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A NEW INERTIAL ITERATIVE ALGORITHM FOR SPLIT NULL POINT AND COMMON FIXED POINT PROBLEMS

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Abstract. In this paper, we present a new iterative scheme with a self-adaptive step size for finding a common solution of the split null point and common fixed point problem for an infinite family of multivalued demicontractive mappings between a Banach space and a Hilbert space. We demonstrate strong convergence result with a self-adaptive step size without a priori estimate of the norm of the linear operator under some suitable conditions. A numerical result is also presented to support our main results.

Keywords. Common fixed point; Multivalued demicontractive mapping; Split null point; Self-adaptive step size; Strong convergence.

1. Introduction

Recently, fixed point methods have been investigated for various convex optimization problems; see, e.g., [2, 4, 9, 17, 18] and the references therein. The common future of these problems is we can transfer them into a fixed point problem via their resolvents; see, e.g., [6, 13, 15, 16, 19] and the references therein. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces, $A:\mathscr{H}_1\to 2^{\mathscr{H}_1}$ and $B:\mathscr{H}_2\to 2^{\mathscr{H}_2}$ be multivalued mappings, and $T:\mathscr{H}_1\to\mathscr{H}_2$ be a bounded linear operator. Byrne et al. [3] considered the following Split Null Point Problem (SNPP)

$$x^*\in A^{-1}0\bigcap T^{-1}(B^{-1}0),\quad x^*\in\mathscr{H}_1. \tag{1.1}$$

For solving SNPP (1.1) with two maximal monotone operators A and B in Hilbert spaces, they proposed and studied the following algorithms

$$\begin{cases} x_0\in\mathscr{H}_1,\\ x_{n+1}=J_\mu^A(x_n+\lambda T^*(J_\mu^B-I)Tx_n), \end{cases} \tag{1.2}$$

and

$$\begin{cases} x_0\in\mathscr{H}_1,\\ x_{n+1}=\alpha_n x_0+(1-\alpha_n)J_\mu^A(x_n+\lambda T^*(J_\mu^B-I)Tx_n), \end{cases} \tag{1.3}$$

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where μ is a positive real constant , T^* is the adjoint of T, $\lambda \in (0, \frac{2}{L})$, $L = \|T^*T\|$, and J_μ^A and J_μ^B are the resolvent operators of A and B, respectively. Under some certain conditions, they obtained a weak convergence result for algorithm (1.2) and a strong convergence result for algorithm (1.3).

In 2017, Eslamian et al. [7] introduced an algorithm for solving the Split Common Null Point Problem (SCNPP) and Fixed Point Problem (FPP) between a Banach space E and a Hilbert space $\mathscr{H}$. The proposed algorithm is as follows:

$$\begin{cases} x_1 \in \mathscr{H} \textit{ is chosen arbitrarily}, \\ z_{n,i} = x_n - \lambda_n T^* J_E(Tx_n - Q_{\mu_n}^{B_i} Tx_n), \\ u_n = \beta_{n,0}x_n + \sum_{i=1}^m \beta_{n,i} J_{\lambda_n}^{A_i} z_{n,i}, \\ y_n = \delta_{n,0}u_n + \sum_{i=1}^m \delta_{n,i} S_i u_n, \\ x_{n+1} = \alpha_n \gamma f(y_n) + (I - \alpha_n D)y_n, \end{cases} \tag{1.4}$$

where the step size λ_n satisfies $0 < \lambda_n \|T\|^2 < 2$. Under some conditions, they proved that the sequence generated above converges strongly to a common solution of the SCNPP with two finite families of maximal monotone operators $\{A_i\}_{i=1}^m$ and $\{B_i\}_{i=1}^m$, and FPP with a finite family of single-valued demicontractive mappings $\{S_i\}_{i=1}^m$.

In 2019, Pachara and Suantai [14] considered the follwing Split Common Fixed Point Problem (SCFPP):

$$find\ \ x \in \cap_{i=1}^\infty Fix(S_i) \textit{ such that } Tx \in \cap_{i=1}^\infty Fix(U_i),$$

where $T : \mathscr{H}_1 \to \mathscr{H}_2$ is a bounded linear operator. To solve the SCFPP, they proposed the following algorithm in Hilbert spaces:

$$\begin{cases} y_n = x_n + \sum_{i=1}^n \beta_{n,i} \lambda T^*(\omega_{n,i} - Tx_n), \\ u_n = \alpha_{n,0} y_n + \sum_{i=1}^n \alpha_{n,i} z_{n,i}, \\ x_{n+1} = \xi_n f(x_n) + (1 - \xi_n)u_n, \end{cases} \tag{1.5}$$

where $z_{n,i} \in S_i y_n$, $\omega_{n,i} \in U_i(Tx_n)$ and $\lambda \in (0, \frac{1-\acute{k}}{\|T\|^2})$. They established the strong convergence of algorithm (1.5).

We notice that the step size, $\lambda_n (or\ \lambda)$, of the above algorithms requires prior knowledge of the operator norm, $\|T\|$, which is not easy to implement because they require computation of the operator norm, which is a difficult task.

To avoid this computation, in 2021, Wang et al. [25] introduced an algorithm for solving (SNPP) and (FPP) for multivalued demicontractive mappings on a Hilbert space $\mathscr{H}$. This algorithm can be implemented easily since it has no need to know a priori information about bounded linear operators. The proposed algorithm is as follows:

$$\begin{cases} y_n = J_{\lambda_n}^{B_1}(x_n - \gamma_n T^*(I - J_{\lambda_n}^{B_2})Tx_n), \\ u_n = (1 - \delta_n)y_n + \delta_n \sum_{i=1}^N w_i z_n^{(i)}, \\ x_{n+1} = \alpha_n \tau f(x_n) + (1 - \alpha_n D)u_n,\ n \geq 1, \end{cases} \tag{1.6}$$

where $z_n^{(i)} \in S_i y_n$ and

$$\gamma_n = \rho_n \frac{g_n(x_n)}{F_n(x_n) + G_n(x_n)},$$

where $g_n(x) = \frac{1}{2}\|(I - J^{B_2}_{\lambda_n})Tx\|^2$, $G_n(x) = \|T^*(I - J^{B_2}_{\lambda_n})Tx\|^2$, and $F_n(x) = \|(I - J^{B_1}_{\lambda_n})x\|^2$. Under appropriate conditions, they obtained a strong convergence result without a priori estimate of the norm of the linear operator.

In this paper, inspired and motivated by the works mentioned above, we propose a new algorithm to solve the split null point and common fixed point problem between a Banach space and a Hilbert space. We prove the strong convergence of the sequence generated by our algorithm. A numerical experiment is also provided to demonstrate the efficiency of our proposed algorithm.

2. PRELIMINARIES

In this section, we recall some known definitions and lemmas which will be used for our convergence analysis in the sequel.

Let R be the set of real numbers and N^* the set of positive integers. Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a convex, closed, and nonempty subset of $\mathscr{H}$. We denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in \mathscr{H}$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. The nearest point (metric) projection of $\mathscr{H}$ onto C is denoted by P_C, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in \mathscr{H}$ *and* $y \in C$. P_C is called the metric projection of $\mathscr{H}$ onto C. It is known that P_C is firmly nonexpansive, i.e., $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$, for all $x, y \in \mathscr{H}$. Moreover $P_C x \in C, \langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in \mathscr{H}$, $y \in C$.

We denote by $CB(\mathscr{H})$ the family of all bounded and closed subsets of $\mathscr{H}$. The Pompeiu Hausdorff metric on $CB(\mathscr{H})$ is defined by $H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\}$ for all $A, B \in CB(\mathscr{H})$, where $d(x, B) = \inf_{b \in B} \|x - b\|$.

Let $T : \mathscr{H} \to 2^{\mathscr{H}}$ be a multivalued mapping. An element $p \in \mathscr{H}$ is called a fixed point of T if $p \in Tp$. The set of all fixed points of T is denoted by $Fix(T)$. We say that T satisfies the endpoint condition if $Tp = \{p\}$ for all $p \in Fix(T)$.

Definition 2.1. Let $S : \mathscr{H} \to CB(\mathscr{H})$ be a multivalued. Mapping $I - S$ is said to be demiclosed at zero if, for any sequence $\{x_n\} \subset \mathscr{H}$ which converges weakly to q and the sequence $\{x_n - u_n\}$ converges strongly to 0, where $u_n \in Sx_n$, $q \in Fix(S)$.

Definition 2.2. A multivalued mapping $T : \mathscr{H} \to CB(\mathscr{H})$ is said to be

(i) a contraction if there exists $k \in (0, 1)$ such that

$$H(Tx, Ty) \leq k\|x - y\| \ \ \forall\, x, y \in \mathscr{H};$$

(ii) nonexpansive if

$$H(Tx, Ty) \leq \|x - y\| \ \ \forall\, x, y \in \mathscr{H};$$

(iii) quasi-nonexpansive if $Fix(T) \neq \emptyset$ and

$$H(Tx, Tp) \leq \|x - p\| \ \ \forall\, x \in \mathscr{H}, \ p \in Fix(T);$$

(iv) k-demicontractive [8] if $Fix(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$H(Tx, Tp)^2 \leq \|x - p\|^2 + kd(x, Tx)^2 \ \ \forall\, x \in \mathscr{H}, \ p \in Fix(T).$$

It is known that every multivalued quasi-nonexpansive mapping T with $Fix(T) \neq \emptyset$ is demicontractive, but not all multivalued demicontractive mappings are quasi-nonexpansive.

Example 2.3. Let $\mathscr{H} = R$. For each $i \in N^*$, define

$$S_i x = \begin{cases} \left[-\frac{2i}{i+1}x\ ,\ -\frac{3i}{i+1}x\right], & if\ \ x \le 0; \\ \\ \left[-\frac{3i}{i+1}x\ ,\ -\frac{2i}{i+1}x\right], & if\ \ x > 0. \end{cases}$$

Then $S_i : R \to CB(R)$ is a multivalued demicontractive mapping, which is not quasi-nonexpansive. Moreover, $I - S_i$ is demiclosed at zero.

Proof. It is easy to see that $Fix(S_i) = \{0\}$. For each $0 \ne x \in R$, $H(S_i x, S_i 0)^2 = |-\dfrac{3i}{i+1}x - 0|^2 = |x-0|^2 + (\dfrac{9i^2}{(i+1)^2} - 1)|x|^2 = |x-0|^2 + \dfrac{8i^2-2i-1}{i^2+2i+1}|x|^2$.

Clearly, S_i is not quasi-nonexpansive. We also have

$$d(x, S_i x)^2 = |x - (-\frac{2i}{i+1}x)|^2 = (\frac{3i+1}{i+1})^2|x|^2 = \frac{9i^2+6i+1}{i^2+2i+1}|x|^2.$$

Therefore,

$$H(S_i x, S_i 0)^2 = |x-0|^2 + (\frac{8i^2-2i-1}{9i^2+6i+1})d(x, S_i x)^2.$$

Hence S_i is demicontractive with a constant $k_i = \frac{8i^2-2i-1}{9i^2+6i+1} \in (0,1)$. For any sequence $\{x_n\} \subset R$, which converges weakly to q and the sequence $\{x_n - u_n\}$ converges strongly to 0, where $u_n \in S_i x_n$, $x_n \to q$ and $u_n \to q$. Also

$$-\frac{2i}{i+1}x_n \le u_n \le -\frac{3i}{i+1}x_n$$

or

$$-\frac{3i}{i+1}x_n \le u_n \le -\frac{2i}{i+1}x_n,$$

so

$$-\frac{2i}{i+1}q \le q \le -\frac{3i}{i+1}q$$

or

$$-\frac{3i}{i+1}q \le q \le -\frac{2i}{i+1}q.$$

Therefore, $q = 0 \in Fix(S_i)$. Hence $I - S_i$ is demiclosed at zero. □

Let E be a real Banach space with norm $\|\cdot\|$, and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ *at* $x \in E$ *by* $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ_E of convexity of E is defined by

$$\delta_E(\varphi) = \inf\{1 - \frac{\|x+y\|}{2} : \|x\| = 1 = \|y\|, \|x-y\| \ge \varphi\}.$$

E is called uniformly convex if $\delta_E(\varphi) > 0$ for any $\varphi > 0$. A uniformly convex Banach space is strictly convex and reflexive. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\} \ for\ every\ x \in E.$$

Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists. In this case, E is called smooth. It is known that E is smooth if and only if J is a single-valued mapping of E into E^*. It is also known that E is reflexive if and only if J is surjective, and E is strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J^* on E^*.

Let E be a uniformly convex and smooth Banach space and $B : E \to 2^{E^*}$ be a maximal monotone operator. Now, we consider the metric resolvent of B,

$$Q_\mu^B = (I + \mu J^{-1}B)^{-1},\ \mu > 0.$$

It is known that the operator Q_μ^B is firmly nonexpansive and the fixed points of the operator Q_μ^B are the null points of B. The resolvent plays an essential role in the approximation theory for zero points of maximal monotone operators in Banach spaces. According to the work of Aoyama et al. [1], we have the following property

$$\langle Q_\mu^B x - x^*, J(x - Q_\mu^B x)\rangle \geq 0,\ x \in E,\ x^* \in B^{-1}(0), \tag{2.1}$$

if E is a real Hilbert space, then

$$\langle J_\mu^B x - x^*, x - J_\mu^B x\rangle \geq 0,\ x \in E,\ x^* \in B^{-1}(0),$$

where $J_\mu^B = (I + \mu B)^{-1}$ is the general resolvent, and $B^{-1}(0) = \{z \in E : 0 \in Bz\}$ is the set of null points of B. It is well known that $B^{-1}(0)$ is convex and closed (see [21]). A Hilbert space $\mathscr{H}$ satisfies the Opial's condition, i.e., for any sequence $\{x_n\}$ with $\{x_n\} \rightharpoonup x$, the following inequality holds

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|,$$

for every $y \in \mathscr{H}$ with $y \neq x$, which is also equivalent to $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$.

Definition 2.4. Let E be a Banach space and B be a mapping of E into 2^{E^*}. The effective domain of B denoted by $dom(B)$ is given as $dom(B) = \{x \in E : Bx \neq \emptyset\}$. Let $B : E \to 2^{E^*}$ be a multivalued operator on E. Then

(i) The graph $G(B)$ is defined by $G(B) := \{(x,u) \in E \times E : u \in B(x)\}$.

(ii) The operator B is said to be monotone if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in dom(B)$, $u \in Bx$ *and* $v \in By$.

(iii) A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E.

Definition 2.5. A bounded linear operator G on $\mathscr{H}$ is called strongly positive if there exists a constant $\bar{\gamma} > 0$ such that $\langle Gx, x\rangle \geq \bar{\gamma}\|x\|^2,\ \forall\, x \in \mathscr{H}$.

Lemma 2.6. [12] *Let D be a self-adjoint strongly positive bounded linear operator on a Hilbert space $\mathscr{H}$ with coefficient $\bar{\gamma} > 0$ and $0 < \mu \leq \|D\|^{-1}$. Then $\|I - \mu D\| \leq 1 - \mu\bar{\gamma}$.*

Lemma 2.7. [22] *For $x, y \in \mathscr{H}$, the following statements hold: (i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$; (ii) $\|x+y\|^2 = \|x\|^2 + 2\langle x,y\rangle + \|y\|^2$.*

Lemma 2.8. [5] *Let $\mathscr{H}$ be a real Hilbert space, $x_i \in \mathscr{H}$ $(1 \leq i \leq m)$ and $\{\alpha_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m \alpha_i = 1$. Then*

$$\|\sum_{i=1}^m \alpha_i x_i\|^2 = \sum_{i=1}^m \alpha_i \|x_i\|^2 - \sum_{i,j=1, i\neq j}^m \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.9. [24] *Let C be a convex and closed subset of a real Hilbert space $\mathscr{H}$ and $T : C \to CB(C)$ be a multivalued k-demicontractive mapping. Then,*

(i) $Fix(T)$ is closed;

(ii) If T satisfies the endpoint condition, then $Fix(T)$ is convex.

Lemma 2.10. [26] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the condition $a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\sigma_n$, $n \geq 0$, where $\{\gamma_n\}$ and $\{\sigma_n\}$ are sequences of real numbers such that*
(i) $\{\gamma_n\} \subset [0,1]$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$ or $\sum_{n=1}^{\infty} \gamma_n|\sigma_n| < \infty$.
Then, $\lim_{n\to\infty} a_n = 0$.

Lemma 2.11. [11] *Let $\{a_n\}$, $\{c_n\} \subset R^+$, $\{\sigma_n\} \subset (0,1)$ and $\{b_n\} \subset R$ be sequences such that $a_{n+1} \leq (1-\sigma_n)a_n + b_n + c_n$ for all $n \geq 0$. Assume $\sum_{n=0}^{\infty} |c_n| < \infty$. Then the following results hold:*

(i) If $b_n \leq \beta\sigma_n$ for some $\beta \geq 0$, then $\{a_n\}$ is a bounded sequence.

(ii) If $\sum_{n=0}^{\infty} \sigma_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\sigma_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.12. [10] *Let $\{\Gamma_n\}$ be a real sequence which does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ with $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in N^*$. Define the sequence $\{\phi(n)\}_{n\geq n_0}$ of integers by $\phi(n) = max\{j \leq n_0 : \Gamma_j < \Gamma_{j+1}\}$, where $n_0 \in N^*$ such that $\{j \leq n_0 : \Gamma_j < \Gamma_{j+1}\} \neq \emptyset$. Then, the following hold:*

(i) $\phi(n_0) \leq \phi(n_0+1) \leq \ldots$ and $\phi(n) \to \infty$;

(ii) $\Gamma_{\phi(n)} \leq \Gamma_{\phi(n)+1}$ and $\Gamma_n \leq \Gamma_{\phi(n)+1}$ $\forall\, n \geq n_0$.

3. Main Results

In this section, we present our algorithm and discuss its strong convergence.

We make the following assumptions:

(A_1) Let $\mathscr{H}$ be a Hilbert sapce and E be a uniformly convex and smooth Banach space. Let $T : \mathscr{H} \to E$ be a linear and bounded operator such that $T \neq 0$ and T^* is a adjoint operator of T.

(A_2) For all $i \in N^*$, assume that $S_i : \mathscr{H} \to CB(\mathscr{H})$ is a multivalued k_i-demicontractive mapping such that $I - S_i$ is demiclosed at zero and S_i satisfies the endpoint condition.

(A_3) Let $A : \mathscr{H} \to 2^{\mathscr{H}}$ and $B : E \to 2^{E^*}$ be two maximal monotone operators, and let $J_{\phi_n}^A$ be the resolvent of A for $\liminf_{n\to\infty} \phi_n > 0$, and $Q_{\eta_n}^B$ be the metric resolvent of B for $\liminf_{n\to\infty} \eta_n > 0$.

(A_4) Let $D : \mathscr{H} \to \mathscr{H}$ be a strongly positive, bounded linear self-adjoint operator with coefficient $\bar{\gamma} > 0$ such that $\|D\| \leq 1$ and $f : \mathscr{H} \to \mathscr{H}$ be a contraction with coefficient $\rho \in (0,1)$ and $0 < \gamma < \frac{\bar{\gamma}}{\rho}$.

(A_5) Assume that $\Omega = A^{-1}(0) \cap T^{-1}(B^{-1}0) \cap \bigcap_{i=1}^{\infty} Fix(S_i) \neq \emptyset$.

Algorithm 3.1. *Let $\{\delta_{n,i}\}$ be real sequences in $(0,1)$ for $i = 0,1,2,\cdots, n \geq 2$. Assume that $\{x_n\}$ is a sequence generated iteratively by x_1, $x_2 \in \mathscr{H}$ and*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = w_n - \lambda_n T^* J_E(I - Q_{\eta_n}^B)Tw_n, \\ u_n = J_{\phi_n}^A z_n, \\ y_n = \delta_{n,0}u_n + \sum_{i=1}^{n} \delta_{n,i}v_{n,i}, \\ x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n D)y_n, \end{cases} \tag{3.1}$$

where $v_{n,i} \in S_i u_n$, *and* θ_n *is defined by*

$$\theta_n = \begin{cases} \min\left\{\frac{\xi_n}{\|x_n - x_{n-1}\|}, \frac{n-1}{n+\eta-1}\right\}, & if\, x_n \neq x_{n-1}, \\ \frac{n-1}{n+\eta-1}, & otherwise, \end{cases}$$

where $\eta \geq 3$ *and the step size*

$$\lambda_n = \tau_n \frac{g_n(w_n)}{G_n(w_n) + F_n(w_n)},$$

where $g_n(w_n) = \frac{1}{2}\|J_E(I - Q^B_{\eta_n})Tw_n\|^2, G_n(w_n) = \|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2$ *and* $F_n(w_n) = \|(I - J^A_{\phi_n})w_n\|^2$. *If* $F_n(w_n) = G_n(w_n) = 0$, *then* $w_n = z_n = u_n$, *i.e.* 3.1 *reduces to*

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = \delta_{n,0} w_n + \sum_{i=1}^n \delta_{n,i} v_{n,i}, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n D) y_n. \end{cases}$$

Theorem 3.2. *Let the assumptions* (A_1)-(A_5) *be satisfied and the following conditions hold:*
(C_1) *For all* $n \geq 2$, $\sum_{i=0}^n \delta_{n,i} = 1$ *and* $\liminf_{n\to\infty}(\delta_{n,0} - k)\delta_{n,i} > 0$, $i \in N^*$, *where* $k = sup\{k_i : i \in N^*\} < 1$;
(C_2) $\{\alpha_n\} \subset (0, min\{1, \frac{1}{\bar{\gamma}}, \frac{1}{\bar{\gamma}-\gamma\rho}\})$, $lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=0}^\infty \alpha_n = \infty$ *and* $\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$;
(C_3) $\liminf_{n\to\infty} \tau_n > 0$, $\liminf_{n\to\infty} \tau_n(4 - \tau_n) > 0$.
Then, the sequence $\{x_n\}$ *generated by Algorithm* 3.1 *converges strongly to a point* $p \in \Omega$, *which is the unique solution to the following variational inequality problem:*

$$\langle(\gamma f - D)p, q - p\rangle \leq 0, \;\; \forall\, q \in \Omega. \tag{3.2}$$

Remark 3.3. From the definition of $\{\theta_n\}$ and the condition (C_2), we have

$$\lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0 \; and \; \lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Proof. According to the conditions (C_1)-(C_3), some inequalities in the following proof hold when n is sufficiently large.

Step 1. We show that problem (3.2) has a unique solution $p \in \Omega$.

Since A and B are maximally monotone and T is bounded and linear, we reach the conclusion that $A^{-1}(0)$ and $T^{-1}(B^{-1}0)$ are convex and closed. From Lemma 2.9, we obtain $Fix(S_i)$ $(\forall\, i \in N^*)$ is convex and closed. Hence Ω is convex and closed. (3.2) is equivalent to the following formula

$$\langle(\gamma f + (I - D))p - p, q - p\rangle \leq 0, \;\; \forall\, q \in \Omega,$$

so we just need to prove that exists a unique $p \in \Omega$ such that $p = P_\Omega(\gamma f + (I - D))p$, *i.e.* $P_\Omega(\gamma f + (I - D))$ has a unique fixed point. For all $x, y \in \mathscr{H}$, by Lemma 2.6, we have

$$\begin{aligned} &\|P_\Omega(\gamma f + I - D)x - P_\Omega(\gamma f + I - D)y\| \\ \leq &\|(\gamma f + I - D)x - (\gamma f + I - D)y\| \\ \leq &\gamma\|fx - fy\| + \|(I - D)x - (I - D)y\| \\ \leq &(1 - (\bar{\gamma} - \gamma\rho))\|x - y\|. \end{aligned}$$

Hence, $P_\Omega(\gamma f + I - D)$ is a contraction on $\mathscr{H}$. By the Banach contraction principle, there exists a unique element $p \in \Omega$ such that $p = P_\Omega(\gamma f + (I - D))p$.

Step 2. We prove that $\{x_n\}$ is bounded.

Since $p \in \Omega$, we have $S_i p = \{p\}$, $p = J^A_{\phi_n} p$, and $Tp = Q^B_{\eta_n} Tp$ for all $i \in N^*$. From (3.1), we have

$$\|z_n - p\|^2 = \|w_n - p\|^2 - 2\lambda_n \langle Tw_n - Tp, J_E(I - Q^B_{\eta_n})Tw_n\rangle + \lambda_n^2 \|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2. \quad (3.3)$$

From (2.1), we see that $\langle Q^B_{\eta_n} Tw_n - Tp, J_E(Tw_n - Q^B_{\eta_n} Tw_n)\rangle \geq 0$, $Tp \in B^{-1}(0)$. Hence, we obtain

$$\begin{aligned}
&\langle Tw_n - Tp, J_E(I - Q^B_{\eta_n})Tw_n\rangle \\
=&\|J_E(I - Q^B_{\eta_n})Tw_n\|^2 + \langle Q^B_{\eta_n} Tw_n - Tp, J_E(I - Q^B_{\eta_n})Tw_n\rangle \\
\geq&\|J_E(I - Q^B_{\eta_n})Tw_n\|^2. \qquad (3.4)
\end{aligned}$$

By (3.3), (3.4), and condition (C_3), we have

$$\begin{aligned}
\|z_n - p\|^2 \leq&\|w_n - p\|^2 - 2\lambda_n \|J_E(I - Q^B_{\eta_n})Tw_n\|^2 + \lambda_n^2 \|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 \\
=&\|w_n - p\|^2 - 4\tau_n \frac{g_n^2(w_n)}{F_n(w_n) + G_n(w_n)} + \tau_n^2 \frac{g_n^2(w_n)}{(F_n(w_n) + G_n(w_n))^2} G_n(w_n) \\
\leq&\|w_n - p\|^2 - \tau_n(4 - \tau_n) \frac{g_n^2(w_n)}{F_n(w_n) + G_n(w_n)} \qquad (3.5) \\
\leq&\|w_n - p\|^2. \qquad (3.6)
\end{aligned}$$

From (3.1) and (3.6), we have

$$\begin{aligned}
\|u_n - p\|^2 =&\|J^A_{\phi_n} z_n - J^A_{\phi_n} p\|^2 \\
\leq&\|z_n - p\|^2 \qquad (3.7) \\
\leq&\|w_n - p\|^2. \qquad (3.8)
\end{aligned}$$

By (3.1), Lemma 2.8, and condition (C_1), we have

$$\begin{aligned}
\|y_n - p\|^2 \leq&\delta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^n \delta_{n,i}\|v_{n,i} - p\|^2 - \sum_{i=1}^n \delta_{n,0}\delta_{n,i}\|v_{n,i} - u_n\|^2 \\
\leq&\delta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^n \delta_{n,i} H(S_i u_n, S_i p)^2 - \sum_{i=1}^n \delta_{n,0}\delta_{n,i}\|v_{n,i} - u_n\|^2 \\
\leq&\delta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^n \delta_{n,i}(\|u_n - p\|^2 + k_i d(u_n, S_i u_n)^2) - \sum_{i=1}^n \delta_{n,0}\delta_{n,i}\|v_{n,i} - u_n\|^2 \\
\leq&\delta_{n,0}\|u_n - p\|^2 + \sum_{i=1}^n \delta_{n,i}(\|u_n - p\|^2 + k\|u_n - v_{n,i}\|^2) - \sum_{i=1}^n \delta_{n,0}\delta_{n,i}\|v_{n,i} - u_n\|^2 \\
=&\|u_n - p\|^2 - \sum_{i=1}^n \delta_{n,i}(\delta_{n,0} - k)\|v_{n,i} - u_n\|^2 \qquad (3.9) \\
\leq&\|u_n - p\|^2. \qquad (3.10)
\end{aligned}$$

It follows from (3.1) that

$$\|w_n - p\| \leq \|x_n - p\| + \theta_n \|x_n - x_{n-1}\| = \|x_n - p\| + \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|.$$

By Remark 3.3, $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$, it follows that there exists a constant $M_1^* > 0$ such that

$$\frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq M_1^*$$

for all $n \geq 2$. Hence, we obtain $\|w_n - p\| \leq \|x_n - p\| + \alpha_n M_1^*$, which together with By (3.1), (3.8), and (3.10) yields

$$\begin{aligned}
\|x_{n+1} - p\| \leq & \alpha_n \|\gamma f(x_n) - Dp\| + (1 - \alpha_n \bar{\gamma}) \|u_n - p\| \\
\leq & \alpha_n \|\gamma f(x_n) - \gamma f(p)\| + \alpha_n \|\gamma f(p) - Dp\| + (1 - \alpha_n \bar{\gamma}) \|w_n - p\| \\
\leq & \alpha_n \gamma \rho \|x_n - p\| + \alpha_n \|\gamma f(p) - Dp\| + (1 - \alpha_n \bar{\gamma})(\|x_n - p\| + \alpha_n M_1^*) \\
= & (1 - \alpha_n(\bar{\gamma} - \gamma\rho)) \|x_n - p\| + \alpha_n (\bar{\gamma} - \gamma\rho) \left(\frac{\|\gamma f(p) - Dp\|}{\bar{\gamma} - \gamma\rho} + \frac{(1 - \alpha_n \bar{\gamma})}{\bar{\gamma} - \gamma\rho} M_1^* \right) \\
\leq & (1 - \alpha_n(\bar{\gamma} - \gamma\rho)) \|x_n - p\| + \alpha_n (\bar{\gamma} - \gamma\rho) M_2^*,
\end{aligned}$$

where

$$M_2^* = \sup_{n\geq 2} \left\{ \frac{\|\gamma f(p) - Dp\|}{\bar{\gamma} - \gamma\rho} + \frac{(1 - \alpha_n \bar{\gamma})}{\bar{\gamma} - \gamma\rho} M_1^* \right\}.$$

Set $a_n = \|x_n - p\|$, $b_n = \alpha_n(\bar{\gamma} - \gamma\rho) M_2^*$, $c_n = 0$, and $\sigma_n = \alpha_n(\bar{\gamma} - \gamma\rho)$. By Lemma 2.11, we have that $\{\|x_n - p\|\}$ is bounded. Hence, $\{x_n\}$ is bounded. Additionally, $\{w_n\}$, $\{z_n\}$, $\{u_n\}$, and $\{y_n\}$ are all bounded.

Step 3. We prove that sequence $\{x_n\}$ converges strongly to p.

Using (3.1) and Lemma 2.7, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 \leq & (1 - \alpha_n \bar{\gamma})^2 \|y_n - p\|^2 + 2\alpha_n \langle \gamma f(x_n) - Dp, x_{n+1} - p \rangle \\
= & (1 - \alpha_n \bar{\gamma})^2 \|y_n - p\|^2 + 2\alpha_n \gamma \langle f(x_n) - f(p), x_{n+1} - p \rangle \\
& + 2\alpha_n \langle \gamma f(p) - Dp, x_{n+1} - p \rangle,
\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}
\|w_n - p\|^2 = & \|x_n - p + \theta_n (x_n - x_{n-1})\|^2 \\
= & \|x_n - p\|^2 + \theta_n^2 \|x_n - x_{n-1}\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1} \rangle \\
\leq & \|x_n - p\|^2 + \theta_n \|x_n - x_{n-1}\| (\theta_n \|x_n - x_{n-1}\| + 2\|x_n - p\|) \\
\leq & \|x_n - p\|^2 + 3M_3^* \theta_n \|x_n - x_{n-1}\| \\
= & \|x_n - p\|^2 + 3M_3^* \alpha_n \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\|,
\end{aligned} \tag{3.12}$$

where

$$M_3^* = \sup_{n\geq 2} \{\|x_n - p\|, \theta_n \|x_n - x_{n-1}\|\}.$$

Combining (3.5), (3.7), (3.9), (3.11), and (3.12) we obtain

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le (1-\alpha_n\bar{\gamma})^2(\|u_n-p\|^2-\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|v_{n,i}-u_n\|^2)\\
&+2\alpha_n\gamma\langle f(x_n)-f(p),x_{n+1}-p\rangle+2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\le (1-\alpha_n\bar{\gamma})^2(\|z_n-p\|^2-\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|v_{n,i}-u_n\|^2)\\
&\quad +2\alpha_n\gamma\rho\|x_n-p\|\|x_{n+1}-p\|+2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\le (1-\alpha_n\bar{\gamma})^2(\|w_n-p\|^2-\tau_n(4-\tau_n)\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)}\\
&\quad -\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|v_{n,i}-u_n\|^2)\\
&\quad +\alpha_n\gamma\rho(\|x_n-p\|^2+\|x_{n+1}-p\|^2)+2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\le (1-\alpha_n\bar{\gamma})^2\|x_n-p\|^2+3M_3^*(1-\alpha_n\bar{\gamma})^2\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\
&\quad +\alpha_n\gamma\rho(\|x_n-p\|^2+\|x_{n+1}-p\|^2)\\
&\quad -(1-\alpha_n\bar{\gamma})^2\tau_n(4-\tau_n)\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)}-(1-\alpha_n\bar{\gamma})^2\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|u_n-v_{n,i}\|^2\\
&\quad +2\alpha_n\langle\gamma f(p)-Dp,x_{n+1}-p\rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
&\le \frac{1-2\alpha_n\bar{\gamma}+(\alpha_n\bar{\gamma})^2+\alpha_n\gamma\rho}{1-\alpha_n\gamma\rho}\|x_n-p\|^2+3M_3^*\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\
&-\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\tau_n(4-\tau_n)\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)}-\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|u_n-v_{n,i}\|^2\\
&+\frac{2\alpha_n}{1-\alpha_n\gamma\rho}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle\\
&\le (1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\gamma\rho})\|x_n-p\|^2+\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\gamma\rho}(\frac{(\alpha_n\bar{\gamma})^2}{2\alpha_n(\bar{\gamma}-\gamma\rho)}M^*\\
&\quad +3M_3^*\frac{(1-\alpha_n\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\\
&+\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(p)-Dp,x_{n+1}-p\rangle)-\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\tau_n(4-\tau_n)\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)}\\
&-\frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|u_n-v_{n,i}\|^2,
\end{aligned}$$

where $M^* = \sup_{n\geq 2}\{\|x_n - p\|^2\}$. From conditions (C_1)-(C_3), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 \leq & (1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\gamma\rho})\|x_n-p\|^2 + \frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\gamma\rho}(\frac{(\alpha_n\bar{\gamma})^2}{2\alpha_n(\bar{\gamma}-\gamma\rho)}M^* \\ & + 3M_3^*\frac{(1-\alpha_n\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \frac{1}{\bar{\gamma}-\gamma\rho}\langle \gamma f(p) - Dp, x_{n+1}-p\rangle) \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} & \frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\tau_n(4-\tau_n)\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)} + \frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\delta_{n,i}(\delta_{n,0}-k)\|u_n-v_{n,i}\|^2 \\ \leq & \frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\tau_n(4-\tau_n)\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)} + \frac{(1-\alpha_n\bar{\gamma})^2}{1-\alpha_n\gamma\rho}\sum_{i=1}^{n}\delta_{n,i}(\delta_{n,0}-k)\|u_n-v_{n,i}\|^2 \\ \leq & (1-\frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\gamma\rho})\|x_n-p\|^2 - \|x_{n+1}-p\|^2 + \frac{2\alpha_n(\bar{\gamma}-\gamma\rho)}{1-\alpha_n\gamma\rho}(\frac{(\alpha_n\bar{\gamma})^2}{2\alpha_n(\bar{\gamma}-\gamma\rho)}M^* \\ & + 3M_3^*\frac{(1-\alpha_n\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\| + \frac{1}{\bar{\gamma}-\gamma\rho}\langle \gamma f(p) - Dp, x_{n+1}-p\rangle), \end{aligned} \tag{3.14}$$

for all $1 \leq i \leq n$.

Now we divide the rest of the proof into two cases.

Case 1. Let $\{\|x_n - p\|\}$ be monotonically decreasing. Then $\{\|x_n - p\|\}$ is convergent. Since $\{x_n\}$ is bounded, by (3.14) and conditions (C_1)-(C_3), we get

$$\lim_{n\to\infty}\frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)} = 0. \tag{3.15}$$

Since $\{w_n\}$ is bounded, T is bounded linear, and $J_{\phi_n}^A$ and $Q_{\eta_n}^B$ are firmly nonexpansive, there exists a constant $c > 0$ such that $F_n(w_n) \leq c$ and $G_n(w_n) \leq c$. Thus, from (3.15), we have

$$0 \leq \frac{g_n^2(w_n)}{2c} \leq \frac{g_n^2(w_n)}{F_n(w_n)+G_n(w_n)} \to 0 \ \ (n\to\infty),$$

which implies that

$$\lim_{n\to\infty} g_n(w_n) = \lim_{n\to\infty}\frac{1}{2}\|J_E(I-Q_{\eta_n}^B)Tw_n\|^2 = 0. \tag{3.16}$$

Similarly, from conditions (C_1)-(C_3) and (3.14), we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}\delta_{n,i}\|v_{n,i}-u_n\| = 0 \tag{3.17}$$

and

$$\lim_{n\to\infty}\|v_{n,i}-u_n\| = 0 \ \ (i\in N^*). \tag{3.18}$$

It follows from (3.1), (3.15), (3.17), and Remark 3.3 that

$$\begin{aligned}
&\|z_n - w_n\| \\
=&\|\lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\| \\
=&\|\tau_n \frac{g_n(w_n)}{F_n(w_n)+G_n(w_n)} T^* J_E(I - Q^B_{\eta_n})Tw_n\| \\
=&\tau_n \frac{g_n(w_n)}{\|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 + \|(I - J^A_{\phi_n})w_n\|^2} \|T^* J_E(I - Q^B_{\eta_n})Tw_n\| \\
\leq&\tau_n \frac{g_n(w_n)}{\|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 + \|(I - J^A_{\phi_n})w_n\|^2} \sqrt{\|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 + \|(I - J^A_{\phi_n})w_n\|^2} \\
=&\tau_n \frac{g_n(w_n)}{\sqrt{\|T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 + \|(I - J^A_{\phi_n})w_n\|^2}} \\
=&\tau_n \frac{g_n(w_n)}{\sqrt{F_n(w_n)+G_n(w_n)}} \to 0 \ \ as \ n \to \infty,
\end{aligned} \tag{3.19}$$

$$\lim_{n\to\infty} \|y_n - u_n\| = \lim_{n\to\infty} \sum_{i=1}^{n} \delta_{n,i} \|v_{n,i} - u_n\| = 0, \tag{3.20}$$

and

$$\lim_{n\to\infty} \|w_n - x_n\| = \lim_{n\to\infty} \theta_n \|x_n - x_{n-1}\| = 0. \tag{3.21}$$

Since $J^A_{\phi_n}$ is firmly nonexpansive, from (3.1) and Lemma 2.7, we obtain

$$\begin{aligned}
\|u_n - p\|^2 =&\|J^A_{\phi_n}(w_n - \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n) - J^A_{\phi_n} p\|^2 \\
\leq&\langle J^A_{\phi_n}(w_n - \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n) - J^A_{\phi_n} p, w_n - \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n - p\rangle \\
=&\langle u_n - p, w_n - \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n - p\rangle \\
=&\frac{1}{2}\{\|u_n - p\|^2 + \|w_n - \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n - p\|^2 \\
&- \|u_n - w_n + \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2\} \\
=&\frac{1}{2}\{\|u_n - p\|^2 + \|w_n - p\|^2 + \|\lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 \\
&- 2\langle w_n - p, \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\rangle - \|u_n - w_n\|^2 - \|\lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\|^2 \\
&- 2\langle u_n - w_n, \lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\rangle\} \\
\leq&\frac{1}{2}\{\|u_n - p\|^2 + \|w_n - p\|^2 - \|u_n - w_n\|^2 + 2\|u_n - p\|\|\lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\|\},
\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \leq \|w_n - p\|^2 - \|u_n - w_n\|^2 + 2\|u_n - p\|\|\lambda_n T^* J_E(I - Q^B_{\eta_n})Tw_n\|. \tag{3.22}$$

Combining (3.1), (3.10), (3.12), and (3.22), we arrive at

$$\begin{aligned}
&\|x_{n+1}-p\|^2\\
=&\|\alpha_n(\gamma f(x_n)-Dp)+(I-\alpha_n D)(y_n-p)\|^2\\
\leq&(1-\alpha_n\bar{\gamma})^2\|u_n-p\|^2+\alpha_n^2\|\gamma f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dp\|\|y_n-p\|\\
\leq&(1-\alpha_n\bar{\gamma})^2(\|w_n-p\|^2-\|u_n-w_n\|^2+2\|u_n-p\|\|\lambda_n T^*J_E(I-Q^B_{\eta_n})Tw_n\|)\\
&+\alpha_n^2\|\gamma f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dp\|\|y_n-p\|\\
\leq&(1-\alpha_n\bar{\gamma})^2(\|x_n-p\|^2+3M_3^*\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n+1}\|-\|u_n-w_n\|^2\\
&+2\|u_n-p\|\|\lambda_n T^*J_E(I-Q^B_{\eta_n})Tw_n\|)+\alpha_n^2\|\gamma f(x_n)-Dp\|^2\\
&+2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dp\|\|y_n-p\|. \qquad (3.23)
\end{aligned}$$

Then from (3.19), (3.23), condition (C_2) and Remark 3.3, we have

$$\begin{aligned}
&(1-\alpha_n\bar{\gamma})^2\|u_n-w_n\|^2\\
\leq&(1-\alpha_n\bar{\gamma})^2\|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\alpha_n\bar{\gamma})^2 3M_3^*\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n+1}\|\\
&+2(1-\alpha_n\bar{\gamma})^2\|u_n-p\|\|\lambda_n T^*J_E(I-Q^B_{\eta_n})Tw_n\|+\alpha_n^2\|\gamma f(x_n)-Dp\|^2\\
&+2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dp\|\|y_n-p\|\\
=&(1-\alpha_n\bar{\gamma})^2\|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\alpha_n\bar{\gamma})^2 3M_3^*\alpha_n\frac{\theta_n}{\alpha_n}\|x_n-x_{n+1}\|\\
&+2(1-\alpha_n\bar{\gamma})^2\|u_n-p\|\|z_n-w_n\|+\alpha_n^2\|\gamma f(x_n)-Dp\|^2\\
&+2\alpha_n(1-\alpha_n\bar{\gamma})\|\gamma f(x_n)-Dp\|\|y_n-p\|\to 0,\ as\ n\to\infty.
\end{aligned}$$

Therefore, $\lim_{n\to\infty}\|u_n-w_n\|=0$. Also, from (3.20) and (3.21), we have

$$\|y_n-x_n\|\to 0,\ \ \|u_n-x_n\|\to 0,\ \ \|y_n-w_n\|\to 0,\ \ as\ n\to\infty. \qquad (3.24)$$

By (3.24) and condition (C_2) , we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\|=&\|\alpha_n(\gamma f(x_n)-Dx_n)+(I-\alpha_n D)(y_n-x_n)\|\\
\leq&\alpha_n\|\gamma f(x_n)-Dx_n\|+(1-\alpha_n\bar{\gamma})\|y_n-x_n\|\to 0,\ \ as\ n\to\infty. \qquad (3.25)
\end{aligned}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j}\rightharpoonup x^*$ and

$$\limsup_{n\to\infty}\langle\gamma f(p)-Dp,x_n-p\rangle=\lim_{j\to\infty}\langle\gamma f(p)-Dp,x_{n_j}-p\rangle=\langle\gamma f(p)-Dp,x^*-p\rangle. \qquad (3.26)$$

It easily follows from (3.24) that $w_{n_j}\rightharpoonup x^*$, $u_{n_j}\rightharpoonup x^*$, $y_{n_j}\rightharpoonup x^*$. Since $I-S_i$ $(\forall\ i\in N^*)$ is demiclosed at zero and $u_{n_j}\rightharpoonup x^*$, from (3.18), we obtain $x^*\in\bigcap_{i=1}^{\infty}Fix(S_i)$. Since T is bounded and linear, we get $Tw_{n_j}\rightharpoonup Tx^*$. From (3.16) we have $Q^B_{\eta_{n_j}}Tw_{n_j}\rightharpoonup Tx^*$. Since $Q^B_{\eta_{n_j}}$ is the metric resolvent of B, then we have that $\frac{J_E(I-Q^B_{\eta_{n_j}})Tw_{n_j}}{\eta_{n_j}}\in BQ^B_{\eta_{n_j}}Tw_{n_j}$. By the monotonicity of B, it follows that

$$\langle v-\frac{J_E(I-Q^B_{\eta_{n_j}})Tw_{n_j}}{\eta_{n_j}},u-Q^B_{\eta_{n_j}}Tw_{n_j}\rangle\geq 0$$

$$\Rightarrow \langle v, u - Q^B_{\eta_{n_j}} Tw_{n_j}\rangle \geq -\langle \frac{J_E(I - Q^B_{\eta_{n_j}})Tw_{n_j}}{\eta_{n_j}}, Q^B_{\eta_{n_j}} Tw_{n_j} - u\rangle$$
$$\geq -\frac{\|J_E(I - Q^B_{\eta_{n_j}})Tw_{n_j}\|}{\eta_{n_j}} \|Q^B_{\eta_{n_j}} Tw_{n_j} - u\|,$$

for all $(u,v) \in G(B)$. From (3.16), $\liminf_{j\to\infty} \eta_{n_j} > 0$, and $Q^B_{\eta_{n_j}} Tw_{n_j} \rightharpoonup Tx^*$, one has $\langle v - 0, u - Tx^*\rangle \geq 0$. Since B is maximally monotone, we have that $Tx^* \in B^{-1}(0)$, which concludes that $x^* \in T^{-1}(B^{-1}(0))$. Observe that

$$\begin{aligned}\|u_{n_j} - J^A_{\phi_{n_j}} u_{n_j}\| \leq & \|u_{n_j} - J^A_{\phi_{n_j}} w_{n_j}\| + \|J^A_{\phi_{n_j}} w_{n_j} - J^A_{\phi_{n_j}} u_{n_j}\| \\ \leq & \|J^A_{\phi_{n_j}} z_{n_j} - J^A_{\phi_{n_j}} w_{n_j}\| + \|w_{n_j} - u_{n_j}\| \\ \leq & \|z_{n_j} - w_{n_j}\| + \|w_{n_j} - u_{n_j}\|. \end{aligned} \tag{3.27}$$

By (3.19) and (3.27), we see that $\|u_{n_j} - J^A_{\phi_{n_j}} u_{n_j}\| \to 0$ as $j \to \infty$. Again, we have

$$\begin{aligned}\|u_{n_j} - J^A_{\phi_{n_j}} x^*\| \leq & \|u_{n_j} - J^A_{\phi_{n_j}} u_{n_j}\| + \|J^A_{\phi_{n_j}} u_{n_j} - J^A_{\phi_{n_j}} x^*\| \\ \leq & \|u_{n_j} - J^A_{\phi_{n_j}} u_{n_j}\| + \|u_{n_j} - x^*\|. \end{aligned}$$

Hence, we have $\limsup_{j\to\infty} \|u_{n_j} - J^A_{\phi_{n_j}} x^*\| \leq \limsup_{j\to\infty} \|u_{n_j} - x^*\|$. It follows from the Opial property of the Hilbert space $\mathscr{H}$ that $J^A_{\phi_{n_j}} x^* = x^*$. Therefore, $x^* \in A^{-1}(0)$, which implies that $x^* \in \Omega$. On account of (3.2), (3.25), and (3.26), we obtain that

$$\limsup_{n\to\infty} \frac{\langle \gamma f(p) - Dp, x_{n+1} - p\rangle}{\bar{\gamma} - \gamma\rho} \leq \limsup_{n\to\infty} \frac{\langle \gamma f(p) - Dp, x_n - p\rangle}{\bar{\gamma} - \gamma\rho} = \frac{\langle \gamma f(p) - Dp, x^* - p\rangle}{\bar{\gamma} - \gamma\rho} \leq 0. \tag{3.28}$$

Therefore, from (3.13), (3.28), condition (C_2), Remark 3.3, and Lemma 2.10, one sees that $\{x_n\}$ converges strongly to p.

Case 2. Let $\{\|x_n - p\|\}$ be not monotonically decreasing. Put $\Gamma_n = \|x_n - p\|^2$ and suppose that there exists a subsequence $\{\Gamma_{k_i}\} \subset \{\Gamma_n\}$ such that $\Gamma_{k_i} < \Gamma_{k_i+1}$ for all $i \in N^*$. Let $\psi : N^* \to N^*$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by $\psi(n) = max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}$. Then we have from Lemma 2.12 that $\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$ and $\Gamma_n \leq \Gamma_{\psi(n)+1}$, $\{\psi(n)\}$ is a nondecreasing sequence that $\psi(n) \to \infty$ as $n \to \infty$. From (3.14), conditions (C_2) and (C_3), and $\Gamma_{\psi(n)} \leq \Gamma_{\psi(n)+1}$, we obtain

$$\lim_{n\to\infty} \frac{g^2_{\psi(n)}(w_{\psi(n)})}{F_{\psi(n)}(w_{\psi(n)}) + G_{\psi(n)}(w_{\psi(n)})} = 0.$$

Furthermore, similar to the proof of (3.16), (3.17), and (3.18) we see that

$$\lim_{n\to\infty} g_{\psi(n)}(w_{\psi(n)}) = \lim_{n\to\infty} \frac{1}{2}\|J_E(I - Q^B_{\eta_{\psi(n)}})Tw_{\psi(n)}\|^2 = 0,$$

$$\lim_{n\to\infty} \sum_{i=1}^{\psi(n)} \delta_{\psi(n),i}\|v_{\psi(n),i} - u_{\psi(n)}\| = 0,$$

and

$$\lim_{n\to\infty}\|v_{\psi(n),i}-u_{\psi(n)}\|=0\ \ (i\in N^*).$$

Due to $\Gamma_{\psi(n)}\leq\Gamma_{\psi(n)+1}$, we also have $\lim_{n\to\infty}\|x_{\psi(n)+1}-x_{\psi(n)}\|=0$. According to $\Gamma_{\psi(n)}\leq\Gamma_{\psi(n)+1}$, similar to the proof of Case 1, we obtain

$$\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(p)-Dp,x^*-p\rangle\leq 0. \tag{3.29}$$

From (3.13), we have

$$\begin{aligned}\Gamma_{\psi(n)+1}\leq&(1-\frac{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}{1-\alpha_{\psi(n)}\gamma\rho})\Gamma_{\psi(n)}+\frac{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}{1-\alpha_{\psi(n)}\gamma\rho}(\frac{(\alpha_{\psi(n)}\bar{\gamma})^2}{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}M^*\\&+3M_3^*\frac{(1-\alpha_{\psi(n)}\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}}\|x_{\psi(n)}-x_{\psi(n)-1}\|+\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(p)-Dp,x_{\psi(n)+1}-p\rangle).\end{aligned} \tag{3.30}$$

Set

$$\begin{aligned}\sigma_{\psi(n)}=&\frac{(\alpha_{\psi(n)}\bar{\gamma})^2}{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}M^*+3M_3^*\frac{(1-\alpha_{\psi(n)}\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}}\|x_{\psi(n)}-x_{\psi(n)-1}\|\\&+\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(p)-Dp,x_{\psi(n)+1}-p\rangle.\end{aligned} \tag{3.31}$$

By Condition (C_2), Remark 3.3, (3.29) and (3.31), we conclude that

$$\begin{aligned}\limsup_{n\to\infty}\sigma_{\psi(n)}=&\limsup_{n\to\infty}\frac{(\alpha_{\psi(n)}\bar{\gamma})^2}{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}M^*+\limsup_{n\to\infty}3M_3^*\frac{(1-\alpha_{\psi(n)}\bar{\gamma})^2}{2(\bar{\gamma}-\gamma\rho)}\frac{\theta_{\psi(n)}}{\alpha_{\psi(n)}}\|x_{\psi(n)}-x_{\psi(n)-1}\|\\&+\limsup_{n\to\infty}\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(p)-Dp,x_{\psi(n)+1}-p\rangle\\\leq&\frac{1}{\bar{\gamma}-\gamma\rho}\langle\gamma f(p)-Dp,x^*-p\rangle\\\leq&0.\end{aligned}$$

Thus, from (3.30) and (3.31), we have

$$\frac{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}{1-\alpha_{\psi(n)}\gamma\rho}\Gamma_{\psi(n)}\leq\Gamma_{\psi(n)}-\Gamma_{\psi(n)+1}+\frac{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}{1-\alpha_{\psi(n)}\gamma\rho}\sigma_{\psi(n)}\leq\frac{2\alpha_{\psi(n)}(\bar{\gamma}-\gamma\rho)}{1-\alpha_{\psi(n)}\gamma\rho}\sigma_{\psi(n)},$$

which implies that $\Gamma_{\psi(n)}\leq\sigma_{\psi(n)}$. Since $\limsup_{n\to\infty}\sigma_{\psi(n)}\leq 0$, we obtain $\lim_{n\to\infty}\Gamma_{\psi(n)}=0$. From (3.29), (3.30), Remark 3.3, and Condition (C_2), we have $\lim_{n\to\infty}\Gamma_{\psi(n)+1}=0$, and then $\lim_{n\to\infty}\Gamma_n=0$ due to $\Gamma_n\leq\Gamma_{\psi(n)+1}$, *i.e.* the sequence $\{x_n\}$ converges strongly to p. This completes the proof. □

Remark 3.4. Theorem 3.2 extends and develops [25, Theorem 3.2] from the following acpects:

(a) Inertia techniques are used in our proposed algorithm;
(b) A Hilbert space is extended to a Banach space;
(c) A finite family of multivalued demicontractive mappings is extended to an infinite family of multivalued demicontractive mappings.

4. NUMERICAL EXAMPLE

In this section, we present a numerical example to demonstrate the efficiency of our algorithm.

Example 4.1. Let $\mathscr{H} = R$, $E = R^3$. For $\forall\, i \in N^*$, we define the multivalued mapping $S_i : R \to CB(R)$ as follows:

$$S_i x = \begin{cases} \left[-\frac{2i}{i+1}x\,,\ -\frac{3i}{i+1}x\right], & if\ \ x \leq 0; \\ \left[-\frac{3i}{i+1}x\,,\ -\frac{2i}{i+1}x\right], & if\ \ x > 0. \end{cases}$$

We also define a bounded linear operator $T : R \to R^3$ by $Tx := (2x, -5x, 3x)^T$. For each $i = 0, 1, 2, \cdots,\ n \geq 2$, let

$$\delta_{n,i} = \begin{cases} 1 - \dfrac{n}{n+1}(\dfrac{1-k}{2})\sum_{j=1}^{n} \dfrac{1}{2^j}, & if\ \ i = 0, \\ \dfrac{n}{n+1}(\dfrac{1-k}{2})\dfrac{1}{2^i}, & if\ \ 1 \leq i \leq n, \\ 0, & if\ \ i > n. \end{cases}$$

Let $A : R \to 2^R$ be defined by

$$A(x) = \begin{cases} \{u \in R : z^2 + xz - 2x^2 \geq (z-x)u,\ \forall\, z \in [-9,3]\}, & if\ \ x \in [-9,3], \\ \emptyset, & otherwise. \end{cases}$$

Define a maximal monotone mapping $B : R^3 \to 2^{R^3}$ by $B := \partial g$, where $g : R^3 \to R$ is a function defined by $g(x) = \frac{1}{2}\|Px\|^2$, where

$$P = \begin{pmatrix} -6 & 1 & 5 \\ 2 & -7 & 8 \end{pmatrix}.$$

Define a strongly positive bounded linear operator D by $Dx = x$ with a constant $\bar{\gamma} = 1$ and a contraction f by $f(x) = \frac{1}{8}x$ with $\rho = \frac{1}{8}$. Furthermore, take $\gamma = 2$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$.Take $\eta = 3$, $\xi_n = \frac{1}{(n+1)^2}$, $\phi_n = \eta_n = \frac{8n}{n+1}$, $\tau_n = \frac{3n}{n+1}$, and $\alpha_n = \frac{1}{n+1}$, $\forall\, n \geq 2$. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to 0.

Solution: From Example 2.3, for all $i \in N^*$, we know taht S_i is a multivalued k_i-demicontractive mapping with $k_i = \frac{8i^2-2i-1}{9i^2+6i+1} \in (0,1)$. Thus $k = sup_{i\in N^*} k_i = sup_{i\in N^*} \frac{8i^2-2i-1}{9i^2+6i+1} = \frac{8}{9} < 1$, and $I - S_i$ is demiclosed at zero. By [20, Theorem 4.2], A is maximal monotone. The resolvents of A *and* B can be written by $J^A_{\phi_n} x = \frac{x}{3\phi_n+1}$ and $Q^B_{\eta_n} z = (I + \eta_n P^T P)^{-1} z$, respectively, for all $x \in R$ and $z \in R^3$. From the definition of T, we can obtain $T^* = T^T = (2,-5,3)$. Then, scheme 3.1 reduces to the following form:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = w_n - \frac{3n}{n+1}\frac{g_n(w_n)}{F_n(w_n)+G_n(w_n)} T^*[I - (I+\eta_n P^T P)^{-1}]Tw_n, \\ u_n = \frac{1}{3\phi_n+1} z_n, \\ y_n = [1 - \frac{n}{18(n+1)}(1 - \frac{1}{2^n})]u_n + \frac{n}{18(n+1)}\sum_{i=1}^{n} \frac{1}{2^i} v_{n,i}, \\ x_{n+1} = \frac{1}{4(n+1)} x_n + \frac{n}{n+1} y_n, \end{cases} \tag{4.1}$$

for all $n \geq 2$, where

$$g_n(w_n) = \frac{1}{2}\|(I - (I+\eta_n P^T P)^{-1})Tw_n\|^2,$$

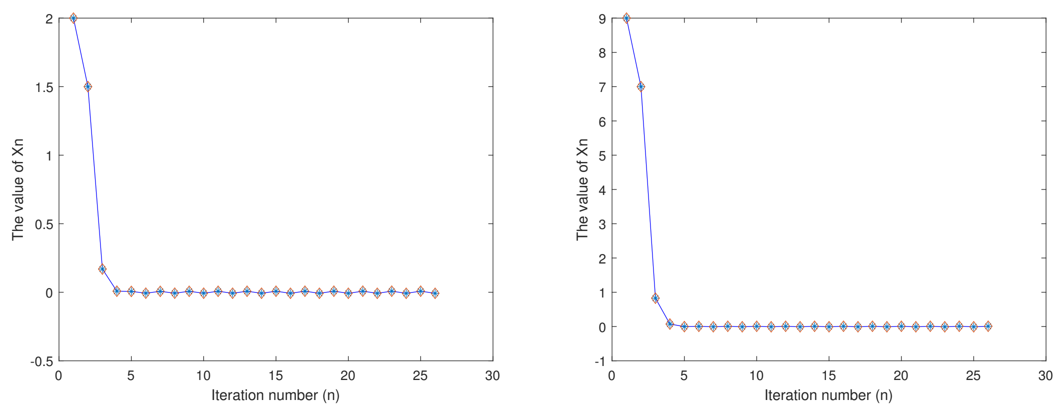
$$G_n(w_n) = \|T^*(I-(I+\eta_n P^T P)^{-1})Tw_n\|^2,$$
$$F_n(w_n) = \|w_n - \frac{w_n}{3\phi_n+1}\|^2,$$

and

$$v_{n,i} = \begin{cases} -\frac{3i}{i+1}u_n, & if \;\; u_n \leq 0, \\ -\frac{2i}{i+1}u_n, & if \;\; u_n > 0. \end{cases}$$

Hence, from Theorem 3.2, the sequence $\{x_n\}$ generated by (4.1) converges strongly to 0.

We choose different initials to demonstrates the efficiency of our algorithm.



(a) Initial values $x_1 = 2, x_2 = 1.5$ (b) Initial values $x_1 = 9, x_2 = 7$

FIGURE 1. Numerical results for Example 4.1

TABLE 1. Computational results for Example 4.1

n	x_n	x_n
$n=1$	2	9
$n=2$	1.5	7
$n=3$	0.007867	0.072633
$n=4$	0.006303	-0.002082
$n=5$	-0.006643	-0.002082
$n=6$	0.006752	0.006173
$n=7$	-0.006650	-0.006657
$n=8$	0.007012	0.006907
$n=9$	-0.006828	0.007094
$n=10$	0.007237	-0.006964
$n=11$	-0.007070	0.007349
$n=12$	0.007442	-0.007158
$n=13$	-0.007232	0.007520
$n=14$	0.007586	-0.007294
$n=15$	0.007520	0.007642
$n=16$	0.007691	-0.007393
$n=17$	-0.007434	0.007734
$n=18$	0.007772	-0.007470
$n=19$	-0.007502	0.007806
$n=20$	0.007836	-0.007530
$n=21$	-0.007555	0.007863

Thus, we can obtain that the sequence $\{x_n\}$ which is generated by (4.1) converges to $0 \in \Omega = \{0\}$. And we can see both Figure 1 and Table 1 that the $\{x_n\}$ converges to 0. Therefore, the iterative algorithm of Theorem 3.2 is well defined and efficient.

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