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ERROR BOUNDS ON THE HERMITE-HADAMARD- AND BULLEN-TYPE INEQUALITIES FOR 2α-LOCAL FRACTIONAL DERIVATIVE

YUPING YU[1], XIAOMAN YUAN[1], TAIGUI QIN[1], TINGSONG DU[1,2,*]

[1]Three Gorges Mathematical Research Center, China Three Gorges University, Yichang 443002, China
[2]Department of Mathematics, College of Science, China Three Gorges University, Yichang 443002, China

Abstract. We present a general integral identity involving 2α-local fractional derivative. Based on the integral identity and the fact that the 2α-local fractional derivative in absolute value is generalized (s,P)-convex, we establish certain integral inequalities that cover each case of the generalized Hermite-Hadamard's and Bullen type for the class of mappings. Certain applications in α-type special means, probability distribution mappings, the trapezoidal and midpoint formulas, and wave equations on Cantor sets are presented to demonstrate the validity of the obtained results as well.

Keywords. Fractal sets; Generalized (s,P)-convexity; Local fractional integrals.

1. Introduction-Preliminaries

Assume that $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ is a convex mapping along with $a<b$. Then, the following double inequalities hold true

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequalities are referred to in the literature as the Hermite-Hadamard's integral inequality, which provides the upper as well as lower bounds of the integral mean value with regard to convex functions. As a matter of fact, inequalities (1.1) hold in the opposite manner if f is concave. Recently, the Hermite-Hadamard's integral inequality was under the spotlight of research. By selecting different methods and linking with generalized convexity, numerous researchers derived a number of classical integral mean inequalities and carried out the research on its generalization, improvement, and various expansion, such as the Bullen-type inequalities for s-convexity [7], the Simpson-type inequalities relating to MT-convex mappings [19], the Fejér- and Hermite–Hadamard-type integral inequalities concerning N-quasiconvex mappings [1], the weighted trapezoidal inequalities in connection with differentiable log-convexity [27],

*Corresponding author.
E-mail address: tingsongdu@ctgu.edu.cn (T. Du).
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the weighted Hermite–Hadamard–Mercer type inequalities with regard to convexity [17], and different Hermite–Hadamard-type integral inequalities for h-convexity and (α,m)-convexity in [9] and [30], respectively. In addition, certain Hermite–Hadamard-type integral inequalities involving conformable fractional integrals were discovered in [20], the Hermite–Hadamard-type integral inequalities utilizing Atangana–Baleanu integral operators for convexity were obtained in [36]. For more meaningful research outcomes on Hermite–Hadamard's integral inequality, interested readers can refer to the references [14, 15, 22, 24, 26] and the references therein.

Tseng et al. [40] derived the succeeding revised Hadamard-type integral inequality, which presents a refinement of the inequality (1.1)

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &\leq \frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

The third inequality above is referred to as the Bullen's inequality in the literature.

In addition, İşcan et al. [16] established the succeeding general identity, which resulted in certain general inequalities that include the Hermite–Hadamard- and Bullen-type with relation to differentiable mappings.

Theorem 1.1. [16] *Suppose that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a differentiable mapping over the interval I°, where $a,b \in I^\circ$ along with $a < b$, and I° is the interior of I. If $f' \in L[a,b]$, then the subsequent equality holds*

$$\begin{aligned} &\sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \\ &=\sum_{i=0}^{n-1}\frac{b-a}{2n^2}\left[\int_0^1(1-2t)f'\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\mathrm{d}t\right]. \end{aligned}$$

Later, in this section, the theoretical knowledge of local fractional operators over $\mathbb{R}^\alpha$ $(0 < \alpha \leq 1)$ proposed by Yang [44] will be employed throughout the research.

Let us introduce certain preliminary concepts of the addition and the multiplication on the α-type set of element sets as below. Among them, α is defined as the fractal dimension of Cantor set and is assumed by default to be $0 < \alpha \leq 1$.

Let Υ_1^α, Υ_2^α and Υ_3^α pertain to the set $\mathbb{R}^\alpha$, then
(1) $\Upsilon_1^\alpha+\Upsilon_2^\alpha$ and $\Upsilon_1^\alpha\Upsilon_2^\alpha$ pertain to the set $\mathbb{R}^\alpha$;
(2) $\Upsilon_1^\alpha+\Upsilon_2^\alpha=\Upsilon_2^\alpha+\Upsilon_1^\alpha=(\Upsilon_1+\Upsilon_2)^\alpha=(\Upsilon_2+\Upsilon_1)^\alpha$;
(3) $\Upsilon_1^\alpha+(\Upsilon_2^\alpha+\Upsilon_3^\alpha)=(\Upsilon_1^\alpha+\Upsilon_2^\alpha)+\Upsilon_3^\alpha$;
(4) $\Upsilon_1^\alpha\Upsilon_2^\alpha=\Upsilon_2^\alpha\Upsilon_1^\alpha=(\Upsilon_1\Upsilon_2)^\alpha=(\Upsilon_2\Upsilon_1)^\alpha$;
(5) $\Upsilon_1^\alpha(\Upsilon_2^\alpha\Upsilon_3^\alpha)=(\Upsilon_1^\alpha\Upsilon_2^\alpha)\Upsilon_3^\alpha$;
(6) $\Upsilon_1^\alpha(\Upsilon_2^\alpha+\Upsilon_3^\alpha)=\Upsilon_1^\alpha\Upsilon_2^\alpha+\Upsilon_1^\alpha\Upsilon_3^\alpha$;
(7) $\Upsilon_1^\alpha+0^\alpha=0^\alpha+\Upsilon_1^\alpha=\Upsilon_1^\alpha$ and $\Upsilon_1^\alpha 1^\alpha=1^\alpha\Upsilon_1^\alpha=\Upsilon_1^\alpha$.
(8) $\Upsilon_1^\alpha \geq \Upsilon_2^\alpha$ when and only when $\Upsilon_1 \geq \Upsilon_2$, $\Upsilon_1,\Upsilon_2 \in \mathbb{R}$;
(9) $(\Upsilon_1^\alpha)^k=(\Upsilon_1^k)^\alpha, k>0$ and $\Upsilon_1>0$;

(10)$\Upsilon_1^\alpha - \Upsilon_2^\alpha = (\Upsilon_1 - \Upsilon_2)^\alpha$;

(11) For any $\Upsilon_1^\alpha \in \mathbb{R}^\alpha$, we note that $(-\Upsilon_1)^\alpha = -\Upsilon_1^\alpha$; for any $\Upsilon_2^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, we also note that $(1/\Upsilon_2)^\alpha = 1^\alpha/\Upsilon_2^\alpha$, but not $1/\Upsilon_2^\alpha$.

Furthermore, we retrospect the coming conceptions of the local fractional derivative and integral which will play a crucial part in the progress of the main outcomes throughout the paper.

Definition 1.2. [44] The mapping $H : \mathbb{R} \to \mathbb{R}^\alpha$ is non-differentiable and $\zeta \to H(\zeta)$ is known as local fractional continuous at ζ_0, if, for every $\varepsilon > 0$, there exists a real number $\kappa > 0$ such that $|H(\zeta) - H(\zeta_0)| < \varepsilon^\alpha$ is valid for $|\zeta - \zeta_0| < \kappa$. $H(\zeta)$ is local continuous defined over the interval (a,b), which is recorded as $H(\zeta) \in C_\alpha(a,b)$.

Definition 1.3. [44] The local fractional derivative of $H(\zeta)$ involved in order α at the point $\zeta = \zeta_0$ is defined by

$$H^{(\alpha)}(\zeta_0) = {}_{\zeta_0}D_\zeta^\alpha H(\zeta) = \left.\frac{\mathrm{d}^\alpha H(\zeta)}{\mathrm{d}\zeta^\alpha}\right|_{\zeta=\zeta_0} = \lim_{\zeta\to\zeta_0} \frac{\Delta^\alpha\big(H(\zeta) - H(\zeta_0)\big)}{(\zeta-\zeta_0)^\alpha}.$$

Among them, $\Delta^\alpha\big(H(\zeta) - H(\zeta_0)\big) = \Gamma(1+\alpha)\big(H(\zeta) - H(\zeta_0)\big)$.

Suppose that $H^{(\alpha)}(\zeta) = D_\zeta^\alpha H(\zeta)$. For every $\zeta \in \Lambda \subseteq \mathbb{R}$, if there exists $H^{((k+1)\alpha)}(\zeta) = \overbrace{D_\zeta^\alpha \cdots D_\zeta^\alpha}^{(k+1)\text{ times}} H(\zeta)$, then it can be recorded as $H \in D_{(k+1)\alpha}(\Lambda)$, where $k = 0,1,2,\cdots$.

Definition 1.4. [44] Let $H(\zeta) \in C_\alpha[a,b]$. If $\Delta = \{\xi_0, \xi_1, \cdots, \xi_N\}$ $(N \in \mathbb{N})$ is a division in connection with the interval $[a,b]$, which meets that $a = \xi_0 < \xi_1 < \cdots < \xi_N = b$, then the local fractional integral of H defined over the interval $[a,b]$ involved in order α is defined by

$${}_a\mathscr{I}_b^{(\alpha)} H(\zeta) = \frac{1}{\Gamma(1+\alpha)} \int_a^b H(\xi)(\mathrm{d}\xi)^\alpha := \frac{1}{\Gamma(1+\alpha)} \lim_{\triangle\xi\to 0} \sum_{k=0}^{N-1} H(\xi_k)(\Delta\xi_k)^\alpha,$$

where $\Delta\xi := \max\{\Delta\xi_1, \Delta\xi_2, \cdots, \Delta\xi_{N-1}\}$, $\Delta\xi_k := \xi_{k+1} - \xi_k$, $k = 0, \cdots, N-1$. If ${}_a\mathscr{I}_b^{(\alpha)} H(\zeta)$ exists for every $\zeta \in [a,b]$, then it can be recorded as $H(\zeta) \in \mathscr{I}_\zeta^{(\alpha)}[a,b]$.

Lemma 1.5. [44] *The following identities hold:*

(a) If $H(\zeta) = w^{(\alpha)}(\zeta) \in C_\alpha[a,b]$, then

$${}_a\mathscr{I}_b^{(\alpha)} H(\zeta) = w(b) - w(a).$$

(b) If $H(\zeta), w(\zeta) \in D_\alpha[a,b]$, and $H^{(\alpha)}(\zeta), w^{(\alpha)}(\zeta) \in C_\alpha[a,b]$, then

$${}_a\mathscr{I}_b^{(\alpha)} H(\zeta) w^{(\alpha)}(\zeta) = H(\zeta)w(\zeta)\Big|_a^b - {}_a\mathscr{I}_b^{(\alpha)} H^{(\alpha)}(\zeta) w(\zeta).$$

(c) The local fractional derivative of the mapping $\zeta^{k\alpha}$ is as below:

$$\frac{\mathrm{d}^\alpha \zeta^{k\alpha}}{\mathrm{d}\zeta^\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k-1)\alpha\big)} \zeta^{(k-1)\alpha},\ k > 0.$$

(d) The local fractional integration in connection with the mapping $\zeta^{k\alpha}$ is as below:

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \zeta^{k\alpha} (\mathrm{d}\zeta)^\alpha = \frac{\Gamma(1+k\alpha)}{\Gamma\big(1+(k+1)\alpha\big)} \Big(b^{(k+1)\alpha} - a^{(k+1)\alpha}\Big),\ k > 0.$$

Recently, the local fractional Hölder integral inequality established by Yang in [44] has received much attention and utilized extensively, as described in the succeeding lemma:

Lemma 1.6. [44] *(Hölder–Yang's inequality). If the mappings $H, w \in C_\alpha[a,b]$, $\phi, \psi > 1$ with $\frac{1}{\phi} + \frac{1}{\psi} = 1$, then we deduce the subsequent integral inequality*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b \left|H(\zeta)w(\zeta)\right|(\mathrm{d}\zeta)^\alpha \leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left|H(\zeta)\right|^\phi (\mathrm{d}\zeta)^\alpha\right)^{\frac{1}{\phi}} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b \left|w(\zeta)\right|^\psi (\mathrm{d}\zeta)^\alpha\right)^{\frac{1}{\psi}}.$$

In 2016, Sarikaya et al. [34] presented a generalized Grüss inequality, which is described as below.

Theorem 1.7. [34] *Suppose that $f, g : [a,b] \to \mathbb{R}^\alpha$ are two mappings and local fractional integrable over $[a,b]$. If $r^\alpha \leq f(x) \leq R^\alpha$ and $n^\alpha \leq g(x) \leq N^\alpha$ for every $x \in [a,b]$, r^α, R^α, n^α, $N^\alpha \in \mathbb{R}^\alpha$, then*

$$\left|\frac{(b-a)^\alpha}{\Gamma(1+\alpha)} {}_a\mathscr{I}_b^{(\alpha)} f(x)g(x) - \left[{}_a\mathscr{I}_b^{(\alpha)} f(x)\right]\left[{}_a\mathscr{I}_b^{(\alpha)} g(x)\right]\right| \leq \frac{(b-a)^{2\alpha}}{4^\alpha \Gamma^2(1+\alpha)} (R^\alpha - r^\alpha)(N^\alpha - n^\alpha).$$

Also, Sarikaya et al. presented the generalized Bullen's inequality in the settings of fractal sets in connection with generalized convex mapping as stated below.

Theorem 1.8. [33] *If the mapping $f(x) \in \mathscr{I}_x^{(\alpha)}[a,b]$ is a generalized convex mapping over the real-valued interval $[a,b]$ along with $a < b$, then we have successive inequality*

$$\frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a\mathscr{I}_b^{(\alpha)} f(x) \leq \left(\frac{1}{2}\right)^\alpha \left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2^\alpha}\right].$$

Yu et al. [45] presented the concept of the generalized (s,P)-convex mappings and gave the generalized Hermite–Hadamard-type integral inequality combined with this class of functions, which are expressed as follows.

Definition 1.9. [45] Suppose that the mapping $f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. For every $x, y \in I, t \in [0,1]$ and certain fixed $s \in (0,1]$, if the successive inequality

$$f(tx + (1-t)y) \leq \left(t^{s\alpha} + (1-t)^{s\alpha}\right)\left[f(x) + f(y)\right]$$

is valid, then f is called as a generalized (s,P)-convex mapping.

Theorem 1.10. [45] *Suppose that the mapping $f : [a,b] \to \mathbb{R}^\alpha$ is a generalized (s,P)-convex mapping for certain fixed $s \in (0,1]$. If the mapping $f \in C_\alpha[a,b]$ along with $a < b$, then one receives the successive Hermite–Hadamard- type integral inequalities*

$$\frac{2^{(s-2)\alpha}}{\Gamma(1+\alpha)} f\left(\frac{a+b}{2}\right) \leq \left(\frac{1}{b-a}\right)^\alpha {}_a\mathscr{I}_b^{(\alpha)} f(x)(\mathrm{d}x)^\alpha \leq \frac{2^\alpha \Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)} [f(a) + f(b)].$$

Local fractal theory is a contemporary science for solving the non-differentiable problems concerning the intricate systems of real-world phenomena. In recent years, driven by the applications of fractal sets in cryptography and other research fields, researchers gradually increased the applications of the local fractal theory to mathematical inequalities. For instance, several generalized Ostrowski type integral inequalities concerning the generalized h-convexity and generalized s-convexity were considered by Sun [37] and Tomar et al. [39], correspondingly. Choi et al. [8] studied the integral inequalities of generalized Ostrowski type from another angle, which involves with 2α-local fractional differentiable functions. Certain generalized Pompeiu-type integral inequalities and its applications were discussed by Erden and Sarikaya [10]. In [4], Al-Sa'di et al. investigated the generalized γ-preinvex mappings and established several generalizations of Hermite–Hadamard type inequalities. Making use of local fractional integrals and generalized (h,m)-convexity, the Hermite–Hadamard- and Fejér–Hermite–Hadamard-type inequalities were generalized by Almutairi and Kiliçman [3], and they also derived certain integral inequalities involving generalized s-convexity through Katugampola fractional integrals on fractal sets [2]. A few different generalized Hermite–Hadamard type integral inequalities pertaining to generalized s-convexity were proposed by Kiliçman and Saleh [21] as well as Mo and Sui [29]. In addition to these outcomes, some generalized Bullen-type inequalities involving with 2α-local fractional differentiable mappings in the settings of fractal space were derived by Erden [11]. In the light of an open-type three points Newton-Cotes quadrature rule, certain local fractional integrals of Maclaurin-type inequalities were established by Meftah [28]. A generalization of the Jensen-type integral inequalities in connection with the strongly convex mapping were introduced by Sánchez and Sanabria [31]. In 2020, Iftikhar et al. [12, 13] explored several Newton-type integral inequalities involving generalized convexity and generalized harmonic convexity by means of local fractional derivatives and integrals, respectively. After that, some generalized Milne-type integral inequalities for generalized m-convexity are achieved, which are further supported by several examples and applications in [5]. For more research results on local fractional calculus, please consult the papers [6, 23, 25, 32] and the references therein.

Moreover, it is necessary to note the generalization of local fractional integrals. For example, with the help of local fractional integrals involving Mittag–Leffler kernels, Sun [38] researched two fractal identities and related fractal Hermite–Hadamard-type inequalities for generalized h-convexity. By virtue of the same integrals, Vivas-Cortez et al. [41] extended Hermite–Hadamard-type inequalities for generalized $(\tilde{h_1},\tilde{h_2})$-preinvexity. Additionally, several applications in different fields like physics, software engineering, pure mathematics and so on, can be discovered in the accessible papers and monographs, for example, in [18, 35, 42] and the bibliographies quoted therein.

The key aim of this paper is to establish the general identity associated with local fractional calculus. With the assistance of the fractal identity, we acquire the local fractional integral inequalities whose 2α-local fractional derivative in modulus are generalized (s,P)-convex. It is worth noting that the general inequalities obtained cover all of the generalized Hermite–Hadamard- and Bullen-type inequalities for the class of mappings. Furthermore, the application of the deduced outcomes in α-type special means, the trapezoidal and midpoint formula, probability distribution mappings as well as wave equation on Cantor sets are also presented.

2. Main Results

In order to present the further inequalities that its absolute value is 2α-local fractional differentiable functions, we commence by establishing general identity associated with 2α-local fractional differentiable functions.

Lemma 2.1. *Let the mapping $f: I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ be local fractional continuously differentiable over the interval I°, where $a,b \in I^\circ$ along with $a<b$. If $f^{(\alpha)} \in D_\alpha(I^\circ)$ and $f^{(2\alpha)} \in C_\alpha[a,b]$,*

then we deduce the following fractal identity

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right]\\
&\quad-\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}\\
&\quad\times f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}.
\end{aligned}\tag{2.1}$$

Proof. Let $n\in N$ be arbitrary. Using the local fractional integration for $i\in\{0,1,2,\cdots,n-1\}$ by parts, it yields that

$$\begin{aligned}
I_i&=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\\
&=\left(\frac{n}{a-b}\right)^{\alpha}(1-2t)^{2\alpha}f^{(\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Bigg|_{0}^{1}\\
&\quad+\left(\frac{2n}{a-b}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{\alpha}\\
&\quad\times f^{(\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\\
&=\left(\frac{n}{a-b}\right)^{\alpha}\left[f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)-f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]\\
&\quad+\left(\frac{2n}{a-b}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{\alpha}\\
&\quad\times f^{(\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\\
&=\left(\frac{n}{a-b}\right)^{\alpha}\left[f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)-f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]\\
&\quad+\left(\frac{2n}{a-b}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\Bigg[\left(\frac{n}{a-b}\right)^{\alpha}(1-2t)^{\alpha}\\
&\quad\times f\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Bigg|_{0}^{1}\\
&\quad+\left(\frac{2n}{a-b}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}\Gamma(1+\alpha)\\
&\quad\times f\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\Bigg]
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n}{a-b}\right)^{\alpha}\left[f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right) - f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\
&\quad + \left(\frac{2n}{a-b}\right)^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}\left\{\left(\frac{n}{b-a}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]\right. \\
&\quad + \left(\frac{2n}{a-b}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^1 \Gamma(1+\alpha) \\
&\quad \left.\times f\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\right\}.
\end{aligned} \tag{2.2}
$$

Taking advantage of the change of variable

$$
x = t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}
$$

for $t \in [0,1]$ and multiplying both the sides of the equality (2.2) by $\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}$, we obtain that

$$
\begin{aligned}
&\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}I_i \\
&= \frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) - f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right] \\
&\quad - \left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\
&\quad + \frac{1}{(b-a)^{\alpha}\Gamma(1+\alpha)}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} \Gamma(1+\alpha)f(x)(\mathrm{d}x)^{\alpha}.
\end{aligned} \tag{2.3}
$$

As a result, we receive that

$$
\begin{aligned}
&\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}I_i \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) - f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right] \\
&\quad - \sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\
&\quad + \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_a\mathscr{I}_b^{(\alpha)}f(x).
\end{aligned} \tag{2.4}
$$

We deduce the desired identity. $\square$

The established identity contains the generalized trapezoid- and Bullen-type outcomes on fractal sets, which are novel and meaningful. Correspondingly, the relevant identities in the

Euclidean spaces can be deduced. Next, we present some special cases of Lemma 2.1.
(i) If $n=1$, then we receive the generalized trapezoid-type identity:

$$\begin{aligned}&\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\\&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha}{4^\alpha}\Big[f^{(\alpha)}(b)-f^{(\alpha)}(a)\Big]\\&\quad-\frac{(b-a)^{2\alpha}}{4^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-2t)^{2\alpha}f^{(2\alpha)}\Big(ta+(1-t)b\Big)(\mathrm{d}t)^\alpha.\end{aligned}$$

(ii) If $n=2$, then we receive the generalized Bullen-type identity:

$$\begin{aligned}&\frac{1^\alpha}{2^\alpha}\left[\frac{f(a)+f(b)}{2^\alpha}+f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\\&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha}{16^\alpha}\Big[f^{(\alpha)}(b)-f^{(\alpha)}(a)\Big]\\&\quad-\frac{(b-a)^{2\alpha}}{8^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left(\frac{1}{2}-t\right)^{2\alpha}\\&\quad\times\left[f^{(2\alpha)}\left(ta+(1-t)\frac{a+b}{2}\right)+f^{(2\alpha)}\left(t\frac{a+b}{2}+(1-t)b\right)\right](\mathrm{d}t)^\alpha.\end{aligned}$$

(iii) If $\alpha=1$, then
for $n=1$,

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\\&=\frac{b-a}{8}\Big[f'(b)-f'(a)\Big]-\frac{(b-a)^2}{8}\int_0^1(1-2t)^2f''\big(ta+(1-t)b\big)\mathrm{d}t,\end{aligned}$$

and for $n=2$,

$$\begin{aligned}&\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\\&=\frac{b-a}{32}\Big[f'(b)-f'(a)\Big]-\frac{(b-a)^2}{16}\int_0^1\left(\frac{1}{2}-t\right)^2\\&\quad\times\left[f''\left(ta+(1-t)\frac{a+b}{2}\right)+f''\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t.\end{aligned}$$

Theorem 2.2. *Suppose that the mapping $f: I\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ is local fractional continuously differentiable over the interval I°, along with $a,b\in I^\circ$, $a<b$, $f^{(\alpha)}\in D_\alpha(I^\circ)$ and $f^{(2\alpha)}\in C_\alpha[a,b]$. If there exist constants $z^\alpha<Z^\alpha$ satisfying that $z^\alpha\le f^{(2\alpha)}\le Z^\alpha$ for each $x\in[a,b]$, then the succeeding inequality holds true*

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right.\\
&\quad\left.-\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}(z^{\alpha}+Z^{\alpha})}{8^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\right|\\
&\le\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}(Z^{\alpha}-z^{\alpha})}{8^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.
\end{aligned}\tag{2.5}$$

Proof. From Lemma 2.1, we know that

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right]\\
&\quad-\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1\left\{(1-2t)^{2\alpha}\right.\\
&\quad\times\left[f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)-\frac{z^{\alpha}+Z^{\alpha}}{2^{\alpha}}\right]\\
&\quad\left.+(1-2t)^{2\alpha}\left(\frac{z^{\alpha}+Z^{\alpha}}{2^{\alpha}}\right)\right\}(\mathrm{d}t)^{\alpha}\\
&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right]\\
&\quad-\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-2t)^{2\alpha}\\
&\quad\times\left[f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)-\frac{z^{\alpha}+Z^{\alpha}}{2^{\alpha}}\right](\mathrm{d}t)^{\alpha}\\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}(z^{\alpha}+Z^{\alpha})}{8^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)},
\end{aligned}$$

which derives that

$$\begin{aligned}
K:=&\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]\\
&-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)-\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}(z^{\alpha}+Z^{\alpha})}{8^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}
\end{aligned}$$

$$= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} \frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}} \left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) - f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right) \right]$$
$$- \sum_{i=0}^{n-1} \frac{(b-a)^{2\alpha}}{4^\alpha n^{3\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 (1-2t)^{2\alpha}$$
$$\times \left[f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right) - \frac{z^\alpha + Z^\alpha}{2^\alpha} \right] (\mathrm{d}t)^\alpha.$$

Hence,

$$|K| \le \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} \frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}} \left| f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) - f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right) \right|$$
$$+ \sum_{i=0}^{n-1} \frac{(b-a)^{2\alpha}}{4^\alpha n^{3\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^{2\alpha}$$
$$\times \left| f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right) - \frac{z^\alpha + Z^\alpha}{2^\alpha} \right| (\mathrm{d}t)^\alpha$$
$$\le \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} \frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}} \left| f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) - f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right) \right|$$
$$+ \sum_{i=0}^{n-1} \frac{(b-a)^{2\alpha}(Z^\alpha - z^\alpha)}{8^\alpha n^{3\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.$$

Since $f^{(2\alpha)}$ satisfies $z^\alpha \le f^{(2\alpha)} \le Z^\alpha$, we deduce that

$$z^\alpha - \frac{z^\alpha + Z^\alpha}{2^\alpha} \le f^{(2\alpha)} - \frac{z^\alpha + Z^\alpha}{2^\alpha} \le Z^\alpha - \frac{z^\alpha + Z^\alpha}{2^\alpha},$$

which can also be expressed as

$$\left| f^{(2\alpha)} - \frac{z^\alpha + Z^\alpha}{2^\alpha} \right| \le \frac{Z^\alpha - z^\alpha}{2^\alpha}.$$

This completes the proof. □

Similarly, we can obtain the generalized trapezoid- and Bullen-type inequalities.

Next, we present special cases of Theorem 2.2.

(i) If $n = 1$, then we receive the generalized trapezoid-type inequality:

$$\left| \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_a\mathscr{I}_b^{(\alpha)} f(x) - \frac{(b-a)^{2\alpha}(z^\alpha + Z^\alpha)}{8^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} \right|$$
$$\le \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{(b-a)^\alpha}{4^\alpha} \left| f^{(\alpha)}(b) - f^{(\alpha)}(a) \right| + \frac{(b-a)^{2\alpha}(Z^\alpha - z^\alpha)}{8^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.$$

(ii) If $n=2$, then we receive the generalized Bullen-type inequality:

$$\left|\frac{1^\alpha}{2^\alpha}\left[\frac{f(a)+f(b)}{2^\alpha}+f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)-\frac{(b-a)^{2\alpha}(Z^\alpha+z^\alpha)}{64^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\right|$$
$$\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha}{16^\alpha}\left[\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)-f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)-f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right]$$
$$+\frac{(b-a)^{2\alpha}(Z^\alpha-z^\alpha)}{64^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.$$

(iii) If $\alpha=1$, then
for $n=1$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t-\frac{(b-a)^2(Z+z)}{48}\right|$$
$$\leq\frac{b-a}{8}\left|f'(b)-f'(a)\right|+\frac{(b-a)^2(Z-z)}{48},$$

and for $n=2$,

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t-\frac{(b-a)^2(Z+z)}{384}\right|$$
$$\leq\frac{b-a}{32}\left[\left|f'\left(\frac{a+b}{2}\right)-f'(a)\right|+\left|f'(b)-f'\left(\frac{a+b}{2}\right)\right|\right]+\frac{(b-a)^2(Z-z)}{384}.$$

Theorem 2.3. *Under the hypotheses of Theorem 2.2, we deduce*

$$\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^\alpha\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|$$
$$\leq\sum_{i=0}^{n-1}\frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma^2(1+\alpha)}{n^\alpha\Gamma(1+3\alpha)}\right]$$
$$\times\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|+\sum_{i=0}^{n-1}\frac{(b-a)^{3\alpha}}{16^\alpha n^{3\alpha}}\frac{Z^\alpha-z^\alpha}{\Gamma(1+2\alpha)}. \tag{2.6}$$

Proof. By making use of the generalized Grüss inequality, we infer that

$$\left|\frac{(b-a)^\alpha}{\Gamma^2(1+\alpha)}\int_0^1(1-2t)^{2\alpha}f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^\alpha\right.$$
$$-\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-2t)^{2\alpha}(\mathrm{d}t)^\alpha\right] \tag{2.7}$$
$$\left.\times\left[\frac{1}{\Gamma(1+\alpha)}\int_0^1 f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^\alpha\right]\right|$$
$$\leq\frac{(b-a)^{2\alpha}}{4^\alpha\Gamma^2(1+\alpha)}(Z^\alpha-z^\alpha)(N^\alpha-n^\alpha),$$

where $N^{\alpha}=\sup_{t\in(0,1)}(1-2t)^{2\alpha}=1^{\alpha}$, and $n^{\alpha}=\inf_{t\in(0,1)}(1-2t)^{2\alpha}=0^{\alpha}$.

Using inequality (2.7) and taking the module of the integral identity in Lemma 2.1, we find that

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\right| \\
&=\left|\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right]\right. \\
&\quad-\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left\{\frac{(b-a)^{\alpha}}{\Gamma^{2}(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}\right. \\
&\quad\times f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha} \\
&\quad-\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}(\mathrm{d}t)^{\alpha}\right] \\
&\quad\times\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\right] \\
&\quad+\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}(\mathrm{d}t)^{\alpha}\right] \\
&\quad\left.\left.\times\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\right]\right\}\right| \\
&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left\{\left|\frac{(b-a)^{\alpha}}{\Gamma^{2}(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}\right.\right. \\
&\quad\times f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha} \\
&\quad-\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}(\mathrm{d}t)^{\alpha}\right] \\
&\quad\left.\left.\times\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\right]\right|\right\} \\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma^{2}(1+\alpha)}{\Gamma(1+2\alpha)}\left|\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}(1-2t)^{2\alpha}(\mathrm{d}t)^{\alpha}\right]\right. \\
&\quad\left.\times\left[\frac{1}{\Gamma(1+\alpha)}\int_{0}^{1}f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)(\mathrm{d}t)^{\alpha}\right]\right| \\
&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma^2(1+\alpha)}(Z^{\alpha}-z^{\alpha})+\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma^2(1+\alpha)}{\Gamma(1+2\alpha)}\\
&\times\left|\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left[f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)-f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]\right|\\
=&\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma^2(1+\alpha)}{n^{\alpha}\Gamma(1+3\alpha)}\right]\\
&\times\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\
&+\sum_{i=0}^{n-1}\frac{(b-a)^{3\alpha}}{16^{\alpha}n^{3\alpha}}\frac{Z^{\alpha}-z^{\alpha}}{\Gamma(1+2\alpha)}.
\end{aligned}
$$

This ends the proof. □

Theorem 2.4. *Suppose that the mapping $f: I\subseteq\mathbb{R}\to\mathbb{R}^{\alpha}$ is local fractional continuously differentiable over the interval I°, along with $a,b\in I^{\circ}$, $a<b$, $f^{(\alpha)}\in D_{\alpha}(I^{\circ})$ and $f^{(2\alpha)}\in C_{\alpha}[a,b]$. If $|f^{(2\alpha)}|$ is a generalized (s,P)-convex mapping for certain fixed $s\in(0,1]$. If $f^{(2\alpha)}$ is bounded over the interval $[a,b]$, which is represented by $||f^{(2\alpha)}||_{\infty}=\sup_{x\in(a,b)}|f^{(2\alpha)}|<\infty^{\alpha}$, then*

$$
\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|\\
&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\
&\quad+\sum_{i=0}^{n-1}\frac{M(b-a)^{2\alpha}||f^{(2\alpha)}||_{\infty}}{n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)},
\end{aligned}
\tag{2.8}
$$

where

$$
M=\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-4^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+4^{\alpha}\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}. \tag{2.9}
$$

Proof. Taking the modulus of the integral identity in lemma 2.1 and taking advantage of the generalized (s,P)-convexity as well as bounded of $f^{(2\alpha)}$, we obtain that

$$
\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|\\
&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}\left|t^{s\alpha}+(1-t)^{s\alpha}\right|
\end{aligned}
$$

$$\times\left[\left|f^{(2\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|+\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\right](\mathrm{d}t)^{\alpha}$$
$$\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \quad (2.10)$$
$$+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}||f^{(2\alpha)}||_{\infty}}{2^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}Y.$$

Here, by applying the local fractional integral theory to compute integral Y, we can obtain

$$\begin{aligned}Y:&=\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}t^{s\alpha}(\mathrm{d}t)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}(1-t)^{s\alpha}(\mathrm{d}t)^{\alpha}\\&=2^{\alpha}\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}-8^{\alpha}\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}+8^{\alpha}\frac{\Gamma(1+(s+2)\alpha)}{\Gamma(1+(s+3)\alpha)}.\end{aligned} \quad (2.11)$$

Substituting the integral value Y into inequality (2.10), we obtain the desired conclusion. This ends the proof. □

Next, we present some special cases of Theorem 2.4.

(i) If $n=1$, then we receive the generalized trapezoid-type inequality:

$$\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|$$
$$\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^{\alpha}}{4^{\alpha}}\left|f^{(\alpha)}(b)-f^{(\alpha)}(a)\right|+M(b-a)^{2\alpha}||f^{(2\alpha)}||_{\infty}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.$$

(ii) If $n=2$, then we receive the generalized Bullen-type inequality:

$$\left|\frac{1^{\alpha}}{2^{\alpha}}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|$$
$$\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^{\alpha}}{16^{\alpha}}\left[\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)-f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)-f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right]$$
$$+\frac{M(b-a)^{2\alpha}||f^{(2\alpha)}||_{\infty}}{2^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.$$

(iii) If $\alpha=1$, then
for $n=1$,

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\right|$$
$$\leq\frac{b-a}{8}\left|f'(b)-f'(a)\right|+\left(\frac{1}{2(s+1)}-\frac{2}{s+2}+\frac{2}{s+3}\right)(b-a)^2||f''||_{\infty},$$

and for $n=2$,

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\right|\\&\leq\frac{b-a}{32}\left[\left|f'\left(\frac{a+b}{2}\right)-f'(a)\right|+\left|f'(b)-f'\left(\frac{a+b}{2}\right)\right|\right]\\&\quad+\left(\frac{1}{16(s+1)}-\frac{1}{4(s+2)}+\frac{1}{4(s+3)}\right)(b-a)^2||f''||_\infty.\end{aligned}$$

Theorem 2.5. *Suppose that the mapping $f: I\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ is local fractional continuously differentiable over the interval I°, along with $a,b\in I^\circ$, $a<b$, $f^{(\alpha)}\in D_\alpha(I^\circ)$ and $f^{(2\alpha)}\in C_\alpha[a,b]$. If there exists the real line numbers $L^\alpha\in\mathbb{R}^\alpha$, where $L^\alpha>0^\alpha$, such that $f^{(2\alpha)}$ satisfies the local Lipschitz condition, then*

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^\alpha\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|\\&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^\alpha n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|\\&\quad-\sum_{i=0}^{n-1}\frac{(b-a)^{3\alpha}L^\alpha}{8^\alpha n^{4\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.\end{aligned}\tag{2.12}$$

Proof. Taking the modulus of the integral identity in lemma 2.1, we achieve that

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^\alpha\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|\\&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^\alpha n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}\\&\quad\times\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|(\mathrm{d}t)^\alpha\\&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^\alpha n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}+\frac{a-b}{n}t\right)\right.\\&\quad\left.-f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)+f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|(\mathrm{d}t)^\alpha\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \sum_{i=0}^{n-1} \frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}} \left| f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) - f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right) \right| \\
&+ \sum_{i=0}^{n-1} \frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^{2\alpha} \right. \\
&\times \left| f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n} + \frac{a-b}{n}t\right) - f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right| (\mathrm{d}t)^{\alpha} \\
&\left. + \frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^{2\alpha} \left| f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right| (\mathrm{d}t)^{\alpha} \right].
\end{aligned} \tag{2.13}
$$

Considering $f^{(2\alpha)}$ satisfies a local Lipschitz condition, we deduce that

$$
\begin{aligned}
&\left| f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n} + \frac{a-b}{n}t\right) - f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right) \right| \\
&\leq L^{\alpha} \left| \left(\frac{(n-i-1)a+(i+1)b}{n} + \frac{a-b}{n}t\right)^{\alpha} - \left(\frac{(n-i-1)a+(i+1)b}{n}\right)^{\alpha} \right| \\
&= \left(\frac{(a-b)L}{n}\right)^{\alpha} t^{\alpha}.
\end{aligned} \tag{2.14}
$$

Moreover, we know that

$$
\frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^{2\alpha} (\mathrm{d}t)^{\alpha} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}, \tag{2.15}
$$

and

$$
\frac{1}{\Gamma(1+\alpha)} \int_0^1 |1-2t|^{2\alpha} t^{\alpha} (\mathrm{d}t)^{\alpha} = \frac{\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+3\alpha)}. \tag{2.16}
$$

Applying inequality (2.14), equalities (2.15) and (2.16) into (2.13) achieves the desired result. This completes the proof. □

Next, we present some special cases of Theorem 2.5.

(i) If $n=1$, then we receive the generalized trapezoid-type inequality:

$$
\begin{aligned}
&\left| \frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}} {}_a\mathscr{I}_b^{(\alpha)} f(x) \right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{(b-a)^{\alpha}}{4^{\alpha}} \left| f^{(\alpha)}(b) - f^{(\alpha)}(a) \right| + \frac{(b-a)^{2\alpha}}{4^{\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} \left| f^{(2\alpha)}(b) \right| \\
&\quad - \frac{(b-a)^{3\alpha}L^{\alpha}}{8^{\alpha}} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.
\end{aligned}
$$

(ii) If $n=2$, then we receive the generalized Bullen-type inequality:

$$\begin{aligned}&\left|\frac{1^\alpha}{2^\alpha}\left[\frac{f(a)+f(b)}{2^\alpha}+f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|\\&\le\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha}{16^\alpha}\left[\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)-f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)-f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right]\\&\quad+\frac{(b-a)^{2\alpha}}{32^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}\left[\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|+\left|f^{(2\alpha)}(b)\right|\right]-\frac{(b-a)^{3\alpha}L^\alpha}{128^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}.\end{aligned}$$

(iii) If $\alpha=1$, then
for $n=1$,

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\right|\\&\le\frac{b-a}{8}\left|f'(b)-f'(a)\right|+\frac{(b-a)^2}{24}\left|f''(b)\right|-\frac{(b-a)^3L}{48},\end{aligned}$$

and for $n=2$,

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\right|\\&\le\frac{b-a}{32}\left[\left|f'\left(\frac{a+b}{2}\right)-f'(a)\right|+\left|f'(b)-f'\left(\frac{a+b}{2}\right)\right|\right]\\&\quad+\frac{(b-a)^2}{192}\left[\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right]-\frac{(b-a)^3L}{768}.\end{aligned}$$

Theorem 2.6. *Suppose that the mapping $f:I\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ is local fractional continuously differentiable over the interval I°, along with $a,b\in I^\circ$, $a<b$, $f^{(\alpha)}\in D_\alpha(I^\circ)$ and $f^{(2\alpha)}\in C_\alpha[a,b]$. If $|f^{(2\alpha)}|^q$ is a generalized (s,P)-convex mapping for certain fixed $s\in(0,1]$ along with $\frac{1}{p}+\frac{1}{q}=1(q>1)$, then*

$$\begin{aligned}&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^\alpha\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}{}_a\mathscr{I}_b^{(\alpha)}f(x)\right|\\&\le\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^\alpha}{4^\alpha n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|\\&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^\alpha n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\\&\quad\times\left(\frac{2^\alpha\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q+\left|f^{(2\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|^q\right]\right)^{\frac{1}{q}}.\end{aligned}\tag{2.17}$$

Proof. Using properties of the modulus and taking advantage of the generalized Hölder–Yang's inequality in (2.1), we find that

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha} \\
&\quad\times\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|(\mathrm{d}t)^{\alpha} \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2p\alpha}(\mathrm{d}t)^{\alpha}\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^{q}(\mathrm{d}t)^{\alpha}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.18}$$

Since $|f^{(2\alpha)}|^q$ is a generalized (s,P)-convex mapping, we can exploit the generalized Hermite–Hadamard-type integral inequality for generalized (s,P)-convex mapping to derive that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^{q}(\mathrm{d}t)^{\alpha} \\
&=-\left(\frac{n}{a-b}\right)^{\alpha}\frac{1}{\Gamma(1+\alpha)}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}}|f^{(2\alpha}(x)|^{q}(\mathrm{d}x)^{\alpha} \\
&\leq\frac{2^{\alpha}\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^{q}+\left|f^{(2\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|^{q}\right].
\end{aligned} \tag{2.19}$$

Also, we observe that

$$\frac{1}{\Gamma(1+\alpha)}\int_0^1\left|1-2t\right|^{2p\alpha}(\mathrm{d}t)^{\alpha}=\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}. \tag{2.20}$$

Applying (2.19) and (2.20) to (2.18) yields inequality (2.17). This completes the proof. □

Next, we present some special cases of Theorem 2.6.
(i) If $n=1$, then we receive the generalized trapezoid-type inequality:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\right|\\&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^{\alpha}}{4^{\alpha}}\left|f^{(\alpha)}(b)-f^{(\alpha)}(a)\right|+\frac{(b-a)^{2\alpha}}{4^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\\&\quad\times\left(\frac{2^{\alpha}\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\left|f^{(2\alpha)}(b)\right|^{q}+\left|f^{(2\alpha)}(a)\right|^{q}\right]\right)^{\frac{1}{q}}.\end{aligned}$$

(ii) If $n=2$, then we receive the generalized Bullen-type inequality:

$$\begin{aligned}&\left|\frac{1^{\alpha}}{2^{\alpha}}\left[\frac{f(a)+f(b)}{2^{\alpha}}+f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\right|\\&\leq\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^{\alpha}}{16^{\alpha}}\left[\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)-f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)-f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right]\\&\quad+\frac{(b-a)^{2\alpha}}{32^{\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{2^{\alpha}\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\right)^{\frac{1}{q}}\\&\quad\times\left\{\left[\left|f^{(2\alpha)}(a)\right|^{q}+\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}+\left[\left|f^{(2\alpha)}(b)\right|^{q}+\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

(iii) If $\alpha=1$, then
for $n=1$,

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_{a}^{b}f(x)\mathrm{d}t\right|\\&\leq\frac{b-a}{8}\left|f'(b)-f'(a)\right|+\frac{(b-a)^{2}}{8}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\left[\left|f''(b)\right|^{q}+\left|f''(a)\right|^{q}\right]\right)^{\frac{1}{q}},\end{aligned}$$

and for $n=2$,

$$\begin{aligned}&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)\mathrm{d}t\right|\\&\leq\frac{b-a}{32}\left[\left|f'\left(\frac{a+b}{2}\right)-f'(a)\right|+\left|f'(b)-f'\left(\frac{a+b}{2}\right)\right|\right]+\frac{(b-a)^{2}}{64}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\\&\quad\times\left\{\left[\left|f''(a)\right|^{q}+\left|f''\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}+\left[\left|f''(b)\right|^{q}+\left|f''\left(\frac{a+b}{2}\right)\right|^{q}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Theorem 2.7. *Suppose that the mapping $f:I\subseteq\mathbb{R}\to\mathbb{R}^{\alpha}$ is local fractional continuously differentiable over the interval I°, along with $a,b\in I^{\circ}$, $a<b$, $f^{(\alpha)}\in D_{\alpha}(I^{\circ})$ and $f^{(2\alpha)}\in C_{\alpha}[a,b]$.*

If $|f^{(2\alpha)}|^q$ is a generalized (s,P)-convex mapping over the interval $[a,b]$, then, for all $q \geq 1$,

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \\
&\quad+\sum_{i=0}^{n-1}\frac{M^{\frac{1}{q}}(b-a)^{2\alpha}}{2^{(2-\frac{1}{q})\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}} \\
&\quad\times\left[\left|f^{(2\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|^{q}+\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^{q}\right]^{\frac{1}{q}},
\end{aligned} \tag{2.21}$$

where M is given as in (2.9).

Proof. By making use of properties of the modulus and the generalized power-mean inequality in (2.1), it yields that

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}{}_{a}\mathscr{I}_{b}^{(\alpha)}f(x)\right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha} \\
&\quad\times\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|(\mathrm{d}t)^{\alpha} \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left|f^{(\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-f^{(\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right| \\
&\quad+\sum_{i=0}^{n-1}\frac{(b-a)^{2\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}(\mathrm{d}t)^{\alpha}\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_0^1|1-2t|^{2\alpha}\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^{q}(\mathrm{d}t)^{\alpha}\right)^{\frac{1}{q}}.
\end{aligned} \tag{2.22}$$

Taking advantage of the generalized (s,P)-convexity of $|f^{(2\alpha)}|^q$ and calculating the integral on the right-side of the inequality (2.22), we infer that

$$\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)}\int_0^1 |1-2t|^{2\alpha}\left|f^{(2\alpha)}\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q (\mathrm{d}t)^\alpha \\
&\leq \frac{1}{\Gamma(1+\alpha)}\int_0^1 |1-2t|^{2\alpha}|t^{s\alpha}+(1-t)^{s\alpha}|\left[\left|f^{(2\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|^q\right. \\
&\quad \left.+\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right](\mathrm{d}t)^\alpha \\
&=\left[\left|f^{(2\alpha)}\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left|f^{(2\alpha)}\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right](\mathrm{d}t)^\alpha Y,
\end{aligned} \tag{2.23}$$

where Y is mentioned in (2.11). Substituting (2.23) into inequalities (2.22), we can derive the desired conclusion. This terminates the proof. □

Next, we present some special cases of Theorem 2.7.

(i) If $n=1$, then we receive the generalized trapezoid-type inequality:

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2^\alpha}-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_a\mathscr{I}_b^{(\alpha)}f(x)\right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha}{4^\alpha}\left|f^{(\alpha)}(b)-f^{(\alpha)}(a)\right| \\
&\quad +\frac{M^{\frac{1}{q}}(b-a)^{2\alpha}}{2^{(2-\frac{1}{q})\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}} \\
&\quad \times\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}(b)\right|^q\right]^{\frac{1}{q}}.
\end{aligned}$$

(ii) If $n=2$, then we receive the generalized Bullen-type inequality:

$$\begin{aligned}
&\left|\frac{1^\alpha}{2^\alpha}\left[\frac{f(a)+f(b)}{2^\alpha}+f\left(\frac{a+b}{2}\right)\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_a\mathscr{I}_b^{(\alpha)}f(x)\right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha}{16^\alpha}\left[\left|f^{(\alpha)}\left(\frac{a+b}{2}\right)-f^{(\alpha)}(a)\right|+\left|f^{(\alpha)}(b)-f^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right] \\
&\quad +\frac{M^{\frac{1}{q}}(b-a)^{2\alpha}}{2^{(5-\frac{1}{q})\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)^{1-\frac{1}{q}} \\
&\quad \times\left\{\left[\left|f^{(2\alpha)}(a)\right|^q+\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}+\left[\left|f^{(2\alpha)}(b)\right|^q+\left|f^{(2\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

(iii) If $\alpha = 1$, then
for $n = 1$,

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\right|$$
$$\leq \frac{b-a}{8}\left|f'(b)-f'(a)\right| + \frac{(b-a)^2}{24}\left(\frac{6}{s+1} - \frac{24}{s+2} + \frac{24}{s+3}\right)^{\frac{1}{q}}\left[\left|f''(b)\right|^q + \left|f''(a)\right|^q\right]^{\frac{1}{q}},$$

and for $n = 2$,

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}t\right|$$
$$\leq \frac{b-a}{32}\left[\left|f'\left(\frac{a+b}{2}\right) - f'(a)\right| + \left|f'(b) - f'\left(\frac{a+b}{2}\right)\right|\right] + \frac{(b-a)^2}{192}\left(\frac{6}{s+1} - \frac{24}{s+2} + \frac{24}{s+3}\right)^{\frac{1}{q}}$$
$$\times\left\{\left[\left|f''(a)\right|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}} + \left[\left|f''(b)\right|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{\frac{1}{q}}\right\}.$$

3. Examples

In this section, we provide two examples to support our main results.

Example 3.1. Let the hypotheses of Theorem 2.6 hold. If one chooses $|f^{(2\alpha)}(t)|^q = t^{s\alpha}$, which is a generalized (s,P)-convex mapping along with each $t \in (0,\infty)$, $q > 1$, and $s \in (0,1]$, and puts $a = 0, b = 1$, then

$$\begin{aligned}
&\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\frac{\Gamma(1+\frac{s}{q}\alpha)}{\Gamma(1+(\frac{s}{q}+2)\alpha)}\left[\left(\frac{i}{n}\right)^{(\frac{s}{q}+2)\alpha} + \left(\frac{i+1}{n}\right)^{(\frac{s}{q}+2)\alpha}\right] - \frac{\Gamma(1+\frac{s}{q}\alpha)}{\Gamma(1+(\frac{s}{q}+3)\alpha)}\right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{1^{\alpha}}{4^{\alpha}n^{2\alpha}}\frac{\Gamma(1+\frac{s}{q}\alpha)}{\Gamma(1+(\frac{s}{q}+1)\alpha)}\left|\left(\frac{i+1}{n}\right)^{(\frac{s}{q}+1)\alpha} - \left(\frac{i}{n}\right)^{(\frac{s}{q}+1)\alpha}\right| \\
&\quad + \sum_{i=0}^{n-1}\frac{1^{\alpha}}{4^{\alpha}n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right)^{\frac{1}{p}} \\
&\quad \times\left(\frac{2^{\alpha}\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}\left[\left|\left(\frac{i+1}{n}\right)\right|^{s\alpha} + \left|\left(\frac{i}{n}\right)\right|^{s\alpha}\right]\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.1}$$

Next, let us take values for certain variables, $\alpha = 1$ and $q \in (1,10]$ with $p = \frac{q}{q-1}$. For three different cases, $n = 1$, $n = 2$, and $n = 3$, we plot the corresponding surfaces to represent the functional graph of the left, middle, and right side, which can obviously verify the correctness of inequality (3.1).

Example 3.2. If one chooses $|f^{(2\alpha)}(t)|^q = t^{s\alpha}$, which is a generalized (s,P)-convex mapping along with each $t \in (0,\infty)$, and takes special values for partial variables, which are $a = 0, b = 1$,

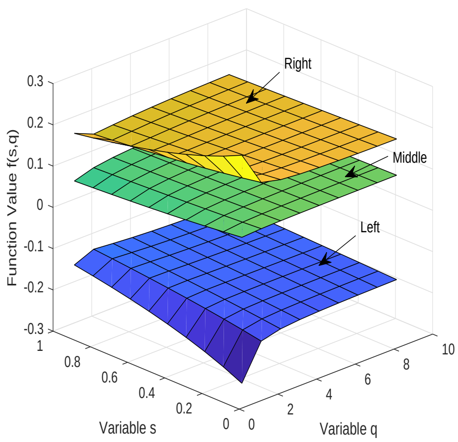

FIGURE 1. The functional surface corresponding to Example 3.1 for $s \in [0.1, 1]$, $n = 1$

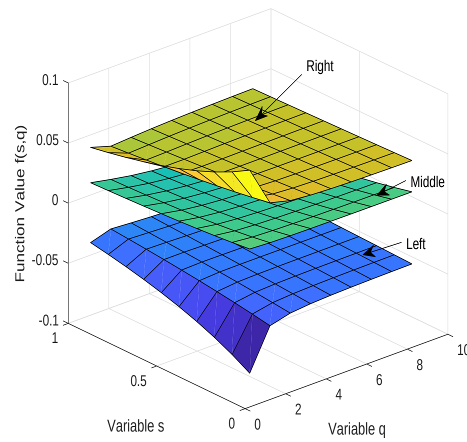

FIGURE 2. The functional surface corresponding to Example 3.1 for $s \in [0.1, 1]$, $n = 2$

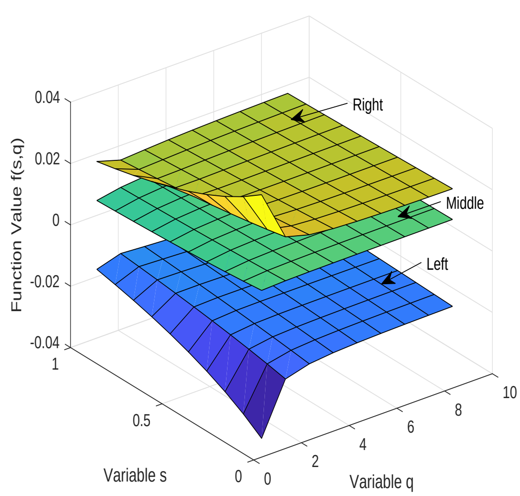

FIGURE 3. The functional surface corresponding to Example 3.1 for $s \in [0.1, 1]$, $n = 3$

$\alpha = 1$, but $s \in (0, 1]$ and $q \in (1, 10]$, then, for $n = 1$, inequality (2.21) is represented as follows

$$\begin{aligned} &\left| \frac{1}{2(\frac{s}{q}+1)(\frac{s}{q}+2)} - \frac{1}{(\frac{s}{q}+1)(\frac{s}{q}+2)(\frac{s}{q}+3)} \right| \\ &\leq \frac{1}{8(\frac{s}{q}+1)} + \frac{1}{24}\left(\frac{6}{s+1} - \frac{24}{s+2} + \frac{24}{s+3} \right)^{\frac{1}{q}}. \end{aligned} \tag{3.2}$$

and for $n=2$, the inequality is represented as below

$$\begin{aligned}&\left|\frac{1}{4(\frac{s}{q}+1)(\frac{s}{q}+2)}\left(1+\left(\frac{1}{2}\right)^{(\frac{s}{q}+1)}\right)-\frac{1}{(\frac{s}{q}+1)(\frac{s}{q}+2)(\frac{s}{q}+3)}\right|\\&\leq\frac{1}{32(\frac{s}{q}+1)}+\frac{(\frac{1}{s+1}-\frac{4}{s+2}+\frac{4}{s+3})^{\frac{1}{q}}}{32\cdot 6^{1-\frac{1}{q}}}\left[\left(\frac{1}{2}\right)^{\frac{s}{q}}+\left(\left(\frac{1}{2}\right)^{s}+1\right)^{\frac{1}{q}}\right].\end{aligned}\tag{3.3}$$

Drawing the corresponding surfaces with MATLAB software to indicate the functional graph of the left, middle, and right side, which can apparently illustrate the correctness of inequalities (3.2) and (3.3).

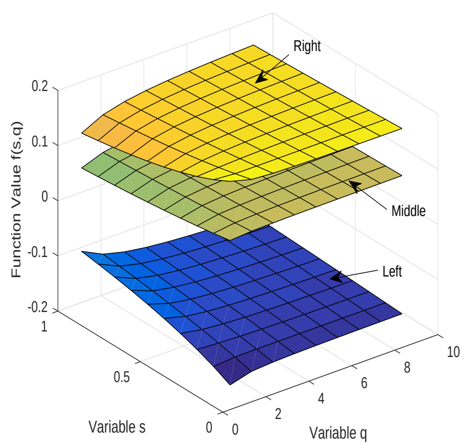

FIGURE 4. The functional surface corresponding to Example 3.2 for $s\in[0.1,1]$, $n=1$

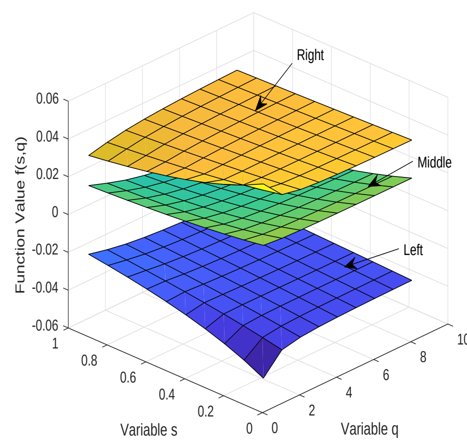

FIGURE 5. The Functional surface corresponding to Example 3.2 for $s\in[0.1,1]$, $n=2$

4. SOME APPLICATIONS

4.1. **Special means.** We consider the application of our inequality to several generalized α-type special means with arbitrary real numbers a,b $(a\neq b)$
(i) Arithmetic mean:

$$A_\alpha(a,b)=\left(\frac{a+b}{2}\right)^\alpha=\frac{a^\alpha+b^\alpha}{2^\alpha}.$$

(ii) Generalized m-logarithmic mean:

$$L_{m\alpha}(a,b)=\left[\frac{\Gamma(1+m\alpha)}{\Gamma(1+(m+1)\alpha)}\frac{b^{(m+1)\alpha}-a^{(m+1)\alpha}}{(b-a)^\alpha}\right]^{\frac{1}{m}},m\in\mathbb{R}\setminus\{-1,0\}.$$

Proposition 4.1. *Let $a,b \in \mathbb{R}, 0 < a < b$, for certain fixed $s \in (0,1]$. Then*

$$\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+2)\alpha)}\left|\sum_{i=0}^{n-1}\left(\frac{1}{n}\right)^{\alpha} A_{\alpha}\left(\left(\frac{(n-i)a+ib}{n}\right)^{s+2}, \left(\frac{(n-i-1)a+(i+1)b}{n}\right)^{s+2}\right)\right.$$
$$\left. -\Gamma(1+\alpha)L^{s+2}_{(s+2)\alpha}(a,b)\right|$$
$$\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}$$
$$\times\left|\left(\frac{(n-i-1)a+(i+1)b}{n}\right)^{(s+1)\alpha} - \left(\frac{(n-i)a+ib}{n}\right)^{(s+1)\alpha}\right|$$
$$+\sum_{i=0}^{n-1}\frac{M(b-a)^{2\alpha}}{n^{3\alpha}}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}.$$

Proof. The outcome follows from Theorem 2.4, with $f:[0,1]\to[0^{\alpha},1^{\alpha}]$, $f^{(2\alpha)}(x) = x^{s\alpha}$ and $f^{(\alpha)}(x) = \frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}x^{(s+1)\alpha}$. Besides, it is obvious that $||f^{(2\alpha)}||_{\infty} = \sup_{x\in(a,b)}|f^{(2\alpha)}| = 1^{\alpha}$. This completes the proof. □

4.2. **Probability distribution function.** Assume that X is a continuous random variable involving generalized probability density mapping $f:[a,b]\to\mathbb{R}^{\alpha}$, and the cumulative distribution mapping is defined as below

$$\mathrm{Pr}_{\alpha}(X\leq x) = \frac{1}{\Gamma(1+\alpha)}\int_a^x g(\lambda)(\mathrm{d}\lambda)^{\alpha}.$$

The following inequality closely connecst the cumulative distribution mapping and the generalized expectation $E^{\alpha}(X)$.

Proposition 4.2. *If $s = 1$ in Theorem 2.3, then*

$$\left|\sum_{i=0}^{n-1}\left(\frac{1}{2n}\right)^{\alpha}\left[\mathrm{Pr}_{\alpha}\left(X\leq\frac{(n-i)a+ib}{n}\right)+\mathrm{Pr}_{\alpha}\left(X\leq\frac{(n-i-1)a+(i+1)b}{n}\right)\right]-\frac{b^{\alpha}-E^{\alpha}(X)}{(b-a)^{\alpha}}\right|$$
$$\leq\sum_{i=0}^{n-1}\frac{(b-a)^{\alpha}}{4^{\alpha}n^{2\alpha}}\left[\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma^2(1+\alpha)}{n^{\alpha}\Gamma(1+3\alpha)}\right]\left|g\left(\frac{(n-i-1)a+(i+1)b}{n}\right)-g\left(\frac{(n-i)a+ib}{n}\right)\right|$$
$$+\sum_{i=0}^{n-1}\frac{(b-a)^{3\alpha}}{16^{\alpha}n^{3\alpha}}\frac{Z^{\alpha}-z^{\alpha}}{\Gamma(1+2\alpha)},$$

where M is defined in (2.9).

4.3. **The midpoint and trapezoidal formulas.** We now consider the applications of some inequalities to attain error estimates of the composite quadrature rules. Assume that the partition of interval $[a,b]$ is presented by $I_t : a = \kappa_0 < \kappa_1 < \cdots < \kappa_{t-1} = b$, and define the midpoint and trapezoidal quadrature rule arising from local fraction integrals, respectively, by

$$A_1(f,I_t) = \frac{1}{\Gamma(1+\alpha)}\sum_{j=0}^{t-1} f\left(\frac{x_j+x_{j+1}}{2}\right)(x_{j+1}-x_j)^{\alpha},$$

and

$$A_2(f,I_t) = \frac{1}{\Gamma(1+\alpha)} \sum_{j=0}^{t-1} \frac{f(x_j)+f(x_{j+1})}{2^\alpha} (x_{j+1}-x_j)^\alpha.$$

Besides, the approximate errors of the corresponding ${}_a\mathscr{I}_b^{(\alpha)} f(x)$ are as below

$$B_1(f,I_t) = {}_a\mathscr{I}_b^{(\alpha)} f(x) - A_1(f,I_t),$$

and

$$B_2(f,I_t) = {}_a\mathscr{I}_b^{(\alpha)} f(x) - A_2(f,I_t).$$

Proposition 4.3. *If all the hypotheses in Theorem 2.5 are satisfied, then, for every division I_t of the interval $[a,b]$, the approximation error can be expressed as*

$$\begin{aligned}
&|B_1(f,I_t)+B_1(f,I_t)| \\
&\leq \sum_{j=0}^{t-1} \frac{(x_{j+1}-x_j)^{2\alpha}}{8^\alpha \Gamma(1+2\alpha)} \left| f^{(\alpha)}(x_{j+1}) - f^{(\alpha)}(x_j) \right| \\
&\quad + \sum_{j=0}^{t-1} \frac{(x_{j+1}-x_j)^{3\alpha}}{16^\alpha \Gamma(1+3\alpha)} \left[\left| f^{(2\alpha)}\left(\frac{x_j+x_{j+1}}{2}\right) \right| + \left| f^{(2\alpha)}(x_{j+1}) \right| \right] \\
&\quad - \sum_{j=0}^{t-1} \frac{(x_{j+1}-x_j)^{4\alpha} L^\alpha}{64^\alpha \Gamma(1+3\alpha)}.
\end{aligned}$$

Proof. If $n=2$ in Theorem 2.5 on the subinterval $[x_j,x_{j+1}]$ $(j=0,1,\cdots,t-1)$, then

$$\begin{aligned}
&\left| \left(\frac{1}{2}\right)^\alpha \left[\frac{f(x_j)+f(x_{j+1})}{2^\alpha} + f\left(\frac{x_j+x_{j+1}}{2}\right) \right] - \frac{\Gamma(1+\alpha)}{(x_{j+1}-x_j)^\alpha} {}_{x_j}\mathscr{I}_{x_{j+1}}^{(\alpha)} f(x) \right| \\
&\leq \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \frac{(x_{j+1}-x_j)^\alpha}{16^\alpha} \left| f^{(\alpha)}(x_{j+1}) - f^{(\alpha)}(x_j) \right| + \frac{(x_{j+1}-x_j)^{2\alpha}}{32^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)} \\
&\quad \times \left[\left| f^{(2\alpha)}\left(\frac{x_j+x_{j+1}}{2}\right) \right| + \left| f^{(2\alpha)}(x_{j+1}) \right| \right] - \frac{(x_{j+1}-x_j)^{3\alpha} L^\alpha}{128^\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+3\alpha)}
\end{aligned}$$

for each $j=0,\cdots,t-1$. Multiplying both sides of the inequality by $\frac{2^\alpha (x_{j+1}-x_j)^\alpha}{\Gamma(1+\alpha)}$ and summing over j from 0 to $t-1$, one derives the error bound. This completes the proof. □

4.4. **The wave equation on Cantor sets.** The wave equation is mainly used to describe the motion phenomena of a fractal waveform, which can be ocean waves, sound waves, light waves, or waves propagating along vibrating piles. With the in-depth research of local fractional calculus theory, some analytical methods were developed to deal with local fractional differential equations. One of them is to take advantage of the local fractional Fourier series method to manage the local fractional wave equation.

The local fractional wave equation was proposed as

$$\frac{\partial^{2\alpha} f_\alpha(x,t)}{\partial t^{2\alpha}} = k^\alpha \frac{\partial^{2\alpha} f_\alpha(x,t)}{\partial x^{2\alpha}}. \tag{4.1}$$

According to equation (4.1), a wave equation on Cantor set was established in [43] as below

$$\frac{\partial^{2\alpha} f_\alpha(x,t)}{\partial t^{2\alpha}} = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \frac{\partial^{2\alpha} f_\alpha(x,t)}{\partial x^{2\alpha}}, \tag{4.2}$$

where $f_\alpha(x,t)$ is with a fractal content. The initial condition of equation (4.2) is indicated by

$$f_\alpha(x,0) = \frac{x^\alpha}{\Gamma(1+\alpha)}.$$

As an application of Lemma 2.1, we derive the following result.

Proposition 4.4. *In accordance with Lemma 2.1, if $n = 2$, then*

$$\begin{aligned}
&\left| \frac{1}{2^\alpha}\left[\frac{f_\alpha(x,a)+f_\alpha(x,b)}{2^\alpha} + f_\alpha\left(x, \frac{a+b}{2}\right)\right] - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\frac{1}{\Gamma(1+\alpha)}\int_a^b f_\alpha(x,t)(\mathrm{d}t)^\alpha \right| \\
&= \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\frac{(b-a)^\alpha S}{16^\alpha} \\
&\quad - \frac{(b-a)^\alpha}{4^\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left[{}_a\mathscr{I}^{(\alpha)}_{\frac{a+b}{2}} \left(\frac{3a+b}{2(a-b)} - \frac{2t}{a-b}\right)^{2\alpha} x^{2\alpha} \frac{\partial^{2\alpha} f_\alpha(x,t)}{\partial x^{2\alpha}} \right. \\
&\quad \left. + {}_a\mathscr{I}^{(\alpha)}_{\frac{a+b}{2}} \left(\frac{3a+b}{2(a-b)} - \frac{2t}{a-b}\right)^{2\alpha} x^{2\alpha} \frac{\partial^{2\alpha} f_\alpha(x,t+\frac{b-a}{2})}{\partial x^{2\alpha}} \right],
\end{aligned}$$

where $S = f^{(\alpha)}(x,b) - f^{(\alpha)}(x,a)$.

5. Conclusions

In this paper, we proposed general inequalities for twice differentiable generalized (s,P)-convex functions via the local fractional calculus, and the results that we obtained in this paper include all cases of the generalized Hermite–Hadamard- and Bullen-type inequalities. In addition, we gave several examples and applications for these inequalities in the settings of fractal sets. We emphasize here that the ideas and methods utilized in this paper can also yield new results in other generalized convex function properties, not only in generalized (s,P)-convexity.

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