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A DECREASING OPERATOR METHOD FOR A FRACTIONAL INITIAL VALUE PROBLEM ON INFINITE INTERVAL

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Abstract. This paper considers the uniqueness of positive solutions for a fractional initial value problem involving Caputo fractional derivative

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = [p(t)+q(t)(f(t,x(t))]^{-1}, \ \ 0 \le t < \infty, \\ x(0) = b_0, x'(0) = b_1, x''(0) = b_2, \end{cases}$$

where $2 < \alpha \le 3$, $f : [0,\infty) \times [0,\infty) \to [0,\infty)$, and $p(t)$ and $q(t)$ are continuous functions. By imposing some suitable conditions on f, p, and q, we obtain the uniqueness of positive solutions for the problem, and we construct an iterative scheme to approximate the unique solution. Our approach is based on a fixed point theorem of decreasing operators on cones. In addition, two simply examples are presented to illustrate our main result.

Keywords. Caputo fractional derivative; Decreasing operator; Positive solution; Infinite interval.

1. Introduction

The fractional calculus is an extension of the traditional integer calculus, which has the properties of an infinity memory and is hereditary. For some fundamental results in the theory of fractional calculus and fractional models, we refer the reader to the [8, 12, 13, 18, 20]. Fractional differential equations arise from a variety of fields in science and engineering. In particular, problems concerning qualitative analysis of the positivity of such solutions for fractional differential equations have received the attention from many authors; see, e.g., [8, 11, 13, 16, 20, 22]. It is known that there have a large of fractional derivatives, in which Caputo fractional derivative is important, and the theory of Caputo fractional differential equations also greatly attracted the attention for their wide applications; see, e.g. [1, 2, 7, 10, 15, 17, 19, 21, 23] and the related references therein.

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In [16], Matar discussed the existence and uniqueness of positive solutions for the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = f(t,x(t)), & 0 < t \leq 1, \\ x(0) = 0, \ x'(0) = \gamma > 0, \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function, and $\gamma > 0$ is a constant. By employing the method of the upper and lower solutions method, Schauder fixed point theorem and Banach contraction mapping principle, the author obtained the existence and uniqueness of positive solutions.

In [4], Bai and Qiu obtained the existence of positive solutions for the singular boundary value problem

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) + f(t,x(t)) = 0, & 0 < t < 1, \\ x(0) = x'(1) = x''(0) = 0, \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $2 < \alpha \leq 3$, $f : (0,1] \times [0,\infty) \to [0,\infty)$ is continuous, and $\lim_{t \to 0^+} f(t,x) = \infty$ for all $x \in [0,\infty)$. By using nonlinear alternative of Leray-Schauder type and Guo-Krasnoselskii's fixed point theorem, the authors obtained the existence of positive solutions.

In [5], Cabana and Wang studied the existence of a positive solution to fractional differential equations with integral boundary conditions

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) + f(t,x(t)) = 0, & 0 < t < 1, \\ x(0) = x''(0) = 0, \ x(1) = \lambda \int_0^1 x(s)ds, \end{cases}$$

where ${}^{C}D_{0^+}^{\alpha}$ denotes the Caputo fractional derivative of order $2 < \alpha < 3$, $0 < \lambda < 2$, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous. By using Guo-Krasnoselskii's fixed point theorem, they obtained the existence of at least one positive solution.

In this paper, we are interested in the uniqueness of positive solutions for fractional differential equations on an infinite interval. Inspired and motivated by the results presented in [3, 6, 11, 13, 20], we concentrate on the positivity of the solutions for the following nonlinear fractional differential equation

$$\begin{cases} {}^{C}D_{0^+}^{\alpha}x(t) = [p(t) + q(t)f(t,x(t))]^{-1}, & 0 \leq t < \infty, \\ x(0) = b_0, \ x'(0) = b_1, \ x''(0) = b_2, \end{cases} \tag{1.1}$$

where ${}^{C}D_{0^+}^{\alpha}$ is the Caputo fractional derivative of order $\alpha \in (2,3]$, $f : [0,\infty) \times [0,\infty) \to [0,\infty)$, $p(t)$ and $q(t)$ are continuous functions. To demonstrate the uniqueness of positive solutions, we use the fixed point of monotone decreasing operator to discuss (1.1). To the best knowledge of the authors, only few papers concerned with the existence of positive solutions for boundary value problems of fractional differential equation on infinite intervals by using the fixed point of monotone decreasing operator up to now. The goal of present paper is to fill the gap in this area. It is interesting and important to study the uniqueness of positive solutions for boundary value problem (1.1).

2. Previous Results and Preliminaries

In this section, we present some basic results about fractional calculus theory which will be used later.

Definition 2.1. [14] Let $f:[a,b]\to \boldsymbol{R}$ be a given function. For $\alpha>0$, the Riemann-Liouville fractional integral of order α of f is defined by

$$(I_a^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,$$

where $\Gamma(\alpha)$ denotes the classical gamma function.

Definition 2.2. [14] Let $f:[a,b]\to \boldsymbol{R}$ be a given function. For $\alpha>0$, the Caputo derivative of fractional order α of f is given by

$${}^C D_a^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}f^n(s)ds,$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes integer part of α.

Lemma 2.3. [14] *Let $\alpha>0$. Suppose $x\in C^{n-1}[0,\infty)$ and $x^{(n)}$ exists almost everywhere on any bounded interval of $[0,\infty)$. Then*

$$(I^{\alpha C}D^\alpha x)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}t^k.$$

In particular, when $2<\alpha<3$, $(I^{\alpha C}D^\alpha x)(t)=x(t)-x(0)-x'(0)t-\frac{1}{2}x''(0)t^2$.

Lemma 2.4. *$x\in C[0,\infty)$ is a solution to (1.1) if and only if*

$$x(t)=b_0+b_1t+\frac{1}{2}b_2t^2+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[p(s)+q(s)f(s,x(s))]^{-1}ds. \tag{2.1}$$

Proof. Let x be a solution to (1.1) . We write this equation as

$$(I^{\alpha C}D^\alpha x)(t)=I^\alpha[p(t)+q(t)f(t,x(t))]^{-1},\ \ 0\le t<\infty.$$

From (1.1) and Lemma 2.3, we obtain

$$x(t)=x(0)+x'(0)t+\frac{1}{2}x''(0)t^2+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[p(s)+q(s)f(s,x(s))]^{-1}ds.$$

By the initial conditions $x(0)=b_0, x'(0)=b_1, x''(0)=b_2$, we obtain equation (2.1). Since each step is reversible, the converse follows easily. □

Suppose that E is a Banach space which is partially ordered by a cone $P\subset E$. We say that $x\le y$ if and only if $y-x\in P$. An operator $A:P\to P$ is decreasing if $x\le y$ implies $Ax\ge Ay$ for all $x,y\in P$. If $x\le y$ and $x\ne y$, we denote $x<y$ or $y>x$. θ is the zero element of E. P is called normal if there exists $N>0$ such that

$$\theta\le x\le y\Rightarrow \| x\|\le N\| y\|,\ x,y\in E.$$

Lemma 2.5. [9] *Let $P\subset E$ be a normal cone, and let A: $P\to P$ be a decreasing operator such that*

(i) there exists $0<\varepsilon<1$ such that $A^2\theta\ge \varepsilon A\theta$;

(ii) for any $\varepsilon\le t^*<1$ and $0<\lambda\le t^*$, there are $\eta=\eta(t^*)>0$ and $\delta=\delta(t^*)>0$ such that

$$A(\lambda x)\le[\lambda(1+\eta)]^{-1}Ax, x\in P.$$

Then there exists $x^*\in P$ such that $Ax^*=x^*$.

3. MAIN RESULTS

For convenience, we give the following notations:
(a) $J = [0, \infty)$;
(b) $C(J) = \{x : x(t) \textit{ is continous in } J\}$;
(c) $C_+(J) = \{x \in C(J) : x(t) \geq 0, \forall t \in J\}$.
In this paper, we use the following space, E, to study (1.1), which is defined by

$$E = \left\{ x \in C(J) : \sup_{t \in J} \frac{|x(t)|}{t^2+1} < \infty \right\}.$$

We know that E is a Banach space equipped with the norm

$$\| x \| = \sup_{t \in J} \frac{|x(t)|}{t^2+1} < \infty.$$

We define a cone P by

$$P = \{x \in E : x(t) \geq 0, \forall t \in J\},$$

so $P \subset C_+(J)$. For $x, y \in P$ with $x \leq y$, we have $0 \leq x(t) \leq y(t)$, $t \in J$ and thus

$$\sup_{t \in J} \frac{|x(t)|}{t^2+1} \leq \sup_{t \in J} \frac{|y(t)|}{t^2+1},$$

that is $\| x \| \leq \| y \|$. Thus P is a normal cone.

Theorem 3.1. *Suppose the following conditions are satisfied:*

(H_1) *$f(t,x)$ is increasing in $x \in [0,\infty)$ for fixed $t \in [0,\infty)$ and for $\lambda \in (0,1)$, $f(t,\lambda x) \geq \lambda f(t,x)$, $x \in [0,\infty)$;*

(H_2) *$f(t,0) = 0$ and there exists a function $a(t)$ such that*

$$f(t,x) \leq a(t)x,\ t \in [0,\infty)\ x \in [0,\infty);$$

(H_3) *there exist $b_i \geq 0 (i = 4,5)$ such that $0 < a(t) < b_4 t^l + b_5$, $l \in \mathbf{N}$;*

(H_4) *there exist $m, r \in \mathbf{N}$ and constants $c_i (i = 1,2,3,4) \geq 0$ such that $p(t) \geq c_1 t^m + c_2 \geq 0$, $0 \leq q(t) \leq c_3 t^r + c_4$ and $m \geq l + r + 2$.*

Then problem (1.1) has a unique positive solution x^* in P. Moreover, for any initial value $x_0 \in P$, the sequence

$$x_{n+1}(t) = b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) f(s, x_n(s))]^{-1} ds,\ \ t \in J,$$

$n = 0,1,2,\cdots$, satisfies $x_n(t) \to x^*(t)$ as $n \to \infty$.

Proof. We are going to apply Lemma 2.5 to solve (1.1) for a positive solution in E. First, for $x \in C_+(J)$, we define an operator A by

$$Ax(t) = b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) f(s, x(s))]^{-1} ds,\ \ t \in J.$$

Then, for $x \in P$,

$$\begin{aligned} 0 \le Ax(t) &\le b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t p(s)^{-1} ds \\ &\le b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t (c_1 s^m + c_2)^{-1} ds \\ &\le b_0 + b_1 t + \frac{1}{2} b_2 t^2 + b_3 t^{\alpha-1}, t \in J, \end{aligned}$$

where

$$b_3 = \frac{1}{\Gamma(\alpha)} \int_0^\infty (c_1 s^m + c_2)^{-1} ds < \infty.$$

Thus $A(P) \subset C_+(J)$ due to

$$\begin{aligned} \parallel Ax \parallel &= \sup_{t \in J} \frac{|b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) f(s, x(s))]^{-1} ds|}{t^2 + 1} \\ &\le \sup_{t \in J} \frac{|b_0 + b_1 t + \frac{1}{2} b_2 t^2 + b_3 t^{\alpha-1}|}{t^2 + 1} \\ &\le b_0 + \frac{b_1}{2} + \frac{b_2}{2} + b_3. \end{aligned}$$

It follows that $\parallel Ax \parallel \le M$ for all $x \in C_+(J)$, where $M = b_0 + \frac{b_1}{2} + \frac{b_2}{2} + b_3$. Hence, $A : P \to P$, obviously, A is a decreasing operator by (H_1).

Next, we verify that A satisfies (i) of Lemma 2.5. If $x \in C_+(J)$ and $x(t) \le b_0 + b_1 t + \frac{1}{2} b_2 t^2 + b_3 t^{\alpha-1}$ for all $t \in J$, we can show that there exists $\varepsilon_0 > 0$ such that

$$p(t) \ge \varepsilon_0 q(t) f(t, x(t)), \quad \forall t \in J. \tag{3.1}$$

In fact, when $x \in C_+(J)$ with $x(t) \le b_0 + b_1 t + \frac{1}{2} b_2 t^2 + b_3 t^{\alpha-1}$, by (H_2), we know that

$$\begin{aligned} q(t) f(t, x(t)) &\le q(t)[a(t)(1 + b_0 + b_1 t + \frac{1}{2} b_2 t^2 + b_3 t^{\alpha-1})] \\ &\le (c_3 t^r + c_4)(b_4 t^l + b_5)(1 + b_0 + b_1 t + \frac{1}{2} b_2 t^2 + b_3 t^{\alpha-1}) \\ &\le b_6 t^{l+r+2} + b_7 t^{l+r+\alpha-1} + b_8 t^{l+r+1} + b_9 t^{l+r} + b_{10}^{r+2} + b_{11}^{l+2} + b_{12} \\ &= b_6 t^{l+r+2} + b_{13}(t) + b_{12}, \ \forall t \in J, \end{aligned} \tag{3.2}$$

where

$$b_{13}(t) = b_7 t^{l+r+\alpha-1} + b_8 t^{l+r+1} + b_9 t^{l+r} + b_{10}^{r+2} + b_{11}^{l+2},$$

and b_i $(i = 6, 7, \cdots, 13)$ are positive constants. Note that $m \ge l + r + 2$, and then there exists a constant $\varepsilon_0 > 0$ such that

$$c_1 t^m + c_2 \ge \varepsilon_0 (b_6 t^{l+r+2} + b_{13}(t) + b_{12}), \quad \forall t \in J. \tag{3.3}$$

Thus (3.1) is true according to (3.2) and (3.3). In (3.2), let $x=0$. By (H_2), we obtain

$$\begin{aligned}0\le A(0)&=b_0+b_1t+\frac{1}{2}b_2t^2+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)^{-1}ds\\&\le b_0+b_1t+\frac{1}{2}b_2t^2+\frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^t p(s)^{-1}ds\\&\le b_0+b_1t+\frac{1}{2}b_2t^2+b_3t^{\alpha-1},\quad \forall t\in J.\end{aligned}\tag{3.4}$$

It follows from (3.1) that $p(t)\ge\varepsilon_0 q(t)f(t,A(0))$ for all $t\in J$, which together with (3.4) yields that

$$\begin{aligned}(A^2(0))(t)&=b_0+b_1t+\frac{1}{2}b_2t^2+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[p(s)+q(s)f(s,A(0)(s))]^{-1}ds\\&\ge b_0+b_1t+\frac{1}{2}b_2t^2+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[p(s)+\varepsilon_0^{-1}p(s)]^{-1}ds\\&\ge\frac{\varepsilon_0}{1+\varepsilon_0}\left\{b_0+b_1t+\frac{1}{2}b_2t^2+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}p(s)^{-1}ds\right\}\\&=\varepsilon(A(0))(t),\quad\forall t\in J,\end{aligned}$$

where $\varepsilon=\frac{\varepsilon_0}{1+\varepsilon_0}$, $0<\varepsilon<1$. Hence, $A^2(0)\ge\varepsilon A(0)$.

Finally, we verify that A the satisfies (ii) of Lemma 2.5 Let $\varepsilon\le t^*<1$, we prove the following inequality

$$p(t)+q(t)f(t,x(t))\le\tau[(t^*)^{-1}p(t)+q(t)f(t,x(t))],\quad\forall\, 0\le x\le A(0),\quad t\in J,\tag{3.5}$$

holds, where

$$\tau=t^*+\frac{(1-t^*)t^*}{t^*+\varepsilon_0},\quad t^*<\tau<1.\tag{3.6}$$

In fact, when $0\le x\le A(0)$, we know by (3.4) that

$$0\le x(t)\le b_0+b_1t+\frac{1}{2}b_2t^2+b_3t^{\alpha-1},\ t\in J.$$

Hence (3.1) holds. Noting (3.6) again, we have

$$\begin{aligned}(\tau-t^*)p(t)&=\frac{(1-t^*)t^*}{t^*+\varepsilon_0}p(t)=(1-\tau)t^*\varepsilon_0^{-1}p(t)\\&\ge(1-\tau)t^*q(t)f(t,x(t)),\ \forall t\in J,\end{aligned}$$

so (3.5) holds. Letting $\eta = \frac{1}{\tau} - 1$, we have $\eta > 0$. For $0 \le x \le A(0)$, $0 < \lambda \le t^*$, by (3.5), (3.6), and (H_1), we have

$$\begin{aligned} A(\lambda x)(t) &= b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) f(s, \lambda x(s))]^{-1} ds \\ &\le b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + \lambda q(s) f(s, x(s))]^{-1} ds, \\ &\le \frac{\tau}{\lambda} (b_0 + b_1 t + \frac{1}{2} b_2 t^2) + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [(\lambda)^{-1} p(s) + q(s) f(s, x(s))]^{-1} ds, \\ &\le \frac{\tau}{\lambda} (b_0 + b_1 t + \frac{1}{2} b_2 t^2) + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [(t^*)^{-1} p(s) + q(s) f(s, x(s))]^{-1} ds, \\ &\le \frac{\tau}{\lambda} \left\{ b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) f(s, x(s))]^{-1} ds \right\} \\ &= [\lambda(1+\eta)]^{-1} (Ax)(t), \quad \forall t \in J. \end{aligned}$$

So, the assumption (ii) of Lemma 2.5 is satisfied. An application of Lemma 2.5 implies that the operator equation $Ax = x$ has a unique solution x^* in P. That is, problem (1.1) has a unique positive solution x^* in P. Moreover, for any initial value $x_0 \in P$, constructing the sequence

$$x_{n+1}(t) = b_0 + b_1 t + \frac{1}{2} b_2 t^2 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [p(s) + q(s) f(s, x_n(s))]^{-1} ds,$$

$n = 0, 1, 2, \cdots$, we have $x_n(t) \to x^*(t)$ as $n \to \infty$. □

Now, we give some examples to illustrate our result.

Example 3.2. Consider the following initial value problem of a nonlinear fractional differential equation

$$\begin{cases} {}^C D_{0^+}^{\frac{5}{2}} x(t) = [100t^5 + 2 + (20t^2 + 5) \frac{2t^2 x(t)}{t+300}]^{-1}, & 0 \le t < \infty, \\ x(0) = 1, \ x'(0) = 0, \ x''(0) = 1, \end{cases}$$

where $\alpha = \frac{5}{2}$, $p(t) = 100t^5 + 2$, $q(t) = 20t^2 + 5$, $r = 2, m = 5$,

$$f(t, x) = \frac{2t^2 x}{t + 300},$$

$b_0 = 1, b_2 = 0$, and $b_3 = 1$. Thus $f(t, x)$ is increasing in $x \in [0, \infty)$ for fixed $t \in [0, \infty)$ and for $\lambda \in (0, 1)$,

$$f(t, \lambda x) = \frac{2t^2 \lambda x}{t + 300} = \lambda \frac{2t^2 x}{t + 300} = \lambda f(t, x), x \in [0, \infty),$$

and then condition (H_1) is satisfied.

Clearly, $f(t, 0) = \frac{2t^2 \times 0}{t+300} = 0$, $t \in [0, \infty)$, and condition (H_2) is satisfied. In the following, let $a(t) = 2t + 100$, it is obvious that

$$f(t, x) = \frac{2t^2 x}{t + 300} \le (2t + 100)x = a(t)x, (t, x) \in [0, \infty) \times [0, \infty),$$

so condition (H_3) is satisfied. Evidently, there exist $b_4 = 3, b_5 = 100 > 0$ such that $0 < a(t) < 3t + 100$, where $l = 1$; and $m = 5, r = 2$, and $c_1 = 50$, $c_2 = 2$, $c_3 = 100$, $c_4 = 5$ such that $p(t) \ge 50t^5 + 2 \ge 0$, $0 \le q(t) \le 100t^2 + 5$, and we know $l + r + 2 = 5$, so $m \ge 5$ holds, it is clear that condition (H_4) is satisfied. All the hypotheses of Theorem 3.1 are fulfilled. Therefore, it follows that the boundary value problem has a unique positive solution.

Example 3.3. Consider the following initial value problem of a fractional differential equation

$$\begin{cases} {}^C D_{0^+}^{\frac{5}{2}} x(t) = [t^6 + 1 + (t^2+1)\frac{x(t)}{1+x(t)} e^{-t}, & 0 \le t < \infty, \\ x(0) = 1,\ x'(0) = 0,\ x''(0) = 1, \end{cases}$$

where $\alpha = \frac{5}{2}$, $p(t) = t^6 + 1$, $q(t) = t^2 + 1$, $r = 2, m = 6$,

$$f(t,x) = \frac{x}{1+x} e^{-t},$$

$b_0 = 1, b_2 = 0$, and $b_3 = 1$. Thus $f(t,x)$ is increasing in $x \in [0,\infty)$ for fixed $t \in [0,\infty)$ and for $\lambda \in (0,1)$,

$$f(t,\lambda x) = \frac{\lambda x}{1+\lambda x} e^{-t} \le \lambda \frac{\lambda}{x} 1 + x e^{-t} = \lambda f(t,x), x \in [0,\infty),$$

and then condition (H_1) satisfied. Further, $f(t,0) = 0$, $t \in [0,\infty)$, so condition (H_2) is satisfied. Let $a(t) = e^{-t}$. Then

$$f(t,x) \le a(t)x, (t,x) \in [0,\infty) \times [0,\infty),$$

and then (H_3) is satisfied. Moreover, we can see that $0 < a(t) < t+1$ and it is easy to prove that (H_4) is satisfied. So, by Theorem 3.1 that the problem has a unique positive solution.

4. Conclusion

In this paper, we gave a result dealing with the uniqueness of positive solutions for the inital value problems of nonlinear fractional differential equations involving Caputo fractional derivatives (1.1). Recently, most of the unique results were obtained by the Banach contractive theorem or the fixed point theorems of increasing operators. In this paper, we used a different method, a fixed point theorem of decreasing operators, and then we established the unique positive solution. Moreover, we presented a sequence to approximate the unique solution.

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