J. Nonlinear Funct. Anal. 2023 (2023) 37 https://doi.org/10.23952/jnfa.2023.37

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

ALGORITHM FOR SOLVING NONLINEAR MONOTONE OPERATOR EQUATIONS WITH APPLICATIONS

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Abstract. Systems of nonlinear equations are frequently encountered in science and engineering, and finding practical approaches to solve them is important. In this paper, an algorithm for solving systems of nonlinear equations is proposed. The proposed method is obtained by combing a modified spectral gradient method and the projection technique. Under standard assumptions, the global convergence of the proposed method is established. The reported numerical results show that the method is efficient.

Keywords. Conjugate gradient method; Derivative-free technique; Nonlinear monotone operator equations.

1. INTRODUCTION

Consider the constrained nonlinear equation:

$$F(x) = 0, x \in \Omega, \tag{1.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone, and Ω is a nonempty, convex, and closed subset of $\mathbb{R}^n$. Here, F is monotone if $(F(x) - F(y))^T (x - y) \geq 0$ for all $x, y \in \mathbb{R}^n$. When $\Omega = \mathbb{R}^n$, it is known as the unconstrained nonlinear equation and was widely studied by using a variety of techniques; see, e.g., [19, 21, 23, 24, 26, 30]. In compressive sensing, monotone equations can

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Received June 15, 2023; Accepted November 14, 2023.

be formulated as an l_1-norm regularized optimization problem; see, e.g., [8, 29]. Contrarily, convex constrained monotone equations have been employed in various scientific fields, such as the chemical equilibrium system and the economic equilibrium problem [22, 31]. Recently, numerous researchers proposed a variety of methods to solving problem (1.1); see, e.g., [1, 3, 4, 5, 11, 12, 13, 14, 16]. These approaches have highly effective numerical performance and achieve global convergence under certain suitable conditions. In particular, Awwal et al. [6] modified the method from [28] by proposing a three-term conjugate gradient technique for restricted monotone nonlinear equations. Under some assumptions, the convergence of the method was established, and the method, according to numerical results, was superior to those in [25]. The search direction achieves the sufficient descent requirement. Specifically,

$$F_k^T d_k \leq -c\|F_k\|^2, c > 0, \tag{1.2}$$

where d_k denotes the direction of the search and $F_k := F(x_k)$. The numerical outcomes demonstrated its effectiveness. It is known that not all conjugate gradient (CG) algorithms provide the sufficient descent property (1.2) for global convergence. Many researchers overcame this problem and demonstrated the efficiency of their algorithms in obtaining descent and convergence as in [10, 18, 27]. In this paper, we propose another efficient algorithm that has a descent direction and global convergence properties. The search direction comprises both spectral and conjugate gradient parameters, which is likely the reason for its efficiency compared with other algorithms.

The remainder of the paper is structured as follows. in Section 2, the algorithms and details are presented. in Section 3, the convergence analysis of the proposed algorithms is presented. In Section 4, a computational experiment is provided to demonstrate the effectiveness of our algorithms in real-world situations. In Section 5, the last section, concluding observations are presented.

2. Direction and Algorithm

This section describes our algorithm step by step. The projection, denoted by $P_\Omega(.)$, is a mapping from $\mathbb{R}^n$ to Ω defined as: $P_\Omega(x) = \arg\min\{\|x-y\| : y \in \Omega\}$ for all $x \in \mathbb{R}$, and it has the nonexpansive property, that is, $\|P_\Omega(x) - P_\Omega(y)\| \leq \|x-y\|$ for all $x, y \in \mathbb{R}$. As a result, we have $\|P_\Omega(x) - y\| \leq \|x-y\|$ for all $x, y \in \Omega$.

We now propose a search direction based on a new spectral CG parameter to solve for monotonic nonlinear equation (1.1) as well as a signal recovery problem arising from compressive sensing. The new formula is defined as:

$$d_k = \begin{cases} -F_k, & \text{if } k = 0, \\ -\vartheta_k F_k + \beta_k^{new} s_{k-1}, & \text{if } \quad k \geq 1, \end{cases} \tag{2.1}$$

where

$$\vartheta_k = 1 + \frac{F_k^T s_{k-1}}{\|F_{k-1}\|^2}; \; \beta_k^{new} = \frac{\varphi_k\|F_k\|^2 - |F_k^T F_{k-1}|}{|F_k^T F_{k-1}| + \varphi_k\|F_{k-1}\|^2}$$

$$\varphi_k = \frac{\|d_{k-1} + F_k\|}{\|d_{k-1}\|}; \; s_{k-1} = z_{k-1} - x_{k-1}; \; z_{k-1} = x_{k-1} + \alpha_{k-1} d_{k-1}.$$

In what follows, we present the steps of our algorithm for solving nonlinear monotone equations.

Algorithm 2.1. AR-New Algorithm
Step 0 *Give* $x_0 \in \mathbb{R}^n$, $r, \sigma \in (0,1)$, $d_0 = -F_0$, $\varepsilon > 0$ *and set* $k = 0$.
Step 1 *Compute* $F_k = F(x_k)$ *and check* $\|F_k\| \leq \varepsilon$ *stop, else go to* **Step 2**.
Step 2 *Calculate* $z_k = x_k + \alpha_k d_k$ *with the step-size* α_k, *where* $\alpha_k = r^w$ *with* w *being the smallest positive integer such that:*

$$-F(x_k + r^w d_k)^T d_k \geq \sigma r^w \|F(x_k + r^w d_k)\| \|d_k\|^2 \tag{2.2}$$

is satisfied.
Step 3 *If* $z_k \in \Omega$ *and* $\|F(z_k)\| \leq \varepsilon$, *then stop. Else, calculate the next iteration by:*

$$x_{k+1} = P_\Omega \left(x_k - \frac{F(z_k)^T (x_k - z_k)}{\|F(z_k)\|^2} F(z_k) \right).$$

Step 4 *Update the search direction by using Eq.* (2.1).
Step 5 *Set* $k = k+1$ *and go back to* **Step 2**.

3. Convergence Analysis

In this section, we establish global convergence of the new algorithm by making use of the following assumptions:

Assumption 3.1. Suppose that F fulfills the following assumptions:
i. The solution set of $F(x) = 0$ is non-empty.
ii. The function F is Lipschitz continuous, i.e., there exists a positive constant L such that:

$$\|F(x) - F(y)\| \leq L\|x - y\|, \forall x, y \in R^n. \tag{3.1}$$

iii. F is monotone.

Lemma 3.2. *The search direction* d_k *given by Eq.* (2.1) *satisfies* $F_k^T d_k \leq -\|F_k\|^2$.

Proof. If $k = 0$, then $F_0^T d_0 = -F_0^T d_0 = -\|F_0\|^2$. For $k > 0$, we obtain by multiplying both sides of Eq. (2.1) by F_k^T that

$$\begin{aligned}
F_k^T d_k &= -\vartheta_k \|F_k\|^2 + \frac{\varphi_k \|F_k\|^2 - |F_k^T F_{k-1}|}{|F_k^T F_{k-1}| + \varphi_k \|F_{k-1}\|^2} F_k^T s_{k-1} \\
&\leq -\vartheta_k \|F_k\|^2 + \frac{\varphi_k \|F_k\|^2}{\varphi_k \|F_{k-1}\|^2} F_k^T s_{k-1} \\
&\leq -\left(1 + \frac{F_k^T s_{k-1}}{\|F_{k-1}\|^2}\right) \|F_k\|^2 + \frac{\|F_k\|^2}{\|F_{k-1}\|^2} F_k^T s_{k-1} \\
&\leq -\|F_k\|^2 - \frac{\|F_k\|^2}{\|F_{k-1}\|^2} F_k^T s_{k-1} + \frac{\|F_k\|^2}{\|F_{k-1}\|^2} F_k^T s_{k-1} \\
&\leq -\|F_k\|^2.
\end{aligned}$$

□

Lemma 3.3. [2] *Let Assumption* (3.1) *hold and* $\bar{x}$ *be a solution of problem* $F(x)$. *Let* $\{x_k\}$ *be generaed by the AR-New Algorithm. Then* $\{x_k\}$ *and* $\{F_k\}$ *are bounded, that is, there exist* $a, b > 0$ *such that* $\|x_k\| \leq a$ *and* $\|F_k\| \leq b$. *Moreover,* $\{\|x_k - \bar{x}\|\}$ *converges and* $\lim_{k\to\infty} \alpha_k \|d_k\| = 0$.

Lemma 3.4. *Let Assumptions (3.1) hold. Then*

(1) There exists a step-size α_k satisfying the line search.
(2) The step-size satisfies the inequality

$$\alpha_k > \min\left\{1, \frac{r\|F_k\|^2}{(L+\sigma)b\|d_k\|}\right\}. \tag{3.2}$$

Proof. Since the proof of (1) is similar to the one given in [2], we here only prove (2). If $\alpha_k \neq 1$, then $\frac{\alpha_k}{r}$ does not satisfy (2.2), that is,

$$-F\left(x_k + \frac{\alpha_k}{r}d_k\right)^T d_k \geq \sigma r^w \left\|F(x_k + \frac{\alpha_k}{r}d_k)\right\| \|d_k\|^2.$$

From Lemma 3.2, we conclude that

$$\begin{aligned}\|F_k\|^2 &= \left(F\left(x_k + \frac{\alpha_k}{r}d_k\right) - F(x_k)\right)^T d_k - F\left(x_k + \frac{\alpha_k}{r}d_k\right)^T d_k \\ &\leq L\frac{\alpha_k}{r}\left\|F\left(x_k + \frac{\alpha_k}{r}d_k\right)\right\| \|d_k\|^2 + \sigma\frac{\alpha_k}{r}\left\|F\left(x_k + \frac{\alpha_k}{r}d_k\right)\right\| \|d_k\|^2 \\ &< \frac{L+\sigma}{r}\alpha_k b\|d_k\|^2.\end{aligned}$$

It follows that

$$\alpha_k > \frac{r\|F_k\|^2}{(L+\sigma)b\|d_k\|^2}.$$

which concludes the proof. □

Theorem 3.5. *Let Assumption (3.1) hold and $\{x_k\}$ be generated by Algorithm 2.1 (AR-New Algorithm). Then $\liminf_{k\to\infty}\|F_k\| = 0$. Furthermore, $\{x_k\}$ converges to a solution of $F(x) = 0$.*

Proof. Suppose by contradiction that $\liminf_{k\to\infty}\|F_k\| \neq 0$. Then there exist a positive constant υ such that, for all $k \geq 0$, $\|F(x_k)\| \geq \upsilon$.

Next, we show that the direction d_k is bounded. For $k = 0$, $\|d_0\| = \|F_0\| \leq b$. For $k \geq 1$,

$$|\vartheta_k| = \left|1 + \frac{F_k^T s_{k-1}}{\|F_{k-1}\|^2}\right| \leq 1 + \frac{\|F_k\|\|s_{k-1}\|}{\|F_{k-1}\|^2} \tag{3.3}$$

and

$$|\varphi_k| = \frac{\|d_{k-1} + F_k\|}{\|d_{k-1}\|} \leq \frac{\|d_{k-1}\| + \|F_k\|}{\|d_{k-1}\|} = 1 + \frac{\|F_k\|}{\|d_{k-1}\|}. \tag{3.4}$$

Also,

$$|\varphi_k| = \frac{\|d_{k-1} + F_k\|}{\|d_{k-1}\|} \geq \frac{\|F_k\| - \|d_{k-1}\|}{\|d_{k-1}\|},$$

which implies,

$$\frac{1}{|\varphi_k|} \leq \frac{\|d_{k-1}\|}{\|F_k\| - \|d_{k-1}\|}. \tag{3.5}$$

Now,

$$
\begin{aligned}
|\beta_k^{new}| &= \left| \frac{\varphi_k\|F_k\|^2 - |F_k^T F_{k-1}|}{|F_k^T F_{k-1}| + \varphi_k\|F_{k-1}\|^2} \right| \\
&\leq \frac{|\varphi_k|\|F_k\|^2 + \|F_k\|\|F_{k-1}\|}{|\varphi_k|\|F_{k-1}\|^2} = \frac{\|F_k\|^2}{\|F_{k-1}\|^2} + \frac{\|F_k\|}{|\varphi_k|\|F_{k-1}\|} \\
&= \frac{\|F_k\|^2}{\|F_{k-1}\|^2} + \frac{\|F_k\|}{\left(\frac{\|d_{k-1}\|}{\|F_k\| - \|d_{k-1}\|}\right)\|F_{k-1}\|} \\
&= \frac{\|F_k\|^2}{\|F_{k-1}\|^2} + \frac{\|F_k\|\left(\|F_k\| - \|d_{k-1}\|\right)}{\|d_{k-1}\|\|F_{k-1}\|} \\
&\leq \frac{\|F_k\|^2}{\|F_{k-1}\|^2} + \frac{\|F_k\|^2}{\|d_{k-1}\|\|F_{k-1}\|} \\
&= 2\frac{\|F_k\|^2}{\|F_{k-1}\|^2}.
\end{aligned} \tag{3.6}
$$

From $\lim_{k\to\infty}\alpha_k\|d_k\| = 0$ and Cauchy-Schwarz inequality, we have for all k

$$
\|d_k\| \geq \|F_k\|, \implies \frac{1}{\|d_k\|} \leq \frac{1}{\|F_k\|} \leq \frac{1}{\upsilon}.
$$

From (2.1), (3.3), (3.4), (3.5), and (3.6), we have

$$
\begin{aligned}
\|d_k\| &= \|-\vartheta_k F_k + \beta_k^{new} s_{k-1}\| \\
&\leq |\vartheta_k|\|F_k\| + |\beta_k^{new}|\|s_{k-1}\| \\
&\leq \left(1 + \frac{\|F_k\|\|s_{k-1}\|}{\|F_{k-1}\|^2}\right)\|F_k\| + 2\frac{\|F_k\|^2}{\|F_{k-1}\|^2}\|s_{k-1}\| \\
&= \|F_k\| + \frac{\|F_k\|^2}{\|F_{k-1}\|^2}\|s_{k-1}\| + 2\frac{\|F_k\|^2}{\|F_{k-1}\|^2}\|s_{k-1}\| \\
&= \|F_k\| + 3\frac{\|F_k\|^2}{\|F_{k-1}\|^2}\|s_{k-1}\| \\
&= b + 3\frac{b^2}{\upsilon^2}\alpha_{k-1}\|d_{k-1}\|.
\end{aligned}
$$

From $\lim_{k\to\infty}\alpha_k\|d_k\| = 0$, we have, for all $\varepsilon > 0$, that there exists $k_0$such that $\alpha_{k-1}\|d_{k-1}\| < \varepsilon$ for all $k > k_0$. Letting $\varepsilon = \upsilon^2$ and $M = \max\{\|d_0\|, \|d_1\|, \ldots, \|d_{k_0}\|, \|M_1\|\}$, where $M_1 = b(1+3b)$, we have $\|d_k\| \leq M$. Multiplying both sides of Eq. (3.2) by $\|d_k\|$, we arrive at

$$
\alpha_k\|d_k\| > \min\left\{\|d_k\|, \frac{r\upsilon}{(L+\sigma)bM}\right\} \geq \left\{\upsilon, \frac{r\upsilon}{(L+\sigma)bM}\right\} > 0.
$$

This contradicts $\lim_{k\to\infty}\alpha_k\|d_k\| = 0$. Thus $\liminf_{k\to\infty}\|F_k\| = 0$. □

TABLE 1. Detailed information for the initial points (I.P.)

I.P.	Value
$x1$	$(0.1, 0.1, 0.1, \dots, 0.1)^T$
$x2$	$(0.2, 0.2, 0.2, \dots, 0.2)^T$
$x3$	$(1/2^n, 1/2^n, 1/2^n, \dots, 1/2^n)^T$
$x4$	$(5, 5, 5, \dots, 5)^T$
$x5$	$(0.5, 0.5, 0.5, \dots, 0.5)^T$
$x6$	$(1/n, 1/n, 1/n, \dots, 1/n)^T$

4. NUMERICAL EXPERIMENTS

This section compares the results of the proposed algorithm (AR-New) with two algorithms (P-HS) and (P-CG) regarding the number of iterations (I.Niter), the number of function evaluations (F.NFun), the simulation time (C.Time), and the norm of the function (N.Norm) with the following algorithms

- For the (AR-New) algorithm, we set: $\sigma = 0.0001$ and $r = 0.8$.
- For the (P-HS) [20] algorithm, we set: $\sigma = 0.0001$, $r = 0.55$, and $\zeta = 0.01$.
- For the norm descent derivative-free algorithm (P-CG) [19], we assigned: $\sigma = 0.0001$, $r = 0.55$, and $\zeta = 0.1$.

All the algorithms are coded in Matlab R2018b on a personal computer with the specification of 4 GB RAM CPU and 2.50 GHz. The simulation stops if I.Niter exceeds 1000 or $\|F(x_k)\| \leq 10^{-6}$. The mapping F is taking as $F(x) = (f_1(x), f_2(x), f_3(x), \dots, f_n(x))^T$, and $x = (x_1, x_2, x_3, \dots, x_n)^T$ for $i = 1, 2, 3, \dots, n$. We solved 8 different test problems by using 6 different initial points (I.P.) (See, Table 1). Indeed, the following test problems are used in our comparison:

Problem 1 [17] $f(x_1) = e^{x_1} - 1$, $f(x_i) = e^{x_i} + x_i - 1$, $i = 2, \dots, n$, where $\Omega = \mathbb{R}^n_+$.

Problem 2 [7] $f(x_i) = 2x_i - \sin|x_i|$, $i = 1, \dots, n$, where $\Omega = \mathbb{R}^n_+$.

Problem 3 [32] $f(x_i) = e^{x_i} - 1$ $i = 1, \dots, n$, where $\Omega = \mathbb{R}^n_+$.

Problem 4 [7] $f(x_1) = x_1 - e^{\cos(h(x_1+x_2))}$, $f(x_i) = x_i - e^{\cos(h(x_{i-1}+x_i+x_{i+1}))}$, $i = 2, \dots, n-1$, and $f(x_n) = x_n - e^{\cos(h(x_{n-1}+x_n))}$ for $i = 2, 3, \dots, n-1$, where $h = \frac{1}{n+1}$ and $\Omega = \mathbb{R}^n_+$.

Problem 5 [7] $f(x_i) = x_i - sin|x_i - 1|$, $i = 1, \dots, n$, where $\Omega = \mathbb{R}^n_+$.

Problem 6 $f(x_1) = 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2)$,

$$f(x_i) = 3x_i^3 + 2x_{i+1} - 5 + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1}) + 4x_i - x_{i-1}e^{(x_{i-1}-x_i)} - 3, \; i = 2, \dots, n-1,$$

$$f(x_n) = -x_{n-1}e^{(x_{n-1}+x_n)} + 4x_n - 3$$

for $i = 2, 3, \dots, n-1$, where $\Omega = \mathbb{R}^n_+$.

Problem 7 $f(x_i) = 8^{0.5}x_i - 1$, $i = 1, \dots, n$, where $\Omega = \mathbb{R}^n_+$.

Problem 8 [17] $f(x_i) = \log(x_i + 1) - \frac{x_i}{n}$, $i = 1, \dots, n$, where $\Omega = \mathbb{R}^n_+$.

In the tables (Tables 2 to 9), we show the performance of the proposed new algorithm AR-New

TABLE 2. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 1

		AR-New algorithm				P-HS algorithm				P-CG algorithm			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	9	31	0.02833	7.79E-07	44	133	0.05517	7.13E-07	22	46	0.02293	7.58E-07
	$x2$	10	35	0.00544	3.94E-07	45	136	0.01697	9.78E-07	23	48	0.00894	8.00E-07
	$x3$	13	44	0.00601	6.63E-07	857	2577	0.20229	9.89E-07	38	78	0.01407	9.03E-07
	$x4$	500	1515	0.17451	9.89E-07	648	1949	0.20882	9.86E-07	501	1008	0.12172	9.86E-07
	$x5$	11	37	0.00524	1.76E-07	48	145	0.0224	7.50E-07	23	48	0.00649	7.65E-07
	$x6$	13	43	0.00568	4.84E-07	41	124	0.01073	7.71E-07	41	84	0.01077	8.53E-07
5000	$x1$	9	31	0.01311	8.25E-07	46	139	0.04214	7.69E-07	21	44	0.01891	9.88E-07
	$x2$	10	35	0.01168	3.82E-07	48	145	0.05168	7.40E-07	22	46	0.02008	9.35E-07
	$x3$	13	44	0.0136	6.63E-07	857	2577	0.67459	9.89E-07	38	78	0.03335	9.03E-07
	$x4$	499	1512	0.55266	9.91E-07	677	2036	0.55391	9.75E-07	501	1008	0.49231	9.82E-07
	$x5$	11	37	0.01621	3.31E-07	50	151	0.04079	8.09E-07	23	48	0.02319	6.67E-07
	$x6$	13	43	0.01586	4.85E-07	41	124	0.03375	7.72E-07	41	84	0.03335	8.52E-07
10000	$x1$	9	31	0.02609	9.58E-07	47	142	0.07281	7.56E-07	22	46	0.03607	5.62E-07
	$x2$	10	35	0.02518	3.78E-07	49	148	0.06769	7.28E-07	22	46	0.03154	9.85E-07
	$x3$	13	44	0.0258	6.63E-07	857	2577	1.1468	9.89E-07	38	78	0.05549	9.03E-07
	$x4$	499	1512	0.90188	9.80E-07	689	2072	1.0057	9.85E-07	501	1008	0.73829	9.80E-07
	$x5$	11	37	0.02411	4.57E-07	51	154	0.07644	7.95E-07	23	48	0.03722	7.49E-07
	$x6$	13	43	0.02478	4.85E-07	41	124	0.06267	7.72E-07	41	84	0.0633	8.52E-07
50000	$x1$	10	35	0.09884	2.65E-07	49	148	0.31317	8.09E-07	22	46	0.14253	8.63E-07
	$x2$	10	35	0.10095	3.87E-07	51	154	0.32635	7.82E-07	23	48	0.14756	7.43E-07
	$x3$	13	44	0.11302	6.63E-07	857	2577	4.9648	9.89E-07	38	78	0.23177	9.03E-07
	$x4$	498	1509	3.867	9.82E-07	718	2159	4.4973	9.75E-07	501	1008	3.1862	9.75E-07
	$x5$	11	37	0.10701	1.00E-06	53	160	0.33518	8.53E-07	24	50	0.15771	6.11E-07
	$x6$	13	43	0.11203	4.85E-07	41	124	0.25437	7.72E-07	41	84	0.26153	8.52E-07
100000	$x1$	10	35	0.18942	3.45E-07	50	151	0.65757	7.92E-07	23	48	0.31583	5.31E-07
	$x2$	10	35	0.2	4.24E-07	52	157	0.69527	7.67E-07	23	48	0.31792	9.72E-07
	$x3$	13	44	0.23667	6.63E-07	857	2577	10.8748	9.89E-07	38	78	0.50969	9.03E-07
	$x4$	497	1506	7.7783	9.98E-07	730	2195	11.9544	9.85E-07	501	1008	6.525	9.73E-07
	$x5$	12	41	0.22678	2.00E-07	54	163	0.83682	8.36E-07	24	50	0.32946	8.19E-07
	$x6$	13	43	0.23149	4.85E-07	41	124	0.78256	7.72E-07	41	84	0.54709	8.52E-07

in comparison with the two algorithms (P-HS, P-CG) and each table shows the comparison with respect to one of the functions included in this paper, meaning that (from function 1 to function 8), respectively.

The performance of all methods was assessed using Dolan and Moré [9] as shown in the following figures:

The performance of the new algorithm (AR-New) over the existing algorithms (P-HS and P-

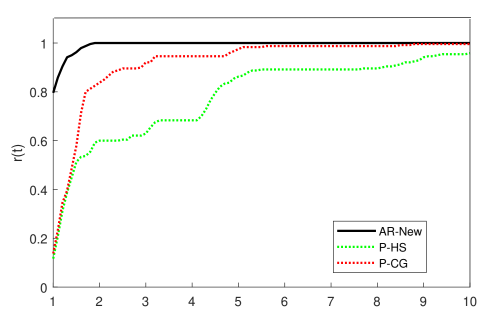

FIGURE 1. The measurement of the iteration's rate (I.Niter).

CG) regarding the estimated number of iterations is shown in Figure 1.

In Figure 2, the new algorithm (AR-New) outperforms the existing algorithms regarding performance in terms of the number of function evaluations.

TABLE 3. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 2

		AR-New algorithm				**P-HS algorithm**				**P-CG algorithm**			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	13	41	0.01608	9.60E-07	20	60	0.00729	4.62E-07	22	45	0.00748	7.06E-07
	$x2$	15	37	0.00931	6.68E-07	20	60	0.00742	9.27E-07	23	47	0.00831	7.02E-07
	$x3$	17	33	0.01015	7.38E-07	17	51	0.00846	9.13E-07	19	39	0.00638	9.71E-07
	$x4$	48	147	0.01717	8.71E-07	51	154	0.0204	9.57E-07	57	116	0.01618	8.88E-07
	$x5$	14	34	0.00906	7.82E-07	22	66	0.01057	4.92E-07	24	49	0.0131	8.27E-07
	$x6$	15	37	0.00603	6.26E-07	18	54	0.00641	9.56E-07	20	41	0.00664	9.66E-07
5000	$x1$	15	37	0.03488	8.72E-07	21	63	0.02616	4.70E-07	23	47	0.02604	7.91E-07
	$x2$	16	40	0.03059	9.52E-07	21	63	0.02822	9.44E-07	24	49	0.02686	7.86E-07
	$x3$	17	33	0.03811	7.38E-07	17	51	0.02059	9.13E-07	19	39	0.01911	9.71E-07
	$x4$	50	113	0.06165	9.92E-07	54	163	0.07395	7.35E-07	60	122	0.05988	7.89E-07
	$x5$	26	80	0.02652	7.11E-07	23	69	0.02639	5.01E-07	25	51	0.0359	9.27E-07
	$x6$	15	47	0.01695	6.26E-07	18	54	0.021	9.57E-07	20	41	0.01944	9.67E-07
10000	$x1$	16	40	0.06147	7.86E-07	21	63	0.06094	6.65E-07	24	49	0.03481	5.61E-07
	$x2$	17	33	0.06128	8.58E-07	22	66	0.03573	6.08E-07	25	51	0.05131	5.57E-07
	$x3$	17	34	0.0481	7.38E-07	17	51	0.03034	9.13E-07	19	39	0.05389	9.71E-07
	$x4$	52	119	0.11006	7.15E-07	55	166	0.09379	7.28E-07	61	124	0.23116	8.21E-07
	$x5$	17	33	0.06308	6.40E-07	23	69	0.03384	7.08E-07	26	53	0.05209	6.57E-07
	$x6$	15	37	0.03422	6.26E-07	18	54	0.04711	9.57E-07	20	41	0.04294	9.67E-07
50000	$x1$	18	46	0.31158	7.14E-07	22	66	0.15899	6.77E-07	25	51	0.33173	6.28E-07
	$x2$	19	49	0.26738	7.80E-07	23	69	0.18617	6.19E-07	26	53	0.29686	6.24E-07
	$x3$	17	33	0.22353	7.38E-07	17	51	0.13218	9.13E-07	19	39	0.27055	9.71E-07
	$x4$	54	115	0.48365	8.14E-07	57	172	0.59471	7.99E-07	63	128	0.73566	9.92E-07
	$x5$	18	46	0.24339	9.13E-07	24	72	0.23441	7.21E-07	27	55	0.31369	7.36E-07
	$x6$	15	37	0.13213	6.26E-07	18	54	0.14823	9.57E-07	20	41	0.18554	9.67E-07
100000	$x1$	19	49	0.52752	6.44E-07	22	66	0.44368	9.57E-07	25	51	0.41349	8.88E-07
	$x2$	20	45	0.56519	7.03E-07	23	69	0.39914	8.76E-07	26	53	0.41656	8.82E-07
	$x3$	20	38	0.50564	7.38E-07	17	51	0.35943	9.13E-07	19	39	0.31451	9.71E-07
	$x4$	53	123	1.3399	8.03E-07	58	175	1.5624	7.92E-07	65	132	1.0193	7.59E-07
	$x5$	23	49	0.53108	8.23E-07	25	75	0.59937	4.64E-07	28	57	0.43574	5.21E-07
	$x6$	15	37	0.27163	6.26E-07	18	54	0.48361	9.57E-07	20	41	0.29978	9.67E-07

TABLE 4. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 3

		AR-New algorithm				**P-HS algorithm**				**P-CG algorithm**			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	19	59	0.00779	8.04E-07	20	60	0.0089	4.78E-07	22	45	0.00569	6.25E-07
	$x2$	14	44	0.00513	6.38E-07	20	60	0.00589	9.73E-07	23	47	0.00617	5.51E-07
	$x3$	12	39	0.00443	4.75E-07	17	51	0.00466	9.67E-07	19	40	0.00576	5.91E-07
	$x4$	618	1869	0.18562	9.85E-07	639	1922	0.15048	9.93E-07	616	1238	0.1322	9.91E-07
	$x5$	24	74	0.00768	8.78E-07	22	66	0.00637	4.56E-07	23	47	0.00673	8.99E-07
	$x6$	26	80	0.00829	8.32E-07	38	115	0.01668	9.61E-07	20	41	0.00568	7.70E-07
5000	$x1$	21	65	0.02067	7.30E-07	21	63	0.01582	4.87E-07	23	47	0.01816	7.01E-07
	$x2$	15	47	0.0152	9.03E-07	21	63	0.01817	9.91E-07	24	49	0.01847	6.18E-07
	$x3$	12	39	0.01952	4.75E-07	17	51	0.01402	9.67E-07	19	40	0.01337	5.91E-07
	$x4$	648	1959	0.58177	9.77E-07	668	2009	0.60181	9.92E-07	646	1298	0.47848	9.78E-07
	$x5$	26	80	0.02446	7.98E-07	23	69	0.02707	4.65E-07	25	51	0.01771	5.05E-07
	$x6$	26	80	0.02634	8.33E-07	38	115	0.04476	9.61E-07	20	41	0.01712	7.71E-07
10000	$x1$	22	68	0.03654	6.58E-07	21	63	0.03364	6.89E-07	23	47	0.03147	9.91E-07
	$x2$	16	50	0.03067	8.12E-07	22	66	0.03672	6.38E-07	24	49	0.02717	8.73E-07
	$x3$	12	39	0.02886	4.75E-07	17	51	0.03108	9.67E-07	19	40	0.02647	5.91E-07
	$x4$	660	1995	0.99926	9.98E-07	681	2048	1.0923	9.77E-07	658	1322	0.79615	9.97E-07
	$x5$	27	83	0.04472	7.19E-07	23	69	0.11293	6.57E-07	25	51	0.02749	7.14E-07
	$x6$	26	80	0.03942	8.33E-07	38	115	0.0891	9.61E-07	20	41	0.0271	7.71E-07
50000	$x1$	23	71	0.15229	9.38E-07	22	66	0.23935	7.01E-07	25	51	0.13759	5.56E-07
	$x2$	18	56	0.12004	7.37E-07	23	69	0.2204	6.50E-07	25	51	0.1328	9.79E-07
	$x3$	12	39	0.08054	4.75E-07	17	51	0.09483	9.67E-07	19	40	0.09302	5.91E-07
	$x4$	690	2085	4.4749	9.91E-07	710	2135	3.8951	9.76E-07	688	1382	3.8074	9.83E-07
	$x5$	29	89	0.194	6.53E-07	24	72	0.13008	6.69E-07	26	53	0.14551	7.99E-07
	$x6$	26	80	0.1622	8.33E-07	38	115	0.18031	9.62E-07	20	41	0.1043	7.72E-07
100000	$x1$	24	74	0.30325	8.46E-07	22	66	0.2203	9.92E-07	25	51	0.42273	7.87E-07
	$x2$	19	59	0.24226	6.64E-07	23	69	0.26497	9.19E-07	26	53	0.28348	6.93E-07
	$x3$	12	39	0.17483	4.75E-07	17	51	0.19516	9.67E-07	19	40	0.20272	5.91E-07
	$x4$	703	2124	8.5847	9.85E-07	722	2171	7.9899	9.89E-07	701	1408	7.8647	9.75E-07
	$x5$	29	89	0.37615	9.24E-07	24	72	0.25835	9.46E-07	27	55	0.30498	5.66E-07
	$x6$	26	80	0.32315	8.33E-07	38	115	0.39557	9.62E-07	20	41	0.22079	7.72E-07

TABLE 5. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 4

		AR-New algorithm				**P-HS algorithm**				**P-CG algorithm**			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	21	41	0.01784	7.17E-07	38	113	0.01859	7.85E-07	22	45	0.01509	9.19E-07
	$x2$	21	41	0.01553	6.89E-07	38	113	0.01489	7.56E-07	22	45	0.00925	8.84E-07
	$x3$	21	41	0.01599	7.44E-07	40	119	0.02652	7.47E-07	22	45	0.0101	8.26E-07
	$x4$	20	42	0.02871	9.78E-07	38	113	0.01426	7.36E-07	22	45	0.00871	8.01E-07
	$x5$	20	42	0.01526	9.53E-07	37	110	0.01387	9.55E-07	22	45	0.01071	7.79E-07
	$x6$	21	45	0.01525	7.42E-07	40	119	0.01462	7.96E-07	22	45	0.00916	7.84E-07
5000	$x1$	19	33	0.05363	6.54E-07	20	58	0.02647	4.77E-07	21	43	0.02978	5.45E-07
	$x2$	19	98	0.07238	9.87E-07	20	58	0.02836	4.59E-07	21	43	0.04693	5.24E-07
	$x3$	17	41	0.05446	6.79E-07	19	55	0.02903	9.01E-07	20	41	0.04479	9.69E-07
	$x4$	17	40	0.05941	8.94E-07	19	55	0.02878	9.13E-07	20	41	0.02706	9.47E-07
	$x5$	17	40	0.05689	8.70E-07	19	55	0.03112	8.88E-07	20	41	0.027	9.21E-07
	$x6$	16	41	0.05933	6.79E-07	19	55	0.04271	8.45E-07	20	41	0.03052	9.21E-07
10000	$x1$	16	41	0.11161	9.25E-07	19	55	0.05157	9.24E-07	21	43	0.05032	5.36E-07
	$x2$	16	41	0.11051	8.90E-07	19	55	0.05318	8.89E-07	21	43	0.04953	5.16E-07
	$x3$	16	41	0.1133	9.61E-07	19	55	0.04929	7.68E-07	20	41	0.05118	9.57E-07
	$x4$	16	41	0.10721	8.06E-07	19	55	0.04957	8.05E-07	20	41	0.06882	9.33E-07
	$x5$	16	41	0.11335	7.84E-07	19	55	0.05698	7.83E-07	20	41	0.0556	9.07E-07
	$x6$	16	41	0.11242	9.60E-07	19	55	0.052	7.12E-07	20	41	0.04986	9.12E-07
50000	$x1$	15	40	0.45746	8.41E-07	19	55	0.20704	5.26E-07	21	43	0.21821	5.29E-07
	$x2$	15	40	0.4656	8.08E-07	19	55	0.19141	5.06E-07	21	43	0.22809	5.09E-07
	$x3$	15	40	0.48044	8.73E-07	18	52	0.19425	9.15E-07	20	41	0.22347	9.48E-07
	$x4$	15	40	0.46723	7.33E-07	19	55	0.21715	4.59E-07	20	41	0.23651	9.20E-07
	$x5$	15	40	0.47817	7.12E-07	18	52	0.19166	9.79E-07	20	41	0.21425	8.95E-07
	$x6$	15	40	0.47742	8.73E-07	18	52	0.19228	8.37E-07	20	41	0.22295	9.04E-07
100000	$x1$	17	39	1.0647	7.58E-07	18	52	0.41049	8.49E-07	21	43	0.50314	5.28E-07
	$x2$	17	39	1.0605	7.29E-07	18	52	0.43843	8.17E-07	21	43	0.49476	5.08E-07
	$x3$	17	39	1.0678	7.87E-07	18	52	0.42165	6.67E-07	20	41	0.45407	9.47E-07
	$x4$	17	39	1.0647	6.60E-07	18	52	0.43611	7.40E-07	20	41	0.54576	9.19E-07
	$x5$	17	39	1.0523	6.42E-07	18	52	0.42917	7.19E-07	20	41	0.50242	8.93E-07
	$x6$	17	39	1.0568	7.87E-07	18	52	0.44534	6.09E-07	20	41	0.48286	9.03E-07

TABLE 6. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 5

		AR-New algorithm				**P-HS algorithm**				**P-CG algorithm**			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	12	40	0.02106	5.92E-07	19	57	0.00643	4.61E-07	20	42	0.00666	8.06E-07
	$x2$	12	40	0.00516	7.77E-07	17	51	0.00567	7.85E-07	19	40	0.00691	5.57E-07
	$x3$	15	49	0.00656	9.64E-07	19	57	0.00616	9.34E-07	21	44	0.00833	8.37E-07
	$x4$	13	43	0.00593	8.58E-07	23	69	0.00731	5.66E-07	25	52	0.01129	7.84E-07
	$x5$	12	40	0.00534	3.32E-07	37	112	0.03506	7.13E-07	19	40	0.00648	6.69E-07
	$x6$	16	52	0.00645	6.83E-07	144	435	0.04598	9.25E-07	23	48	0.00714	9.38E-07
5000	$x1$	13	43	0.01762	4.24E-07	20	60	0.02221	4.67E-07	21	44	0.027	8.78E-07
	$x2$	13	43	0.01853	5.54E-07	18	54	0.01873	7.95E-07	20	42	0.0204	6.06E-07
	$x3$	16	52	0.03179	4.52E-07	20	60	0.02299	9.06E-07	22	46	0.0248	8.70E-07
	$x4$	14	46	0.02123	6.14E-07	24	72	0.02607	5.73E-07	26	54	0.0294	8.54E-07
	$x5$	12	40	0.0188	7.41E-07	39	118	0.03707	7.80E-07	20	42	0.01955	7.29E-07
	$x6$	17	55	0.02461	3.62E-07	20	60	0.02061	8.78E-07	24	50	0.02624	9.38E-07
10000	$x1$	13	43	0.03375	5.99E-07	20	60	0.03912	6.60E-07	22	46	0.04203	6.05E-07
	$x2$	13	43	0.03369	7.84E-07	19	57	0.04891	5.09E-07	20	42	0.03719	8.57E-07
	$x3$	16	52	0.03421	5.53E-07	21	63	0.03633	5.77E-07	23	48	0.05636	5.96E-07
	$x4$	14	46	0.04742	8.68E-07	24	72	0.03451	8.10E-07	27	56	0.05198	5.89E-07
	$x5$	13	43	0.03173	3.43E-07	40	121	0.06498	7.71E-07	21	44	0.048	5.02E-07
	$x6$	17	55	0.03616	4.66E-07	21	63	0.043	5.94E-07	24	50	0.05368	6.34E-07
50000	$x1$	14	46	0.13412	4.32E-07	21	63	0.25713	6.68E-07	23	48	0.18287	6.59E-07
	$x2$	14	46	0.12965	5.64E-07	20	60	0.19039	5.16E-07	21	44	0.17081	9.34E-07
	$x3$	16	52	0.1542	8.13E-07	22	66	0.18953	5.82E-07	24	50	0.18114	6.46E-07
	$x4$	15	49	0.15927	6.27E-07	25	75	0.18651	8.21E-07	28	58	0.22305	6.41E-07
	$x5$	13	43	0.12568	7.68E-07	42	127	0.32077	8.42E-07	22	46	0.1683	5.47E-07
	$x6$	17	55	0.18006	6.18E-07	22	66	0.22505	5.93E-07	24	50	0.19387	6.77E-07
100000	$x1$	14	46	0.29106	6.11E-07	21	63	0.35486	9.45E-07	23	48	0.35583	9.32E-07
	$x2$	14	46	0.27274	7.97E-07	20	60	0.33266	7.29E-07	22	46	0.38408	6.43E-07
	$x3$	16	52	0.32093	9.56E-07	22	66	0.36127	8.23E-07	24	50	0.38332	9.13E-07
	$x4$	15	49	0.29021	8.86E-07	26	78	0.41259	5.26E-07	28	58	0.4419	9.07E-07
	$x5$	14	46	0.28474	3.57E-07	43	130	0.68268	8.33E-07	22	46	0.35672	7.73E-07
	$x6$	17	55	0.35207	6.82E-07	22	66	0.39048	8.31E-07	24	50	0.40473	9.35E-07

TABLE 7. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 6

		AR-New algorithm				P-HS algorithm				P-CG algorithm			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	25	87	0.03692	8.92E-07	192	579	0.15193	9.71E-07	31	67	0.02657	7.63E-07
	$x2$	20	72	0.02744	9.28E-07	190	573	0.14847	9.24E-07	18	41	0.01685	6.81E-07
	$x3$	24	85	0.03221	7.41E-07	194	585	0.21256	9.61E-07	118	243	0.0947	8.85E-07
	$x4$	324	992	0.42174	9.79E-07	1001	3011	1.0844	0.00225	339	686	0.27082	9.95E-07
	$x5$	22	78	0.04209	6.59E-07	183	553	0.16794	9.17E-07	111	229	0.10254	9.69E-07
	$x6$	45	149	0.05227	8.63E-07	194	585	0.15909	9.56E-07	379	767	0.30394	9.92E-07
5000	$x1$	20	74	0.11367	4.82E-07	201	606	0.79668	9.13E-07	26	57	0.09326	9.17E-07
	$x2$	20	72	0.11835	9.80E-07	198	597	0.84984	9.50E-07	19	43	0.07465	6.81E-07
	$x3$	23	81	0.12614	6.28E-07	202	609	0.81377	9.96E-07	117	241	0.43788	8.97E-07
	$x4$	324	992	1.4787	9.80E-07	1001	3011	3.6793	0.00225	339	686	1.2862	9.99E-07
	$x5$	22	78	0.11876	7.29E-07	186	562	0.72376	9.30E-07	108	223	0.38872	9.03E-07
	$x6$	39	131	0.21008	8.64E-07	202	609	1.0069	9.95E-07	708	1426	2.4861	9.90E-07
10000	$x1$	41	136	0.37627	8.42E-07	204	615	1.6268	9.68E-07	25	55	0.16822	7.79E-07
	$x2$	20	72	0.21959	9.94E-07	202	609	1.5586	9.12E-07	20	45	0.13205	3.38E-07
	$x3$	24	85	0.23982	7.65E-07	206	621	1.2937	9.60E-07	115	237	0.73419	9.87E-07
	$x4$	324	992	2.6249	9.80E-07	1001	3011	6.2698	0.00225	339	686	2.159	9.99E-07
	$x5$	22	78	0.22245	7.59E-07	188	568	1.2956	9.45E-07	107	221	0.73303	8.71E-07
	$x6$	39	131	0.44577	8.59E-07	206	621	1.6178	9.59E-07	218	444	1.3713	9.50E-07
50000	$x1$	40	133	1.8277	9.62E-07	213	642	7.2623	9.12E-07	23	51	0.7498	7.01E-07
	$x2$	22	79	1.0766	6.55E-07	210	633	6.7299	9.29E-07	21	47	0.69827	3.33E-07
	$x3$	24	85	1.1686	7.67E-07	214	645	7.1719	9.96E-07	113	233	3.6406	9.43E-07
	$x4$	411	1255	15.922	9.79E-07	1001	3011	30.3519	0.00225	339	686	10.0524	1.00E-06
	$x5$	22	78	1.0721	8.05E-07	195	589	5.7617	9.22E-07	107	221	3.1961	9.03E-07
	$x6$	39	131	1.7632	8.78E-07	214	645	6.5117	9.95E-07	124	255	3.72	8.88E-07
100000	$x1$	40	133	3.8203	9.45E-07	216	651	14.5922	9.67E-07	23	51	1.5958	5.44E-07
	$x2$	22	79	2.3421	6.61E-07	213	642	16.3622	9.71E-07	21	47	1.629	5.22E-07
	$x3$	24	85	2.5072	7.67E-07	218	657	14.9596	9.59E-07	113	233	7.9657	8.92E-07
	$x4$	411	1255	34.5768	9.79E-07	1001	3011	64.5747	0.00225	339	686	22.4225	1.00E-06
	$x5$	22	78	2.3151	8.17E-07	198	598	13.0849	9.53E-07	192	392	12.9772	9.95E-07
	$x6$	44	146	4.1933	9.85E-07	218	657	14.3533	9.59E-07	124	255	8.199	8.73E-07

TABLE 8. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 7

		AR-New algorithm				P-HS algorithm				P-CG algorithm			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	10	22	0.00333	3.63E-07	43	130	0.00949	9.37E-07	16	54	0.00649	4.54E-07
	$x2$	10	22	0.0036	2.20E-07	42	127	0.00851	8.13E-07	15	51	0.00444	6.83E-07
	$x3$	10	22	0.00283	5.05E-07	44	133	0.00882	9.12E-07	16	54	0.00526	6.32E-07
	$x4$	12	26	0.0029	1.89E-07	51	154	0.00938	9.74E-07	19	63	0.00542	5.41E-07
	$x5$	10	22	0.00323	2.10E-07	42	127	0.0088	7.75E-07	15	51	0.00477	6.51E-07
	$x6$	10	22	0.00248	4.99E-07	44	133	0.00812	9.00E-07	16	54	0.00459	6.23E-07
5000	$x1$	10	22	0.00924	8.12E-07	46	139	0.0263	7.14E-07	17	57	0.01411	4.08E-07
	$x2$	10	22	0.00799	4.92E-07	44	133	0.03147	8.87E-07	16	54	0.01442	6.14E-07
	$x3$	11	24	0.00884	1.91E-07	46	139	0.02307	9.96E-07	17	57	0.01269	5.69E-07
	$x4$	12	26	0.0094	4.23E-07	54	163	0.02959	7.43E-07	20	66	0.01604	4.87E-07
	$x5$	10	22	0.00774	4.69E-07	44	133	0.0232	8.46E-07	16	54	0.0125	5.86E-07
	$x6$	11	24	0.0079	1.90E-07	46	139	0.02306	9.92E-07	17	57	0.01359	5.67E-07
10000	$x1$	11	24	0.0141	1.94E-07	47	142	0.0556	7.06E-07	17	57	0.02913	5.77E-07
	$x2$	10	22	0.01353	6.95E-07	45	136	0.04407	8.76E-07	16	54	0.02258	8.69E-07
	$x3$	11	24	0.01344	2.70E-07	47	142	0.04994	9.84E-07	17	57	0.02477	8.05E-07
	$x4$	12	26	0.01409	5.99E-07	55	166	0.06055	7.34E-07	20	66	0.02383	6.89E-07
	$x5$	10	22	0.0133	6.63E-07	45	136	0.04685	8.35E-07	16	54	0.02841	8.29E-07
	$x6$	11	24	0.01364	2.69E-07	47	142	0.04177	9.82E-07	17	57	0.02249	8.03E-07
50000	$x1$	11	24	0.05054	4.33E-07	49	148	0.22351	7.70E-07	18	60	0.10111	5.19E-07
	$x2$	11	24	0.04924	2.62E-07	47	142	0.2109	9.56E-07	17	57	0.10583	7.82E-07
	$x3$	11	24	0.05182	6.04E-07	50	151	0.22037	7.50E-07	18	60	0.11378	7.24E-07
	$x4$	13	28	0.06668	2.26E-07	57	172	0.26904	8.00E-07	21	69	0.11772	6.20E-07
	$x5$	11	24	0.05268	2.50E-07	47	142	0.21029	9.12E-07	17	57	0.09813	7.45E-07
	$x6$	11	24	0.05425	6.04E-07	50	151	0.27948	7.50E-07	18	60	0.11263	7.24E-07
100000	$x1$	11	24	0.11121	6.12E-07	50	151	0.59091	7.61E-07	18	60	0.21846	7.34E-07
	$x2$	11	24	0.10838	3.71E-07	48	145	0.48987	9.44E-07	18	60	0.22853	4.45E-07
	$x3$	11	24	0.11325	8.54E-07	51	154	0.50858	7.41E-07	19	63	0.24917	4.12E-07
	$x4$	13	28	0.13002	3.20E-07	58	175	0.56647	7.91E-07	21	69	0.26795	8.76E-07
	$x5$	11	24	0.10466	3.54E-07	48	145	0.48107	9.01E-07	18	60	0.22023	4.24E-07
	$x6$	11	24	0.10892	8.54E-07	51	154	0.51599	7.41E-07	19	63	0.22845	4.12E-07

TABLE 9. Numerical Comparisons of the AR-New versus P-HS, P-CG for problem 8

		AR-New algorithm				P-HS algorithm				P-CG algorithm			
dim.	**I.P.**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**	**I.Niter**	**F.NFun**	**C.Time**	**N.Norm**
1000	$x1$	12	38	0.02096	7.65E-07	17	50	0.00652	4.86E-07	19	39	0.00838	5.44E-07
	$x2$	23	71	0.00998	8.87E-07	18	53	0.00712	7.67E-07	15	31	0.00707	5.74E-07
	$x3$	18	55	0.00843	9.26E-07	16	47	0.00665	7.62E-07	18	37	0.00694	7.44E-07
	$x4$	38	115	0.01619	9.73E-07	22	65	0.00775	6.70E-07	21	43	0.00801	8.98E-07
	$x5$	28	85	0.01242	7.14E-07	20	59	0.00741	6.49E-07	20	41	0.00763	8.13E-07
	$x6$	14	41	0.00655	5.57E-07	35	105	0.01309	1.00E-06	21	43	0.00831	5.68E-07
5000	$x1$	13	41	0.02009	7.83E-07	18	53	0.02489	4.87E-07	20	41	0.0386	5.90E-07
	$x2$	25	77	0.03652	7.24E-07	19	56	0.02231	7.75E-07	16	33	0.02149	5.55E-07
	$x3$	18	55	0.02692	9.36E-07	16	47	0.01925	7.63E-07	19	39	0.02315	6.59E-07
	$x4$	24	72	0.03655	9.50E-07	23	68	0.03013	6.83E-07	22	45	0.02522	9.59E-07
	$x5$	29	88	0.04862	9.34E-07	21	62	0.02632	6.59E-07	21	43	0.0232	8.79E-07
	$x6$	14	41	0.02088	4.37E-07	37	111	0.04225	8.32E-07	21	43	0.02384	5.81E-07
10000	$x1$	14	44	0.03937	6.58E-07	18	53	0.03633	6.88E-07	20	41	0.04129	8.31E-07
	$x2$	26	80	0.06723	6.44E-07	20	59	0.03825	4.98E-07	16	33	0.03489	7.69E-07
	$x3$	18	55	0.04379	9.37E-07	16	47	0.03207	7.63E-07	19	39	0.03799	6.58E-07
	$x4$	25	75	0.0709	7.92E-07	23	68	0.07052	9.66E-07	23	47	0.05267	6.76E-07
	$x5$	30	91	0.08192	8.33E-07	21	62	0.04611	9.31E-07	22	45	0.04495	6.20E-07
	$x6$	14	41	0.03851	3.16E-07	37	111	0.07915	8.29E-07	21	43	0.04418	5.83E-07
50000	$x1$	15	47	0.18517	9.00E-07	19	56	0.17339	6.99E-07	21	43	0.19458	9.27E-07
	$x2$	27	83	0.31098	9.07E-07	21	62	0.20073	5.07E-07	17	35	0.15893	8.48E-07
	$x3$	18	55	0.18298	9.39E-07	16	47	0.12969	7.63E-07	19	39	0.16992	6.57E-07
	$x4$	27	81	0.30438	6.75E-07	24	71	0.23443	9.84E-07	24	49	0.21921	7.53E-07
	$x5$	32	97	0.37128	7.50E-07	22	65	0.23421	9.48E-07	23	47	0.21809	6.92E-07
	$x6$	14	41	0.16754	2.39E-07	37	111	0.36623	8.26E-07	21	43	0.20329	5.84E-07
100000	$x1$	16	50	0.37382	8.04E-07	19	56	0.378	9.88E-07	22	45	0.43779	6.57E-07
	$x2$	28	86	0.65209	8.16E-07	21	62	0.4233	7.17E-07	18	37	0.34962	6.00E-07
	$x3$	18	55	0.3709	9.39E-07	16	47	0.28935	7.63E-07	19	39	0.33309	6.57E-07
	$x4$	27	81	0.63193	9.46E-07	25	74	0.50883	6.34E-07	25	51	0.50212	5.33E-07
	$x5$	33	100	0.7614	6.75E-07	23	68	0.45019	6.11E-07	23	47	0.45063	9.78E-07
	$x6$	14	41	0.31143	2.30E-07	37	111	0.71247	8.26E-07	21	43	0.38919	5.85E-07

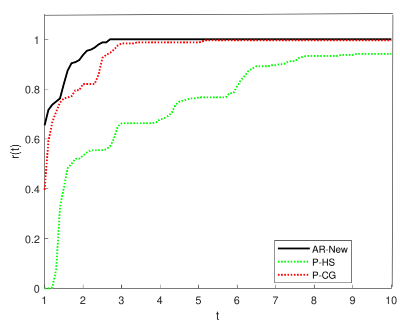

FIGURE 2. The rate of the function's calculation (F.NFun).

The new algorithm (AR-New) is the most effective of the algorithms compared to it inside the figure in relation to the time spent implementing the specified algorithms, according to Figure 3.

4.1. **Signal Recovery.** A common problem in signal processing and statistical inference is obtaining sparse solutions to ill-conditioned linear systems of equations. Some of which involve minimizing a quadratic function with (ℓ_2) error term and a sparse ℓ_1-regularization term,

$$\min_{h} \frac{1}{2}\|v-Ax\|_2^2+\omega\|x\|_1, \tag{4.1}$$

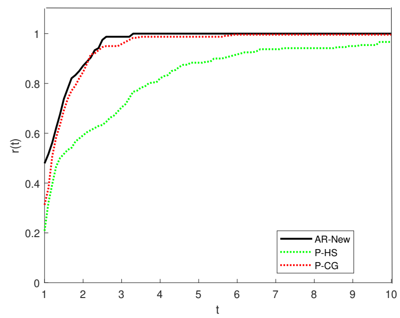

FIGURE 3. The length of time it takes for the CPU to execute (C.Time).

where $x \in \mathbb{R}^n$, $v \in \mathbb{R}^k$ is an observation, $A \in \mathbb{R}^{k\times n}$ ($k << n$) is linear, and $\omega > 0$ is a parameter. Note that (4.1) is an unconstrained convex optimization problem. This problem has been shown to be equivalent to $F(x) = 0$; see [15, 28] for more detail. Hence AR-New can solve problem (4.1).

All numerical tests were run on a PC with Intel Core(TM) i3 processor running at 2.3 GHz and 8GB of RAM. The proposed AR-NEW algorithm is compared with two other methods, CGD by Liu et al. [19] and PCG by Liu and Feng [18]. The purpose is to recover a length-n sparse signal from k observations, with the mean squared error (MSE) being used to evaluate the recovery. The MSE is defined as:

$$MSE := \frac{1}{n}\|x_a - x_*\|^2.$$

The original signal is x_a, while the recovered signal is x_*. The signal has a size of $n = 2^{11}$ and a size of $k = 2^9$, with 2^7 randomly non-zero elements. The $randn(k,n)$ program in MATLAB is used to generate the matrix A in (4.1), and the noise (Gaussian noise) is normally distributed with a mean of 0 and a variance 10^{-3}.

The following parameters are used for the AR-NEW implementation: $r = 0.8$, $\sigma = 10^{-4}$ with a merit function defined as follows:

$$f(x) = \frac{1}{2}\|Ax - v\|_2^2 + \omega\|x\|_1.$$

To achieve fairness in comparison, each code was run from the same initial point, same continuation technique on the parameter ω. That is,

$$\omega = 0.005\left\|A^T v\right\|_\infty.$$

$x_0 = A^T v$ is used to start the experiment, and

$$\frac{\|f_k - f_{k-1}\|}{\|f_{k-1}\|} < 10^{-5},$$

where f_k is the function value at x_k.

The experiment was repeated a number of times and the performance of each algorithm was recorded. Figure 4 shows the original, disturbed, and recovery achieved by each algorithm. In addition, Figure 5 highlight the convergence behavior of CGD, PCG, and AR-NEW based on the pattern of MSE and objective function values, as well as the number of iterations and CPU processing time.

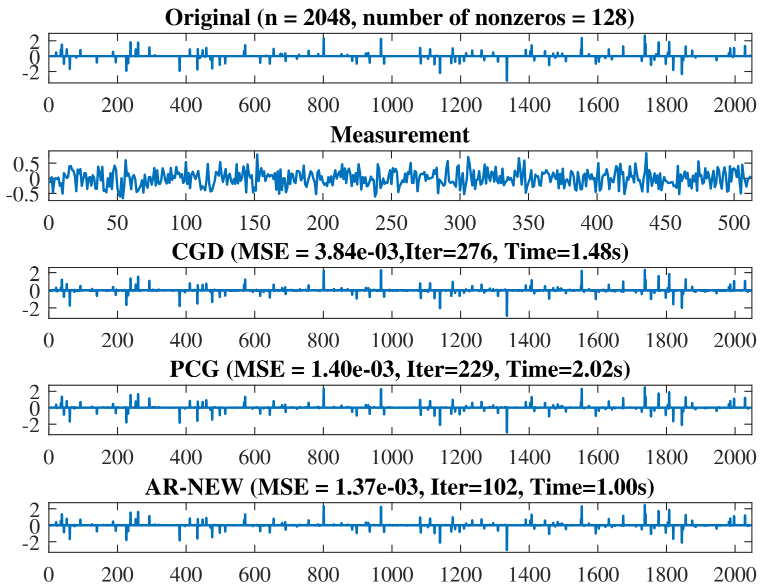

FIGURE 4. The signal reconstruction.

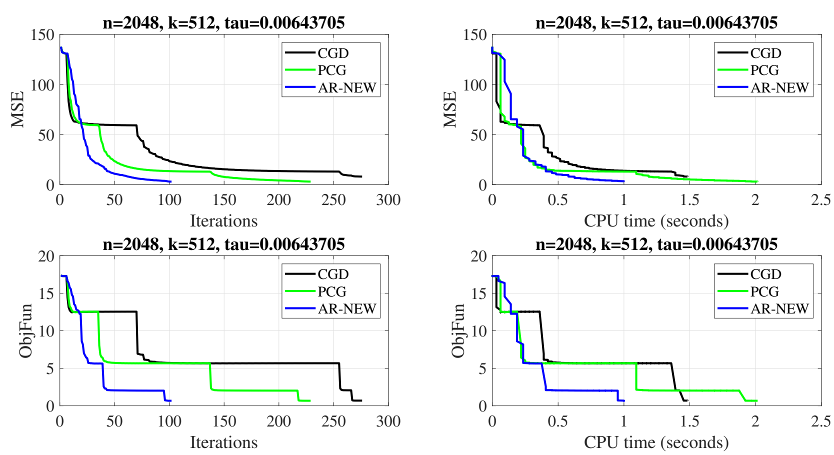

FIGURE 5. The performance profile for the signal reconstruction.

AR-NEW can be seen to have better performance based on Figure 4 as well Table 10. In summary, AR-NEW recovered the sparse signal with the least mean squared error, iterations, and CPU time.

5. CONCLUSIONS

In this paper, we integrated the projection technique with the spectral conjugate gradient to present an AR-New projection algorithm. The constrained system of monotone nonlinear equations can be solved by using this innovative conjugated projective proportional algorithm. The adequate ratio criterion, a prerequisite for a system to produce outcomes of global convergence,

TABLE 10. Results of recovery by each algorithm

	CGD			PCG			AR-NEW		
SN	ITER	CPU	MSE	ITER	CPU	MSE	ITER	CPU	MSE
1	276	1.48	3.84E-03	229	2.02	1.40E-03	102	1	1.37E-03
2	183	1.25	6.60E-03	188	1.2	3.60E-03	159	1.38	1.80E-03
3	234	1.39	4.02E-03	176	1.16	3.21E-03	135	1.42	1.98E-03
4	480	5.8	1.75E-03	166	1.77	3.52E-03	134	1.44	2.47E-03
5	299	2.05	3.30E-03	208	0.91	2.09E-03	100	0.3	2.96E-03
6	530	3.11	2.64E-03	225	2.19	3.95E-03	151	2.72	3.09E-03
7	456	2.81	2.75E-03	137	0.7	5.67E-03	86	0.55	5.02E-03
8	550	3.28	1.02E-03	194	1.08	1.90E-03	156	2.33	1.17E-03
9	245	2.09	3.61E-03	192	1.05	3.45E-03	109	0.69	4.06E-03
10	125	0.75	7.11E-03	281	1.56	1.52E-03	125	0.72	2.09E-03
Average	338	2.401	3.66E-03	200	1.364	3.03E-03	126	1.255	2.60E-03

was met by the proposed algorithm. The performance of our algorithm was contrasted with that of the fundamental P-HS and P-CG algorithms. We can demonstrate that our algorithm outperforms the P-HS and P-CG algorithms by the numerical results. Under the right circumstances, the global convergence was also demonstrated.

Acknowledgements
Auwal Bala Abubakar acknowledges with thanks the Department of Mathematics and Applied Mathematics at the Sefako Makgatho Health Sciences University. Also, Konrawut Khammahawong acknowledges the financial support provided by The Science, Research and Innovation Promotion Funding (TSRI) (Grant No. FRB660012/0168). This research block grant was managed under Rajamangala University of Technology Thanyaburi (FRB66E0654S.3).

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