simulation function in the setting of b-metric spaces. An example to illustrate our main results is presented. As an application, we investigate the existence of a unique solution to homotopy theory.

2. Main Results

In this section, we introduce the new concept of an e_s-simulation function and use this concept to prove the existence of a unique fixed point of generalized contraction mappings.

Definition 2.1. Let (X,d) be a b-metric space with coefficient $s\geq 1$. $\theta_s:[0,\infty)\times[0,\infty)\to\mathbb{R}$ is called an e_s-simulation function if it satisfies the following conditions:

(θ_{s1})**:** $\theta_s(\alpha,\beta)<\beta-\alpha$ for every $\alpha,\beta>0$;

(θ_{s2})**:** for any sequences $\{\alpha_n\}$, $\{\beta_n\}\subset(0,\infty)$, we have

$$l\leq\lim_{n\to\infty}\alpha_n\leq\lim_{n\to\infty}\beta_n\leq sl,\ \ \beta_n>l\in(0,\infty)\ \ \Rightarrow\ \ \limsup_{n\to\infty}\theta_s(s\alpha_n,\beta_n)<0;$$

Definition 2.2. Let (X,d) be a b-metric space with coefficient $s\geq 1$. A mapping $T:X\to X$ is called an e_s-simulation contraction if there exists an e_s-simulation function $\theta_s:[0,\infty)\times[0,\infty)\to\mathbb{R}$ such that

$$\theta_s(sd(Tx,Ty),d(x,y))\geq 0,\quad \forall x,y\in X. \tag{2.1}$$

Remark 2.3. By (θ_{s1}), if θ is an e_s-simulation function, then $\theta_s(s\alpha,\beta)<0$ for all $\alpha,\beta>0$. Hence, if T is an e_s-simulation contraction with respect to θ_s, then $sd(Tx,Ty)<d(x,y)$ for all distinct $x,y\in X$.

Theorem 2.4. *Let (X,d) be a complete b-metric space with coefficient $s\geq 1$, and let $T:X\to X$ be an e_s-simulation contraction with respect to θ_s. Then T has a unique fixed point.*

Proof. Let x be an arbitrary point in X. The proof is split into 5 steps.

Step I: Prove that T is an asymptotically regular mapping at x.

Suppose that $T^n x \neq T^{n-1}x$ for all $n \in \mathbb{N}$. By (2.1) and (θ_{s1}), we have

$$\begin{aligned} 0 &\leq \theta(sd(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) \\ &< d(T^n x, T^{n-1}x) - sd(T^{n+1}x, T^n x) \\ &< d(T^n x, T^{n-1}x) - d(T^{n+1}x, T^n x). \end{aligned}$$

Therefore, $\{d(T^n x, T^{n-1}x)\}$ is a monotonically decreasing sequence of nonnegative reals and $\lim_{n\to\infty} d(T^n x, T^{n+1}x) := r$. Now, we have to prove that $r = 0$. Assume that $r > 0$. By using (θ_{s2}), we have

$$0 \leq \limsup_{n\to\infty} \theta_s(sd(T^{n+1}x, T^n x), d(T^n x, T^{n-1}x)) < 0,$$

which is a contradiction. Then $r = 0$.

On the other hand, we may assume that $T^p x = T^{p-1}x$ for some $p \in \mathbb{N}$. Let $y := T^{p-1}x$. Hence $Ty = y$ and

$$T^n y = T^{n-1}Ty = T^{n-1}y = \ldots = Ty = y$$

for all $n \in \mathbb{N}$. For sufficient large $n \in \mathbb{N}$, we have

$$d(T^n x, T^{n+1}x) = d(T^{n-p+1}T^{p-1}x, T^{n-p+2}T^{p-1}x) = d(T^{n-p+1}y, T^{n-p+2}y) = d(y,y) = 0.$$

Therefore, for each $x \in X$, we have

$$\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0. \tag{2.2}$$

Step II: Prove that the Picard sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$.

Assume to contrary that $\{x_n\}$ is not bounded. Without loss of generality, we can assumes that $x_{n+p} \neq x_n$ for all $n, p \in \mathbb{N}$. Then, there exists a subsequence $\{x_{n_k}\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the smallest number such that $d(x_{n_{k+1}}, x_{n_k}) > 1$ and $d(x_m, x_{n_k}) \leq 1$ for all $m \in \mathbb{N}$ with $n_k \leq m \leq n_{k+1} - 1$. By the triangular inequality, we have

$$1 < d(x_{n_{k+1}}, x_{n_k}) \leq s[d(x_{n_{k+1}}, x_{n_{k+1}-1}) + d(x_{n_{k+1}-1}, x_{n_k})] \leq sd(x_{n_{k+1}}, x_{n_{k+1}-1}) + s.$$

Taking limit as $k \to \infty$ in above inequality and using (2.2), we have

$$1 \leq \liminf_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq \limsup_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) \leq s.$$

From Remark 2.3, we obtain that

$$sd(x_{n_{k+1}}, x_{n_k}) \leq d(x_{n_{k+1}-1}, x_{n_k-1}) \leq s[d(x_{n_{k+1}-1}, x_{n_k}) + d(x_{n_k}, x_{n_k-1})] \leq s + sd(x_{n_k}, x_{n_k-1}).$$

Letting $k \to \infty$ and using (2.2), we deduce that there exist

$$\lim_{k\to\infty} d(x_{n_{k+1}}, x_{n_k}) = 1 \quad \text{and} \quad \lim_{k\to\infty} d(x_{n_{k+1}-1}, x_{n_k-1}) = s. \tag{2.3}$$

By condition (θ_{s2}) with $\alpha_k = d(x_{n_{k+1}}, x_{n_k})$, $\beta_k = d(x_{n_{k+1}-1}, x_{n_k-1})$, and $l = 1$, we can see that

$$0 \leq \limsup_{k\to\infty} \theta_s(sd(x_{n_{k+1}}, x_{n_k}), d(x_{n_{k+1}-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Hence $\{x_n\}$ is bounded.

Step III: Prove that $\{x_n\}$ is a b-Cauchy sequence.

Let $C_n := \sup\{d(x_i,x_j) : i,j \geq n\}$. Thus $\{C_n\}$ is a monotonically non-increasing sequence of nonnegative real numbers. Since $\{x_n\}$ is a bounded sequence, we have that $\{C_n\}$ is a monotonic bounded sequence and $C \geq 0$ is such that $\lim_{n\to\infty} C_n = C$.

Next, we prove that $C = 0$. Let $C > 0$. There exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(x_{m_k}, x_{n_k}) \leq C_k$$

for all $k \in \mathbb{N}$ and hence $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C$. From Remark 2.3 and the definition of C_n, we have $sd(x_{m_k}, x_{n_k}) \leq d(x_{m_k-1}, x_{n_k-1}) \leq C_{k-1}$. Taking limit as $k \to \infty$, and using $\lim_{n\to\infty} C_n = C$ and $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = C$, we have

$$sC \leq \liminf_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq \limsup_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) \leq C.$$

which together with $s > 1$ observes that $C = 0$. In the same way, if $s = 1$, using (θ_{s2}) with $\alpha_k = d(x_{m_k}, x_{n_k})$, $\beta_k = d(x_{m_k-1}, x_{n_k-1})$, and $l = C$, we have

$$0 \leq \limsup_{n\to\infty} \theta_s(sd(x_{m_k}, x_{n_k}), d(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Then $C = 0$. Hence $\{x_n\}$ is a b-Cauchy sequence in X.

Step IV: Prove that T has a fixed point.

Since (X,d) is a complete b-metric space, one sees that there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$, that is, $\lim_{n\to\infty} d(x_n, u) = 0$.

Next, we are going to claim that u is a fixed point of T by contradiction. Assume to contrary that u is not a fixed point of T, that is, $Tu \neq u$. Hence, $d(u,Tu) > 0$. Note that there is $n_1 \in \mathbb{N}$ such that $d(x_n, u) < d(u,Tu)$ for all $n \geq n_1$. In particular, $x_n \neq Tu$ for all $n \geq n_1$, that is,

$$d(Tx_n, Tu) = d(x_{n+1}, Tu) > 0 \tag{2.4}$$

for all $n \geq n_1$.

On the other hand, there exists $n_2 \in \mathbb{N}$ such that $x_n = u$ for all $n \geq n_2$. Hence, there exists a subsequence $\{x_{\sigma(n)}\}$ of $\{x_n\}$ such that $x_{\sigma(n)} \neq u$ for all $n \in \mathbb{N}$. Let $n_3 \in \mathbb{N}$ be such that $\sigma(n_3) \geq n_1$. Then, by (2.4), we have $d(x_{\sigma(n)}, u) > 0$ and $d(Tx_{\sigma(n)}, Tu) > 0$ for all $n \geq n_3$. By using (2.1) and (θ_{s2}), we have

$$\begin{aligned} 0 &\leq \theta_s(sd(Tx_{\sigma(n)}, Tu), d(x_{\sigma(n)}, u)) \\ &< d(x_{\sigma(n)}, u) - sd(Tx_{\sigma(n)}, Tu) \\ &\leq d(x_{\sigma(n)}, u) - d(Tx_{\sigma(n)}, Tu) \end{aligned}$$

for all $n \geq n_3$, which means that $0 \leq d(Tx_{\sigma(n)}, Tu) < d(x_{\sigma(n)}, u)$ for all $n \geq n_3$. In particular, we obtain $x_{\sigma(n)+1} = Tx_{\sigma(n)} \to Tu$. By the unicity of the limit, we get $u = Tu$, which is a contradiction with the fact that we have supposed that $Tu \neq u$. Therefore, u is a fixed point of T.

Step V: Prove that T has a unique fixed point.

Let u, v be two fixed points of T such that $u \neq v$. By using hypothesis (θ_{s1}) and inequality (2.1), we obtain

$$0 \leq \theta_s(sd(Tu,Tv), d(u,v)) = \theta_s(sd(u,v), d(u,v)) < d(u,v) - sd(u,v) \leq 0,$$

which is a contradiction. Therefore, $u = v$ and T has a unique fixed point. □

Example 2.5. Let $X=[0,\infty]$ and $d:X\times X\to[0,\infty)$ be defined by

$$d(x,y)=\begin{cases}(x+y)^2 & \text{if } x\neq y;\\ 0 & \text{if } x=y\end{cases}$$

for all $x,y\in X$. Therefore, (X,d) is a complete b-metric space with $s=2$. Define $T:X\to X$ and $\theta_s:[0,\infty)\times[0,\infty)\to\mathbb{R}$ by

$$Tx=\begin{cases}\frac{1}{2} & \text{if } x=2;\\ \frac{x}{2\sqrt{2}} & \text{if } x\in X\backslash\{2\},\end{cases}$$

and

$$\theta_s(\alpha,\beta)=\begin{cases}1-\alpha & \text{if } \beta=0;\\ \beta-3\alpha & \text{if } \beta>0,\end{cases}$$

for all $\alpha,\beta>0$.

Now, we demonstrate that θ_s is an e_s-simulation function but not simulation function. If $\alpha=\beta=0$, then $\theta_s(\alpha,\beta)=\theta_s(0,0)=1\neq 0$, which implies that $(\zeta 1)$ does not hold, so θ_s is not a simulation function. For any $\alpha,\beta>0$, we have $\theta_s(\alpha,\beta)=\beta-3\alpha<\beta-\alpha$. For any two sequence $\{\alpha_n\},\{\beta_n\}\subset(0,\infty)$ such that there exists $l>0$ with

$$l\leq\lim_{n\to\infty}\alpha_n\leq\lim_{n\to\infty}\beta_n\leq 2l,\ \ \beta_n>l\ \text{ for all }\ n\in\mathbb{N},$$

we have

$$\limsup_{n\to\infty}\theta_s(s\alpha_n,\beta_n)=\limsup_{n\to\infty}(\beta_n-6\alpha_n)\leq\limsup_{n\to\infty}\beta_n-6\liminf_{n\to\infty}\alpha_n\leq 2l-6l<0.$$

Then (θ_{s1}) and (θ_{s2}) hold and hence θ_s is an e_s-simulation function.

Next, we divide the proof that T satisfies inequality (2.1) into 3 cases.

Case I: For $x,y\in X$ with $x=y$, we have $\theta_s(sd(Tx,Ty),d(x,y))=\theta_s(0,0)=1\geq 0$.

Case II: For $x,y\in X\backslash\{2\}$ with $x\neq y$, we obtain that

$$\begin{aligned}\theta_s(sd(Tx,Ty),d(x,y)) &= \theta_s\left(2\left(\frac{x}{2\sqrt{2}}+\frac{y}{2\sqrt{2}}\right)^2,(x+y)^2\right)\\ &= (x+y)^2-\frac{3}{4}(x+y)^2\\ &\geq 0.\end{aligned}$$

Case III: For $(x,y)\in X\backslash\{2\}\cup\{2\}$ or $\{2\}\cup X\backslash\{2\}$, we see that

$$\begin{aligned}\theta_s(sd(Tx,Ty),d(x,y)) &= \theta_s\left(2\left(\frac{x}{2\sqrt{2}}+\frac{1}{2}\right)^2,(x+2)^2\right)\\ &= (x+2)^2-\frac{3}{4}(x+\sqrt{2})^2\\ &\geq (x+2)^2-\frac{3}{4}(x+2)^2\geq 0.\end{aligned}$$

Therefore, (2.1) is satisfied. Hence, all the conditions of Theorem (2.4) hold, so T has a unique fixed point. In this case, 0 is a fixed point of T.

We have the following corollaries from Theorem 2.4.

Corollary 2.6 ([8]). *Let (X,d) be a complete metric space, and let $T : X \to X$ be a $\mathscr{Z}$-contraction with respect to ζ. Then T has a unique fixed point.*

3. Application to Homotopy

In this section, we study the existence of a unique solution to homotopy theory.

Theorem 3.1. *Let (X,d) be a b-metric space, U be an open subset of X, and $\overline{U}$ be a closed subset of X such that $U \subseteq \overline{U}$. Suppose that $H : \overline{U} \times [0,1] \to X$ with the following assumptions:*

1. *$\lambda \in [0,1]$ and $x \neq H(x,\lambda)$ for all $x \in \partial U$ (here ∂U denote the boundary of $U \subseteq X$);*
2. *there exists $L \in [0,1)$ and $M \geq 0$ such that*
$$d(H(x,\lambda),H(y,\lambda)) \leq Ld(x,y)$$
and
$$d(H(x,\lambda),H(x,\mu)) \leq M|\lambda-\mu|$$
for all $x,y \in \overline{U}$ and $\lambda,\mu \in [0,1]$.

Then $H(\cdot,0)$ has a fixed point if and only if $H(\cdot,1)$ has a fixed point.

Proof. Let the set
$$A = \{\lambda \in [0,1] | x = H(x,\lambda) \text{ for some } x \in U\}.$$
Since $H(\cdot,0)$ has a fixed point in U, we have $0 \in A$ and so A is a nonempty set.

Next, we demonstrate that A is both closed and open in $[0,1]$. From the connectedness, we have that $A = [0,1]$. Consequently, $H(\cdot,1)$ has a fixed point in U.

Step I: First, we sprove that A is closed in $[0,1]$. Let $\{\lambda_n\} \subseteq A$ with $\lim_{n\to\infty} \lambda_n = \lambda \in [0,1]$. We prove that $\lambda \in A$. Since $\lambda_n \in A$ for all $n \in \mathbb{N}$, there exists $x_n \in U$ such that $x_n = H(x_n,\lambda_n)$. For any $n,m \in \mathbb{N}$, we have
$$\begin{aligned} d(x_n,x_m) &= d(H(x_n,\lambda_n),H(x_m,\lambda_m)) \\ &\leq s[d(H(x_n,\lambda_n),H(x_n,\lambda_m)) + d(H(x_n,\lambda_m),H(x_m,\lambda_m))] \\ &\leq sM|\lambda_n-\lambda_m| + sLd(x_n,x_m), \end{aligned}$$
which implies that
$$d(x_n,x_m) \leq \frac{sM}{1-sL}|\lambda_n-\lambda_m|. \tag{3.1}$$
Taking limit as $n \to \infty$ in (3.1), we see that $\lim_{n\to\infty} d(x_n,x_m) = 0$. Then $\{x_n\}$ is a Cauchy sequence in X. Since (X,d) is a complete b-metric space, then there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$, that is, $\lim_{n\to\infty} d(x_n,x) = 0$. For any $n \in \mathbb{N}$, we obtain that
$$\begin{aligned} d(x_n,H(x,\lambda)) &= d(H(x_n,\lambda_n),H(x,\lambda)) \\ &\leq s[d(H(x_n,\lambda_n),H(x_n,\lambda)) + d(H(x_n,\lambda),H(x,\lambda))] \\ &\leq sM|\lambda_n-\lambda| + sLd(x_n,x). \end{aligned}$$
Taking limit as $n \to \infty$ in the inequality above, we have $\lim_{n\to\infty} d(x_n,H(x,\lambda)) = 0$, which implies that $x = \lim_{n\to\infty} x_n = H(x,\lambda)$. Therefore, $\lambda \in A$ and hence A is closed in $[0,1]$.

Step II: We prove that A is open in $[0,1]$.

Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since U is open, we have that there exists $r > 0$ such that $B_d\left(x_0, \frac{r}{s}\right) \subseteq U$. Fix $\varepsilon > 0$ with $\varepsilon < \frac{r(1-L)}{sM}$, and let $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$. For $x \in \overline{B_d\left(x_0, \frac{r}{s}\right)}$, we have

$$\begin{aligned} d(H(x,\lambda),x_0) &= d(H(x,\lambda),H(x_0,\lambda_0)) \\ &\leq s[d(H(x,\lambda),H(x,\lambda_0)) + d(H(x,\lambda_0),H(x_0,\lambda_0))] \\ &\leq sM|\lambda - \lambda_0| + sLd(x,x_0) \leq r. \end{aligned}$$

Then, for each fixed $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, we obtain that

$$H(\cdot,\lambda) : \overline{B_d\left(x_0, \frac{r}{s}\right)} \to \overline{B_d\left(x_0, \frac{r}{s}\right)}.$$

Thus $H(\cdot,\lambda)$ has a fixed point in $\overline{U}$. From Assumption 1, we have that the fixed point must be in U. Hence $\lambda \in A$ for all $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$, so A is open in [0,1]. For the reverse implication, we use the same strategy and the desired conclusion follows immediately. □

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