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# INFINITELY MANY HIGH ENERGY RADIAL SOLUTIONS FOR A KIRCHHOFF-SCHRÖDINGER-POISSON SYSTEM

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**Abstract.** This paper is devoted to the following Kirchhoff-Schrödinger-Poisson system

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+u+\phi u=f(u) \ \text{ in } \mathbb{R}^3, \\ -\Delta\phi=u^2 \ \text{ in } \mathbb{R}^3, \end{cases}$$

where $a>0$ and $b\geq 0$ are two constants, and $f\in C(\mathbb{R},\mathbb{R})$. We obtain infinitely many high energy radial solutions by using variational methods and the symmetric mountain pass lemma. The main difficulty in this paper is to obtain the boundedness of the PS-sequence. We use an extra property related to the Pohozaev identity to overcome the difficulty.

**Keywords.** High energy radial solutions; Kirchhoff-Schrödinger-Poisson system; Pohozaev identity; Symmetric mountain pass lemma.

## 1. INTRODUCTION AND MAIN RESULTS

In this paper, we consider the following Kirchhoff-Schrödinger-Poisson system

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+u+\phi u=f(u) \ \text{ in } \mathbb{R}^3, \\ -\Delta\phi=u^2 \ \text{ in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $a>0$ and $b\geq 0$ are two constants, and $f\in C(\mathbb{R},\mathbb{R})$ satisfies the following conditions:

(A1) $f(t)$ is odd;

(A2) $\lim\limits_{t\to 0}\frac{f(t)}{t}=0$;

(A3) there exist $q\in(3,5)$ such that $\lim\limits_{|t|\to\infty}\frac{f(t)}{|t|^q}=0$;

(A4) there exist $\mu>4$ such that $\frac{1}{\mu}f(t)t\geq F(t)>0$ for all $t\in\mathbb{R}\backslash\{0\}$, where $F(t)=\int_0^t f(s)ds$.

We remark that these hypotheses were first studied by Berestycki and Lions in [4].

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When $a=1$ and $b=0$, system (1.1) reduces to the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u+u+\phi u=f(u), & \text{in } \mathbb{R}^3, \\ -\Delta\phi=u^2, & \text{in } \mathbb{R}^3. \end{cases} \tag{1.2}$$

Recently, attention has been paid to problems like (1.2); see, e.g., [6, 7, 8, 9, 10, 12, 13, 14, 19, 21, 24, 26] and the references therein. For autonomous case, papers [8, 24] established the existence of ground state solutions under some suitable conditions on $f$. In [10, 12, 19], it was proved that the system has infinitely many sign-changing solutions by variational methods and some analytical techniques. In [21], the authors obtained infinitely many high energy radial solutions by borrowing the method introduced in [14]. For non-autonomous case, papers [6, 9, 26] proved the existence of solutions for a class of Schrödinger-Poisson system. When $f(u)=a(x)|u|^{p-2}u+\lambda k(x)u, 4<p<6$, Eq. (1.2) was widely considered by many researchers; see, e.g., [7, 13]. For $2<p<4$, Gan et al. [11] used the mountain pass lemma to prove multiple positive solutions.

After Benci and Fortunato [2], system (1.1) has been extensively studied; see, e.g., [15, 16, 17, 20, 23, 25]. In [15, 23], the authors studied the existence of high energy solutions of the Kirchhoff-Schrödinger problem in the following form $-\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(x,u), x\in\mathbb{R}^N$. Liu and He [16] obtained similar result in $\mathbb{R}^3$. By using constraint variational methods and the quantitative deformation lemma, Zhang and Wang [25] obtained a least-energy sign-changing (or nodal) solution to this problem. Using the Nehari manifold and variational methods, Wang et al. [20] proved that this problem had a least energy nodal solution. Without assuming the AR condition on $f$, Lu [17] proved the existence of positive radial solutions. They used variational methods combining a monotonicity approach with a delicate cut-off technique.

Through the above analysis, there are few results on high energy radial solutions for system (1.1). Since there are both nonlocal operators and nonlocal nonlinear terms, the study of system (1.1) becomes more complicated. In this paper, the main difficulty is to obtain the boundedness of PS-sequence. We apply the method in [14, 21] to overcome the difficulty.

The result of this paper is the following:

**Theorem 1.1.** *Assum that $f(t)$ satisfies (A1)-(A4). Then problem (1.1) possesses infinitely many high energy radial solutions.*

In the following, we introduce some notations (we refer to [1]):

• $H^1(\mathbb{R}^3)$ is the usual Sobolev space endowed with norm

$$\|u\|^2=\int_{\mathbb{R}^3}(a|\nabla u|^2+u^2)\,\mathrm{d}x.$$

• $H_r^1(\mathbb{R}^3)$ is the subspace of $H^1(\mathbb{R}^3)$ containing only the radial functions.

• For any $1\le s<\infty, L^s(\mathbb{R}^3)$ is the usual Lebesgue space with the norm $\|u\|_s^s=\int_{\mathbb{R}^3}|u|^s\,\mathrm{d}x$.

• $D^{1,2}(\mathbb{R}^3)$ is completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|u\|^2_{D^{1,2}(\mathbb{R}^3)}=\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x$.

• $\overline{S}$ is optimal Sobolev embedding constant, denoted by

$$\overline{S}=\inf_{u\in D^{1,2}(\mathbb{R}^3)\setminus\{0\}}\frac{\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x}{\left(\int_{\mathbb{R}^3}|u|^6\,\mathrm{d}x\right)^{\frac{1}{3}}}.$$

By the Lax-Milgram theorem, for any $u \in H_r^1(\mathbb{R}^3)$, there exists an unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u = u^2$, we refer to [3]. Furthermore,

$$\phi_u = \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{u^2(y)}{|x-y|}\,\mathrm{d}y. \tag{1.3}$$

Substituting (1.3) into system (1.1), we can rewrite (1.1) in the following equivalent equation

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u + u + \phi_u u = f(u) \ \text{ in } \mathbb{R}^3. \tag{1.4}$$

The corresponding energy functional of problem (1.4) is defined on $H_r^1(\mathbb{R}^3)$ by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+u^2)\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \int_{\mathbb{R}^3}F(u)\,\mathrm{d}x. \tag{1.5}$$

Obviously, energy functional $I(u)$ is well defined and is of $C^1$ with derivative given by

$$\begin{aligned}\langle I'(u),\psi\rangle &= \int_{\mathbb{R}^3}(a\nabla u\cdot\nabla\psi + u\psi)\,\mathrm{d}x + b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\int_{\mathbb{R}^3}\nabla u\nabla\psi\,\mathrm{d}x + \int_{\mathbb{R}^3}\phi_u u\psi\,\mathrm{d}x \\ &- \int_{\mathbb{R}^3}f(u)\psi\,\mathrm{d}x, \end{aligned} \tag{1.6}$$

for any $\psi \in H_r^1(\mathbb{R}^3)$. From (A2) and (A3), for any $\varepsilon > 0$, there exist $C_\varepsilon > 0, q \in (3,5)$ such that

$$|f(t)| \le \frac{\varepsilon}{2}|t| + C_\varepsilon|t|^q.$$

Then we have

$$\begin{aligned} I(u) &\ge \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+u^2)\,\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \frac{\varepsilon}{2}\int_{\mathbb{R}^3}|u|^2\,\mathrm{d}x \\ &- C_\varepsilon\int_{\mathbb{R}^3}|u|^{q+1}\,\mathrm{d}x \\ &\ge \frac{1}{2}(1-\varepsilon)\int_{\mathbb{R}^3}(a|\nabla u|^2+u^2)\,\mathrm{d}x - C_\varepsilon\int_{\mathbb{R}^3}|u|^{q+1}\,\mathrm{d}x. \end{aligned} \tag{1.7}$$

Taking $\varepsilon = \frac{1}{2}$, we define

$$J(u) = \frac{1}{4}\int_{\mathbb{R}^3}(a|\nabla u|^2+u^2)\,\mathrm{d}x - C\int_{\mathbb{R}^3}|u|^{q+1}\,\mathrm{d}x. \tag{1.8}$$

Clearly, $J$ is well defined and is of $C^1$ with derivative given by

$$\langle J'(u),v\rangle = \frac{1}{2}\int_{\mathbb{R}^3}(a\nabla u\cdot\nabla v + uv)\,\mathrm{d}x - C(q+1)\int_{\mathbb{R}^3}|u|^{q-1}uv\,\mathrm{d}x. \tag{1.9}$$

for any $v \in H_r^1(\mathbb{R}^3)$. And, there is

$$I(u) \ge J(u) \ \text{ for all } u \in H_r^1(\mathbb{R}^3). \tag{1.10}$$

Now, as in [8], we define a functional:

$$P(u) = \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x + \frac{3}{2}\int_{\mathbb{R}^3}u^2\,\mathrm{d}x + \frac{b}{2}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)^2 + \frac{5}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - 3\int_{\mathbb{R}^3}F(u)\,\mathrm{d}x,$$

which is the Pohozaev functional associated with (1.1). When $u$ is a weak solution to (1.1), then $P(u) = 0$.

The paper is organized as follow. In Section 1, we introduced the main purpose of this paper. The proof of the Theorem 1 will be given in Section 2. We denote various positive constants as $c, c_i, C$ or $C_i$ $(i = 0, 1, 2, 3, ...)$ for convenience.

## 2. Proof of Theorem 1

**Proof of Theorem 1..1** In this section we divide this proof into four steps.

**Step 1:** Symmetric mountain pass geometry structure. Assume (A1)-(A4) hold, then (1) there exist $\rho > 0$ and $\alpha > 0$ such that

$$I(u) \geq J(u) \geq 0, \text{ if } \|u\| \leq \rho,$$
$$and\ I(u) \geq J(u) \geq \alpha, \text{ if } \|u\| = \rho.$$

(2) for any $n \in \mathbb{N}$, there exists an odd continuous mapping $\tau_{0n} : S^{n-1} = \{\delta = (\delta_1, ..., \delta_n) \in \mathbb{R}^n; |\delta| = 1\} \to H_r^1(\mathbb{R}^3)$ such that

$$J(\tau_{0n}(\delta)) \leq I(\tau_{0n}(\delta)) < 0, \text{ for all } \delta \in S^{n-1},$$

which shows that $I(u)$ and $J(u)$ have symmetric mountain pass geometry structure.

(1) In fact, from (1.8) and Sobolev inequality, we obtain

$$\begin{aligned} J(u) &= \frac{1}{4}\int_{\mathbb{R}^3}(a|\nabla u|^2 + u^2)\,\mathrm{d}x - C\int_{\mathbb{R}^3}|u|^{q+1}\,\mathrm{d}x \\ &\geq \frac{1}{4}\|u\|^2 - C_1\|u\|^{q+1} \\ &= \frac{1}{4}\|u\|^2(1 - 4C_1\|u\|^{q-1}). \end{aligned}$$

Letting $\rho = (\frac{1}{8C_1})^{\frac{1}{q-1}} > 0$, we have

$$J(u) \geq \frac{1}{8}\rho^2 = \alpha > 0, \text{ if } \|u\| = \rho.$$

Obviously, $J(u) \geq 0$ if $\|u\| \leq \rho$.

(2) Following [5, Theorem 10], for any $n \in \mathbb{N}$, we can find an odd continuous mapping $\zeta_n : S^{n-1} \to H_r^1(\mathbb{R}^3)$ such that $\zeta_n(\delta) \neq 0$ for all $\delta \in S^{n-1}$. (A4) implies

$$\lim_{t\to\infty}\frac{F(t)}{|t|^4} = +\infty. \tag{2.1}$$

Then, by (1.5), Fatou's lemma and (2.1), one sees that there is

$$\begin{aligned} &\lim_{t\to\infty}\frac{I(t\zeta_n(\delta))}{t^4} \\ &= \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\zeta_n(\delta)|^2\,\mathrm{d}x\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{\zeta_n(\delta)}\zeta_n^2(\delta)\,\mathrm{d}x - \lim_{t\to\infty}\frac{1}{t^4}\int_{\mathbb{R}^3}F(t\zeta_n(\delta))\,\mathrm{d}x \\ &\leq \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla\zeta_n(\delta)|^2\,\mathrm{d}x\right)^2 + \frac{1}{4}\int_{\mathbb{R}^3}\phi_{\zeta_n(\delta)}\zeta_n^2(\delta)\,\mathrm{d}x - \int_{\mathbb{R}^3}\lim_{t\to\infty}\frac{F(t\zeta_n(\delta))}{|t\zeta_n(\delta)|^4}|\zeta_n(\delta)|^4\,\mathrm{d}x \\ &< 0. \end{aligned}$$

So we take $\tau_{0n}(\delta)(x) = t\zeta_n(\delta)(x) : S^{n-1} \to H_r^1(\mathbb{R}^3)$ for $t$ large such that $I(\tau_{0n})(\delta) < 0$.

**Step 2:** Critical value. $I(u)$ and $J(u)$ have a symmetric mountain pass geometry, and we can define symmetric mountain pass values. Here we follow [18, Chapter 9] essentially and set for $n\in\mathbb{N}$

$$a_n=\inf_{\tau\in\Gamma_n}\max_{\delta\in E_n} I(\tau(\delta)),\ \ b_n=\inf_{\tau\in\Gamma_n}\max_{\delta\in E_n} J(\tau(\delta)),\ n\in\mathbb{N}. \tag{2.2}$$

Here $E_n=\{\delta=(\delta_1,...,\delta_n)\in\mathbb{R}^n:|\delta|\le 1\}$ and a family of mapping $\Gamma_n$ is defined by

$$\Gamma_n=\{\tau\in C(E_n,H_r^1(\mathbb{R}^3)):\tau(\delta)=\tau(-\delta),\forall\delta\in E_n;\tau(\delta)=\tau_{0n}(\delta),\forall\delta\in\partial E_n\}, \tag{2.3}$$

where $\tau_{0n}(\delta):\partial E_n=S^{n-1}\to H_r^1(\mathbb{R}^3)$ is given in Step 1.

**Step 3:** The (PS) condition. We define the map $\Phi:\mathbb{R}\times H_r^1(\mathbb{R}^3)\to H_r^1(\mathbb{R}^3)$ by $\Phi(\theta,v)(x)=v(e^{-\theta}x)$. For $\theta\in\mathbb{R},v\in H_r^1(\mathbb{R}^3)$, we remark $\widetilde{I}=I\circ\Phi$, and have

$$\begin{aligned}\widetilde{I}(\theta,v) &= I(\Phi(\theta,v))\\ &= \frac{ae^{\theta}}{2}\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x+\frac{e^{3\theta}}{2}\int_{\mathbb{R}^3}v^2\,\mathrm{d}x+\frac{be^{2\theta}}{4}\left(\int_{\mathbb{R}^3}|\nabla v|^2\,\mathrm{d}x\right)^2+\frac{e^{5\theta}}{4}\int_{\mathbb{R}^3}\phi_v v^2\,\mathrm{d}x\\ &- e^{3\theta}\int_{\mathbb{R}^3}F(v)\,\mathrm{d}x.\end{aligned}$$

Obviously, by Step 1, $\widetilde{I}$ also has a symmetric mountain pass geometry. Therefor, we define

$$\widetilde{a}_n=\inf_{\widetilde{\tau}\in\widetilde{\Gamma}_n}\max_{\delta\in E_n}\widetilde{I}(\widetilde{\tau}(\delta)), \tag{2.4}$$

where

$$\begin{aligned}&\widetilde{\Gamma}_n=\{\widetilde{\tau}\in C(E_n,H_r^1(\mathbb{R}^3)):\widetilde{\tau}(\delta)=(\theta(\delta),\xi(\delta))\}\text{ satisfies}\\ &\{(\theta(-\delta),\xi(-\delta))=(\theta(\delta),-\xi(\delta)),\forall\delta\in E_n;(\theta(\delta),\xi(\delta))=(0,\tau_{0n}(\delta)),\forall\delta\in\partial E_n\}.\end{aligned}$$

For any $\tau\in\Gamma_n$, we can see that $(0,\tau(\delta))\in\widetilde{\Gamma}_n$ and we may regard $\Gamma_n\subset\widetilde{\Gamma}_n$. Thus, by the definitions of $a_n,\widetilde{a}_n$ and $\widetilde{I}(0,v)=I(v)$, we have $\widetilde{a}_n\le a_n$.

Next, for given $\widetilde{\tau}(\delta)=(\theta(\delta),\xi(\delta))\in\widetilde{\Gamma}_n$, we set $\tau(\delta)=\xi(\delta)(e^{-\theta(\delta)}x)$. We can verify that $\tau(\delta)\in\Gamma_n$ by $\widetilde{I}(\theta,v(x))=I(v(e^{-\theta}x)),I(\tau(\delta))=\widetilde{I}(\widetilde{\tau}(\delta))$ for all $\delta\in E_n$. Then we also have $\widetilde{a}_n\ge a_n$, so $a_n=\widetilde{a}_n$. From the definition of $a_n$, for any $j\in\mathbb{N}$, there exists $\tau_j\in\Gamma_n$ such that

$$\max_{\delta\in E_n} I(\tau_j(\delta))\le a_n+\frac{1}{j}.$$

Since $\widetilde{a}_n=a_n,\widetilde{\tau}_j(\delta)=(0,\tau_j(\delta))\in\widetilde{\Gamma}_n$ satisfies $\max_{\delta\in E_n}\widetilde{I}(\widetilde{\tau}_j(\delta))\le\widetilde{a}_n+\frac{1}{j}$. By [22, Theorem 2.8], we can find a $(\theta_j,v_j)\in\mathbb{R}\times H_r^1(\mathbb{R}^3)$ such that

$$\mathrm{dist}_{H_r^1(\mathbb{R}^3)}((\theta_j,v_j),\widetilde{\tau}_j(E_n))\le\frac{2}{\sqrt{j}}, \tag{2.5}$$

$$\widetilde{I}(\theta_j,v_j)\in[a_n-\frac{1}{j},a_n+\frac{1}{j}], \tag{2.6}$$

$$\|D\widetilde{I}(\theta_j,v_j)\|_{H_r^1(\mathbb{R}^3)}\le\frac{2}{\sqrt{j}}. \tag{2.7}$$

Since $\widetilde{\tau}_j(E_n) \subset \{0\} \times H_r^1(\mathbb{R}^3)$, (2.5) implies $|\theta_j| \le \frac{2}{\sqrt{j}}$, that is, $\theta_j \to 0$. In other words, there exists sequence $(\theta_j, v_j) \in \mathbb{R} \times H_r^1(\mathbb{R}^3)$ such that

$$\widetilde{I}(\theta_j, v_j) \to a_n, \tag{2.8}$$

$$\|\widetilde{I}'(\theta_j, v_j)\|_{H_r^1(\mathbb{R}^3)} \to 0, \tag{2.9}$$

$$\theta_j \to 0. \tag{2.10}$$

Letting $u_j = \Phi(\theta_j, v_j)$, we have $I(u_j) \to a_n$ by (2.8). For any $(h, w) \in \mathbb{R} \times H_r^1(\mathbb{R}^3)$,

$$\langle \widetilde{I}'(\theta_j, v_j), (h, w)\rangle = \langle I'(u_j), \Phi(\theta_j, w)\rangle + H(u_j)h \to 0,$$

where

$$\begin{aligned} H(u_j) &= 2\langle I'(u_j), u_j\rangle - P(u_j) \\ &= \frac{3a}{2}\int_{\mathbb{R}^3} |\nabla u_j|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} u_j^2 \,\mathrm{d}x + \frac{3b}{2}\left(\int_{\mathbb{R}^3} |\nabla u_j|^2 \,\mathrm{d}x\right)^2 + \frac{3}{4}\int_{\mathbb{R}^3} \phi_{u_j} u_j^2 \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} (3F(u_j) - 2f(u_j)u_j) \,\mathrm{d}x. \end{aligned}$$

Letting $h = 1, w = 0$, we get $H(u_j) \to 0$. For any $v \in H_r^1(\mathbb{R}^3)$, we set $w(x) = v(e^{\theta_j} x), h = 0$ and have

$$\langle I'(u_j), v\rangle = \langle \widetilde{I}'(\theta_j, v_j), (0, w)\rangle = o(1)\|v(e^{\theta_j} x)\| = o(1)\|v\|.$$

Thus $I'(u_j) \to 0$ and

$$I(u_j) \to a_n > 0, \tag{2.11}$$

$$I'(u_j) \to 0, \tag{2.12}$$

$$H'(u_j) \to 0. \tag{2.13}$$

From (2.11), (2.13), and (A4), we obtain

$$\begin{aligned} a_n &\ge I(u_j) - \frac{1}{\mu}\langle I'(u_j), u_j\rangle \\ &= (\frac{1}{2} - \frac{1}{\mu})\|u_j\|^2 + b(\frac{1}{4} - \frac{1}{\mu})\left(\int_{\mathbb{R}^3} |\nabla u_j|^2 \,\mathrm{d}x\right)^2 + (\frac{1}{4} - \frac{1}{\mu})\int_{\mathbb{R}^3} \phi_{u_j} u_j^2 \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3} (\frac{1}{\mu} f(u_j)u_j - F(u_j)) \,\mathrm{d}x \\ &\ge \frac{\mu - 2}{2\mu}\|u_j\|^2. \end{aligned}$$

So, $\{u_j^{(n)}\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Going if necessary to a subsequence, still denoted by $\{u_j^{(n)}\}$, we may assume that

$$\begin{aligned} &u_j \rightharpoonup u \text{ in } H_r^1(\mathbb{R}^3), \\ &u_j \to u \text{ in } L^p(\mathbb{R}^3), \text{ where } 2 < p < 6, \\ &u_j(x) \to u(x) \text{ a.e. } x \in \mathbb{R}^3. \end{aligned} \tag{2.14}$$

As $I'(u_j) \to 0$ and $u_j \to u$ in $L^p(\mathbb{R}^3)$, we have

$$\begin{aligned} &I'(u_j)(u_j - u) \to 0, \\ &I'(u)(u_j - u) \to 0. \end{aligned} \tag{2.15}$$

Then, as $j \to \infty$,

$$\begin{aligned} o(1) &= \langle I'(u_j) - I'(u), u_j - u\rangle \\ &= \|u_j - u\|^2 + \int_{\mathbb{R}^3} (\phi_{u_j} u_j - \phi_u u)(u_j - u)\,\mathrm{d}x + \int_{\mathbb{R}^3} (f(u_j) - f(u))(u_j - u)\,\mathrm{d}x \\ &+ b\int_{\mathbb{R}^3} |\nabla u_j|^2\,\mathrm{d}x \int_{\mathbb{R}^3} \nabla u_j \cdot \nabla(u_j - u)\,\mathrm{d}x - b\int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_j - u)\,\mathrm{d}x \\ &= \|u_j - u\|^2 + \int_{\mathbb{R}^3} (\phi_{u_j} u_j - \phi_u u)(u_j - u)\,\mathrm{d}x + \int_{\mathbb{R}^3} (f(u_j) - f(u))(u_j - u)\,\mathrm{d}x \\ &+ b(\int_{\mathbb{R}^3} |\nabla u_j|^2\,\mathrm{d}x - \int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_j - u)\,\mathrm{d}x \\ &+ b\int_{\mathbb{R}^3} |\nabla u_j|^2\,\mathrm{d}x \int_{\mathbb{R}^3} |\nabla(u_j - u)|^2\,\mathrm{d}x. \end{aligned} \tag{2.16}$$

It is easy to see that

$$b\int_{\mathbb{R}^3} |\nabla u_j|^2\,\mathrm{d}x \int_{\mathbb{R}^3} |\nabla(u_j - u)|^2\,\mathrm{d}x \geq 0. \tag{2.17}$$

From (2.14), we have $u_j \rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$. Then it follows that

$$b(\int_{\mathbb{R}^3} |\nabla u_j|^2\,\mathrm{d}x - \int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x) \int_{\mathbb{R}^3} \nabla u \cdot \nabla(u_j - u)\,\mathrm{d}x \to 0. \tag{2.18}$$

Thanks to the definition of $\phi_{u_j}$, we have

$$\|\phi_{u_j}\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla \phi_{u_j}|^2\,\mathrm{d}x = \int_{\mathbb{R}^3} \phi_{u_j}(u_j)^2\,\mathrm{d}x \leq \|\phi_{u_j}\|_{L^6} \|u_j\|_{L^{\frac{12}{5}}}^2 \leq \overline{S}^{-\frac{1}{2}} \|u_j\|_{L^{\frac{12}{5}}}^2 \|\phi_{u_j}\|_{D^{1,2}},$$

which implies that $\|\phi_{u_j}\|_{D^{1,2}} \leq \overline{S}^{-\frac{1}{2}} \|u_j\|_{L^{\frac{12}{5}}}^2$. Combining the Hölder inequality and the Sobolev inequality, there is

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \phi_{u_j} u_j (u_j - u)\,\mathrm{d}x\right| &\leq \|\phi_{u_j}\|_{L^6} \|u_j\|_{L^{\frac{12}{5}}} \|u_j - u\|_{L^{\frac{12}{5}}} \\ &\leq \overline{S}^{-\frac{1}{2}} \|\phi_{u_j}\|_{D^{1,2}} \|u_j\|_{L^{\frac{12}{5}}} \|u_j - u\|_{L^{\frac{12}{5}}} \\ &\leq \overline{S}^{-1} \|u_j\|_{L^{\frac{12}{5}}}^3 \|u_j - u\|_{L^{\frac{12}{5}}} \to 0, j \to \infty. \end{aligned}$$

Similarly, $\left|\int_{\mathbb{R}^3} \phi_u u (u_j - u)\,\mathrm{d}x\right| \to 0$ as $j \to \infty$. Then we obtain

$$\int_{\mathbb{R}^3} (\phi_{u_j} u_j - \phi_u u)(u_j - u)\,\mathrm{d}x \to 0 \text{ as } j \to \infty. \tag{2.19}$$

From the Holder inequality, one has

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(f(u_j)-f(u))(u_j-u)\,\mathrm{d}x\right| &\leq \int_{\mathbb{R}^3}(\varepsilon|u_j|+C_\varepsilon|u_j|^q+\varepsilon|u|+C_\varepsilon|u|^q)(u_j-u)\,\mathrm{d}x\\ &\leq 2\varepsilon[\|u_j\|_{L^2}^2+\|u\|_{L^2}^2]+C_\varepsilon[\|u_j\|_{L^{q+1}}^q+\|u\|_{L^{q+1}}^q]\|u_j-u\|_{L^{q+1}}\\ &\leq c(\varepsilon+C_\varepsilon\|u_j-u\|_{L^{q+1}})\to 0,\, j\to\infty. \qquad (2.20)\end{aligned}$$

Consequently, by (2.16)-(2.20), we obtain $\|u_j-u\|\to 0$. Thus $u_j\to u$ in $H_r^1(\mathbb{R}^3)$ and $u$ satisfies

$$I(u)=a_n \text{ and } I'(u)=0.$$

So, $a_n$ is a critical value of $I(u)$.

**Step 4:** Infinitely many high energy radial solutions. By (1) of Step 1, (1.10), and (2.2), there is $a_n\geq b_n\geq\alpha>0$. In order to prove $a_n\to\infty$ as $n\to\infty$, we prove that $b_n$ is a critical value of $J(u)$ and $b_n\to\infty$ as $n\to\infty$. Let $\{u_j\}\subset H_r^1(\mathbb{R}^3)$ be a sequence satisfying

$$J(u_j)\to b_n, J'(u_j)\to 0.$$

First, we prove that $\{u_j\}$ is bounded. Observe that

$$b_n=J(u_j)-\frac{1}{q+1}\langle J'(u_j),u_j\rangle\geq\left(\frac{1}{4}-\frac{1}{2(q+1)}\right)\|u_j\|^2.$$

So, $\{u_j\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Going if necessary to a subsequence, still denoted by $\{u_j\}$, we may assume that

$$\begin{aligned}&u_j\rightharpoonup u_0 \text{ in } H_r^1(\mathbb{R}^3),\\ &u_j\to u_0 \text{ in } L^p(\mathbb{R}^3), \text{ where } 2<p<6,\\ &u_j(x)\to u_0(x) \text{ a.e. } x\in\mathbb{R}^3. \qquad (2.21)\end{aligned}$$

As $J'(u_j)\to 0$ and $u_j\to u_0$ in $L^p(\mathbb{R}^3)$, we have

$$\begin{aligned}&J'(u_j)(u_j-u_0)\to 0,\\ &J'(u_0)(u_j-u_0)\to 0. \qquad (2.22)\end{aligned}$$

Then, as $j\to\infty$,

$$\begin{aligned}o(1) &= \langle J'(u_j)-J'(u_0),u_j-u_0\rangle\\ &= \frac{1}{4}\|u_j-u_0\|^2-C(q+1)\int_{\mathbb{R}^3}(|u_j|^q-|u_0|^q)(u_j-u_0)\,\mathrm{d}x. \qquad (2.23)\end{aligned}$$

Due to (2.21), we have $u_j\rightharpoonup u$ in $D^{1,2}(\mathbb{R}^3)$. Then it follows that

$$\langle u_0,u_j-u_0\rangle=\int_{\mathbb{R}^3}\nabla u_0\cdot\nabla(u_j-u_0)\,\mathrm{d}x\to 0.$$

From (2.21), we have $u_j\rightharpoonup u_0$ in $H_r^1(\mathbb{R}^3)$. Then it follows that

$$\langle u_0,u_j-u_0\rangle=\int_{\mathbb{R}^3}\nabla u_0\cdot\nabla(u_j-u_0)+u_0\cdot(u_j-u_0)\,\mathrm{d}x\to 0,$$

so

$$\int_{\mathbb{R}^3}u_0\cdot(u_j-u_0)\,\mathrm{d}x\to 0,$$

and

$$\int_{\mathbb{R}^3} |u_0|^{q-1} u_0 (u_j - u_0)\,\mathrm{d}x \to 0. \tag{2.24}$$

Using the Holder inequality, we have

$$\left| \int_{\mathbb{R}^3} |u_j|^q (u_j - u_0)\,\mathrm{d}x \right| \le \int_{\mathbb{R}^3} \left| |u_j|^q (u_j - u_0) \right| \mathrm{d}x \le \|u_j\|^q_{L^{q+1}} \|u_j - u_0\|_{L^{q+1}} \to 0. \tag{2.25}$$

Combining (2.23)-(2.25), we conclude $\|u_j - u_0\| \to 0$, which implies $u_j \to u_0$ in $H^1_r(\mathbb{R}^3)$. Therefore, $J(u)$ satisfies Palais-Smale condition, and then $b_n$ is a critical value of $J(u)$. We apply an argument in [18, Chapter 9]. We set

$$\underline{\Gamma}_n = \{h(\overline{E_m \backslash Y}) : h \in \Gamma_m, m \ge n, Y \in B_m \text{ and genus}(Y) \le m - n\}.$$

where $B_m$ is the family of closed sets $A \in \mathbb{R}^m \backslash \{0\}$ such that $-A = A$ and genus($A$) is the Krasnoselski's genus of $A$. We define another sequence of minimax values by

$$c_n = \inf_{A \in \underline{\Gamma}_n} \max_{u \in A} J(u).$$

Then we have $b_n \ge c_n$ for all $n \in \mathbb{N}, c_1 \le c_2 \le ... \le c_n \le c_{n+1} \le \cdots$. Moreover, since $J(u)$ satisfies the Palais-Smale condition, by modifying the argument in [18, Chapter 9], we have $c_n \to \infty$ as $n \to \infty$. Thus $b_n \to \infty$ as $n \to \infty$. Then we obtain $a_n \to \infty$ as $n \to \infty$. The proof is completed.

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