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ITERATIVE ALGORITHMS FOR VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS OF PSEUDOCONTRACTION WITHOUT LIPSCHITZ ASSUMPTIONS

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Abstract. In this paper, by coupling the modified extra-gradient method with the Mann iteration method, we solve variational inequalities and the fixed point problems of pseudo-contractive mappings without Lipschitz assumptions. Weak and strong convergence theorems are established.

Keywords. Extra-gradient method; Fixed point problem; Pseudo-contractive mapping; Variational inequality problem.

1. Introduction

Finding a point $u^* \in \mathscr{C}$ such that

$$\langle \mathscr{A} u^*, u - u^* \rangle \geq 0, \ \forall u \in \mathscr{C}, \tag{1.1}$$

where $\mathscr{C}$ is a nonempty, convex, and closed subset of a Hilbert space $\mathscr{H}$ and $\mathscr{A} : \mathscr{C} \to \mathscr{H}$ is a nonlinear mapping, this problem is called the variational inequality problem $VIP(\mathscr{C}, \mathscr{A})$ (the solutions of this problem is also represented by this symbol) [13], which was introduced by Stampacchia and further studied by numerous scholars; see, e.g., [2, 5, 6, 9, 12] and the references therein. For solving the $VIP(\mathscr{C}, \mathscr{A})$ (1.1), a practical and effective approach is using the projection

$$u^* = \mathscr{P}_{\mathscr{C}} (u^* - \lambda \mathscr{A} u^*), \ \forall u^* \in VIP(\mathscr{C}, \mathscr{A}), \forall \lambda > 0. \tag{1.2}$$

This equivalent formulation of the $VIP(\mathscr{C}, \mathscr{A})$ plays a pivotal role in structuring various iterative algorithms to solve the $VIP(\mathscr{C}, \mathscr{A})$ and related optimization problems; see, e.g., [1, 3, 8] and the references therein. Korpelevich [7] suggested the following iterative algorithm to solve

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the $VIP(\mathscr{C},\mathscr{A})$ (1.1), the extra-gradient method in $\mathbb{R}^n$,

$$\begin{cases} v_k = \mathscr{P}_{\mathscr{C}}\left(u_k - \lambda \mathscr{A} u_k\right), \\ u_{n+1} = \mathscr{P}_{\mathscr{C}}\left(u_k - \lambda \mathscr{A} v_k\right), \quad k \geq 1, \end{cases}$$

where $\mathscr{C}$ is a nonempty, convex, and closed subset of $\mathbb{R}^n$ and $\mathscr{A}$ is a monotone, Lipschitz continuous mapping of $\mathscr{C}$ into $\mathbb{R}^n$.

Nadezhkina and Takahashi [11] introduced the Mann type extra-gradient method in a Hilbert space $\mathscr{H}$,

$$\begin{cases} v_k = \mathscr{P}_{\mathscr{C}}\left(u_k - \lambda_k \mathscr{A} u_k\right), \\ u_{k+1} = \alpha_k u_k + (1-\alpha_k)\mathscr{T}\mathscr{P}_{\mathscr{C}}\left(u_k - \lambda_k \mathscr{A} v_k\right), \quad k \geq 1, \end{cases} \tag{1.3}$$

where $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is a nonexpansive mapping and $\mathscr{A}:\mathscr{C}\to\mathscr{H}$ is a monotone, Lipschitz continuous mapping. They proved that the sequence $\{x_k\}$ generated by (1.3) converges weakly to a point of $Fix(\mathscr{T})\cap VIP(\mathscr{C},\mathscr{A})$. Subsequently, Yao, Liou, and Yang [15] suggested a new algorithm which coupled the modified extra-gradient method withthe Mann iteration to solve $VIP(\mathscr{C},\mathscr{A})$ (1.1) and the fixed point problem in Hilbert space $\mathscr{H}$

$$\begin{cases} v_k = \mathscr{P}_{\mathscr{C}}\left(u_k - \lambda_k \mathscr{A} u_k\right), \\ w_k = \mathscr{P}_{\mathscr{C}}\left((1-\alpha_k)(v_k - \lambda_k \mathscr{A} v_k)\right), \\ u_{k+1} = \beta u_k + (1-\beta)\mathscr{T} w_k, \quad k \geq 1, \end{cases} \tag{1.4}$$

where $\mathscr{A}$ is an α-inverse strongly monotone mapping, and $\mathscr{T}$ is a nonexpansive mapping. Strong convergence was obtained.

Motivated by iterative algorithms (1.2), (1.3), and (1.4), we in this paper extend these results from nonexpansive mappings to pseudo-contractive mapping incursion in non-Lipschitz assumption and propose two iterative algorithms to solve $VIP(\mathscr{C},\mathscr{A})$ (1.1) and fixed point problem, weak and strong convergence of the suggested algorithms are estimated.

In this paper, by coupling the modified extra-gradient method with the Mann iteration method, we solve variational inequalities and the fixed point problems of pseudo-contractive mappings without Lipschitz assumptions. Weak and strong convergence theorems are established.

2. Preliminaries

The inner product and norm in Hilbert space $\mathscr{H}$ are expressed as $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. The set $Fix(\mathscr{T})=\{u\in\mathscr{H}:\mathscr{T}u=u\}$ means the set of fixed points of the nonlinear mapping $\mathscr{T}$. For any $u\in\mathscr{H}$, the projector $\mathscr{P}_{\mathscr{C}}(\cdot)$, from $\mathscr{H}$ onto $\mathscr{C}$, is defined as

$$\|u-\mathscr{P}_{\mathscr{C}}u\| = \min\{\|u-v\| : v\in\mathscr{C}\}.$$

From the relation between inner product and norm in Hilbert spaces, we summarize the properties of projections as follows.

Proposition 2.1. *Given $u\in\mathscr{H}$ and $w\in\mathscr{C}$.*
(1) $w=\mathscr{P}_{\mathscr{C}}u \Leftrightarrow \langle u-w, v-w\rangle \leq 0$ for all $v\in\mathscr{C}$.
(2) $w=\mathscr{P}_{\mathscr{C}}u \Leftrightarrow \|u-w\|^2 \leq \|u-v\|^2 - \|v-w\|^2$ for all $v\in\mathscr{C}$.
(3) $\langle u-v, \mathscr{P}_{\mathscr{C}}u-\mathscr{P}_{\mathscr{C}}v\rangle \geq \|\mathscr{P}_{\mathscr{C}}u-\mathscr{P}_{\mathscr{C}}v\|^2$ for all $v\in\mathscr{H}$, which hence implies that $\mathscr{P}_{\mathscr{C}}$ is nonexpansive.

A nonlinear mapping $\mathscr{T}:\mathscr{H}\to\mathscr{H}$ is said to be
(1) L-Lipschitzian if there exists $L>0$ such that

$$\|\mathscr{T}u-\mathscr{T}v\|\leq L\|u-v\|,\ \forall u,v\in\mathscr{H};$$

If L=1, we call $\mathscr{T}$ nonexpansive.
(2) monotone if

$$\langle u-v,\mathscr{T}u-\mathscr{T}v\rangle\geq 0,\ \forall u,v\in\mathscr{H};$$

(3) β-strongly monotone with $\beta>0$ if

$$\langle u-v,\mathscr{T}u-\mathscr{T}v\rangle\geq \beta\|u-v\|^2,\ \forall u,v\in\mathscr{H};$$

(4) μ-inverse strongly monotone with $\mu>0$ if

$$\langle u-v,\mathscr{T}u-\mathscr{T}v\rangle\geq \mu\left\|\mathscr{T}u-\mathscr{T}v\right\|^2,\ \forall u,v\in\mathscr{H}.$$

(5) pseudo-contractive if

$$\langle u-v,\mathscr{T}u-\mathscr{T}v\rangle\leq\|u-v\|^2,\ \forall u,v\in\mathscr{C}.$$

From the definition of pseudo-contractive mapping $\mathscr{T}$, one has the following proposition [4]: $\langle\mathscr{T}v-v,\mathscr{T}v-u^*\rangle\leq\left\|\mathscr{T}v-v\right\|^2,\ \forall(u,u^*)\in\mathscr{H}\times Fix(\mathscr{T})$. To avoid the L-Lipschitzian property, the following assumption is needed in this paper

$$\langle\mathscr{T}v-v,\mathscr{T}v-u^*\rangle\leq 0,\ \forall(u,u^*)\in\mathscr{H}\times Fix(\mathscr{T}). \tag{2.1}$$

Lemma 2.2. [16] *Let $\mathscr{H}$ be a real Hilbert space, and let $\mathscr{C}$ a convex and closed subset of $\mathscr{H}$. Let $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ be a continuous pseudo-contractive mapping. Then*
(1) $Fix(\mathscr{T})$ is a closed convex subset of $\mathscr{C}$,
(2) $(I-\mathscr{T})$ is demiclosed at zero.

Lemma 2.3. [14] *Let $\{a_k\}$ be a nonnegative real sequence satisfying the relation*

$$a_{k+1}\leq(1-\gamma_k)a_k+\sigma_k,\ k\geq 0,$$

where $\{\gamma_k\}\subset$ (0,1) and $\{\sigma_k\}$ are such that
(1) $\sum_{k=0}^{\infty}\gamma_k=\infty$;
(2) either $\limsup_{k\to\infty}\frac{\sigma_k}{\gamma_k}\leq 0$ or $\sum_{k=0}^{\infty}|\sigma_k|<\infty$.
Then $\{a_k\}$ converges to zero.

Lemma 2.4. [10] *Let $\{t_k\}$ be a real sequence. Suppose that there exists at least a subsequence $\{t_{k_i}\}$ of $\{t_k\}$ such that $t_{k_i}\leq t_{k_i+1}$ for all $i\geq 0$. For every $k\geq N_0$, one defines an $\{\sigma(k)\}$ as*

$$\sigma(k)=\max\{i\leq k:t_{k_i}<t_{k_i+1}\}.$$

Then $\sigma(k)\to\infty$ as $k\to\infty$ and for all $k\geq N_0$, $\max\{t_{\sigma(k)},t_k\}\leq t_{\sigma(k)+1}$.

The set of solutions of variational inequality and fixed point problems are denoted by $\mathscr{S}=Fix(\mathscr{T})\cap VIP(\mathscr{C},\mathscr{A})\neq\emptyset$.

Next, we present the weak and strong convergence of the proposed algorithms in the following two sections, respectively.

3. WEAK CONVERGENCE ANALYSIS

In this section, we introduce our first algorithm to solve the variational inequality and fixed point problems and analyze its weak convergence.

Theorem 3.1. *Let $\mathscr{C}$ be a nonempty convex, and closed subset of a real Hilbert space $\mathscr{H}$. Let $\mathscr{A}:\mathscr{C}\to\mathscr{H}$ be a μ-inverse strongly monotone mapping, and let $\mathscr{T}$ be a continuous pseudo-contractive mapping without Lipschitz assumption from $\mathscr{C}$ to itself. For $u_1\in\mathscr{C}$ arbitrarily, define a sequence $\{u_k\}$*

$$\begin{cases} v_k = \mathscr{P}_{\mathscr{C}}(u_k-\alpha_k\mathscr{A}u_k), \\ u_{k+1}=(1-\beta_k)v_k+\beta_k\mathscr{T}v_k, \quad k\geq 1, \end{cases} \tag{3.1}$$

where $\{\beta_k\}$ is a sequence in $(0,1)$ satisfying $\inf_k\beta_k(1-\beta_k)>0$ and $\{\alpha_k\}$ is a sequence in $(0,2\mu)$. Then $\{u_k\}$ yielded by (3.1) converges weakly to u, a point belongs to $\mathscr{S}$.

Proof. Taking $u^*\in\mathscr{S}$, we have $u^*\in VIP(\mathscr{C},\mathscr{A})$. We obtain $u^*=\mathscr{P}_{\mathscr{C}}(u^*-\alpha Au^*)$ from (1.2) for all $\alpha>0$. From the definition of $\mathscr{A}$, the definition of $\{\alpha_k\}$, and the nonexpansivity of projector $\mathscr{P}_{\mathscr{C}}$, we find

$$\begin{aligned} \|v_k-u^*\|^2 \leq& \|(I-\alpha_k\mathscr{A})u_k-(I-\alpha_k\mathscr{A})u^*\|^2 \\ =&\|u_k-u^*\|^2-2\alpha_k\langle u_k-u^*,\mathscr{A}u_k-\mathscr{A}u^*\rangle+\alpha_k^2\|\mathscr{A}u_k-\mathscr{A}u^*\|^2 \\ \leq&\|u_k-u^*\|^2-\alpha_k(2\mu-\alpha_k)\|\mathscr{A}u_k-\mathscr{A}u^*\|^2 \\ \leq&\|u_k-u^*\|^2. \end{aligned} \tag{3.2}$$

Thus, the nonexpansivity of $I-\alpha_k\mathscr{A}$ is given by (3.2). By (2.1), (3.1), and condition $\inf_k\beta_k(1-\beta_k)>0$, we have

$$\begin{aligned} \|u_{k+1}-u^*\|^2 =&(1-\beta_k)\|v_k-u^*\|^2+\beta_k\|\mathscr{T}v_k-u^*\|^2-\beta_k(1-\beta_k)\|v_k-\mathscr{T}v_k\|^2 \\ =&(1-\beta_k)\|v_k-u^*\|^2+\beta_k\langle\mathscr{T}v_k-v_k,\mathscr{T}v_k-u^*\rangle \\ &+\beta_k\langle v_k-u^*,\mathscr{T}v_k-u^*\rangle-\beta_k(1-\beta_k)\|v_k-\mathscr{T}v_k\|^2 \\ \leq&\|v_k-u^*\|^2-\beta_k(1-\beta_k)\|v_k-\mathscr{T}v_k\|^2 \\ \leq&\|v_k-u^*\|^2. \end{aligned} \tag{3.3}$$

From (3.2), we attain $\|u_{k+1}-u^*\|\leq\|u_k-u^*\|$. Therefore, $\{u_k\}$ is bounded, so is $\{v_k\}$. Back-tracking to (3.2) and (3.3), we deduce

$$\alpha_k(2\mu-\alpha_k)\|\mathscr{A}u_k-\mathscr{A}u^*\|^2+\beta_k(1-\beta_k)\|v_k-\mathscr{T}v_k\|^2\leq\|u_k-u^*\|^2-\|u_{k+1}-u^*\|^2.$$

By the restrictions, we have

$$\lim_{n\to\infty}\|\mathscr{A}u_k-\mathscr{A}u^*\|=\lim_{n\to\infty}\|v_k-\mathscr{T}v_k\|=0. \tag{3.4}$$

By Proposition 2.1(3), we conclude from (3.1) that

$$\begin{aligned}
&\|v_k - u^*\|^2 \\
\leq &\langle (u_k - \alpha_k \mathscr{A} u_k) - (u^* - \alpha_k \mathscr{A} u^*), v_k - u^* \rangle \\
= &\frac{1}{2}\left(\|(u_k - \alpha_k \mathscr{A} u_k) - (u^* - \alpha_k \mathscr{A} u^*)\|^2 + \|v_k - u^*\|^2 - \|(u_k - v_k) - \alpha_k(\mathscr{A} u_k - \mathscr{A} u^*)\|^2 \right) \\
\leq &\frac{1}{2}\left(\|u_k - u^*\|^2 + \|v_k - u^*\|^2 - \|(u_k - v_k) - \alpha_k(\mathscr{A} u_k - \mathscr{A} u^*)\|^2 \right) \\
\leq &\frac{1}{2}\left(\|u_k - u^*\|^2 + \|v_k - u^*\|^2 - \|u_k - v_k\|^2 + 2\alpha_k \|u_k - v_k\| \|\mathscr{A} u_k - \mathscr{A} u^*\| \right),
\end{aligned}$$

which turns out to be

$$\|v_k - u^*\|^2 \leq \|u_k - u^*\|^2 - \|u_k - v_k\|^2 + 2\alpha_k \|u_k - v_k\| \|\mathscr{A} u_k - \mathscr{A} u^*\|,$$

Again, from (3.3), we attain

$$\|u_{k+1} - u^*\|^2 \leq \|v_k - u^*\|^2 \leq \|u_k - u^*\|^2 - \|u_k - v_k\|^2 + 2\alpha_k \|u_k - v_k\| \|Au_k - Au^*\|.$$

Therefore,

$$\|u_k - v_k\|^2 \leq \|u_k - u^*\|^2 - \|u_{k+1} - u^*\|^2 + 2\alpha_k \|u_k - v_k\| \|\mathscr{A} u_k - \mathscr{A} u^*\|.$$

From (3.4), we obtain that

$$\lim_{k\to\infty} \|u_k - v_k\| = 0. \tag{3.5}$$

Thus, we can choose a subsequence $\{v_{k_i}\}$ of $\{v_k\}$ converges weakly to u for the boundedness of $\{v_k\}$. By (3.4) and Lemma 2.2, we achieve $u \in Fix(\mathscr{T})$.

On the other hand, define $\mathscr{U} x = \begin{cases} \mathscr{A} x + N_{\mathscr{C}} x, & x \in \mathscr{C}; \\ \emptyset, & x \notin \mathscr{C}. \end{cases}$ Then $\mathscr{U}$ is maximal monotone. Putting $(x,y) \in G(\mathscr{U})$, we obtain $\langle x - v_k, y - \mathscr{A} x \rangle \geq 0$ for $y - \mathscr{A} x \in N_{\mathscr{C}} x$ and $v_k \in \mathscr{C}$. Also from $v_k \in \mathscr{C}$, we have

$$\langle x - v_k, v_k - (u_k - \alpha_k \mathscr{A} u_k) \rangle \geq 0,$$

that is, $\left\langle x - v_k, \frac{v_k - u_k}{\alpha_k} + \mathscr{A} u_k \right\rangle \geq 0$. So,

$$\begin{aligned}
\langle x - v_{k_i}, y \rangle &\geq \langle x - v_{k_i}, \mathscr{A} x \rangle \\
&\geq \langle x - v_{k_i}, \mathscr{A} x \rangle - \left\langle x - v_{k_i}, \frac{v_{k_i} - u_{k_i}}{\alpha_{k_i}} + \mathscr{A} u_{k_i} \right\rangle \\
&= \langle x - v_{k_i}, \mathscr{A} x - \mathscr{A} v_{k_i} \rangle + \langle x - v_{k_i}, \mathscr{A} v_{k_i} - \mathscr{A} u_{k_i} \rangle - \left\langle x - v_{k_i}, \frac{v_{k_i} - u_{k_i}}{\alpha_{k_i}} \right\rangle \\
&\geq \langle x - v_{k_i}, \mathscr{A} v_{k_i} - \mathscr{A} u_{k_i} \rangle - \left\langle x - v_{k_i}, \frac{v_{k_i} - u_{k_i}}{\alpha_{k_i}} \right\rangle.
\end{aligned}$$

From (3.5), we reach $\langle x - u, y \rangle \geq 0$. Since $\mathscr{U}$ is maximal monotone, we obtain $u \in \mathscr{U}^{-1}(0)$ and then $u \in VIP(\mathscr{C}, \mathscr{A})$. Thus, we have $u \in \mathscr{S}$. This completes the proof. □

4. Strong Convergence Analysis

We now modify algorithm (3.1) to obtain the strong convergence result.

Theorem 4.1. *Let $\mathscr{C}$ be a nonempty, convex, and closed subset of a real Hilbert space $\mathscr{H}$. Let $\mathscr{A}:\mathscr{C}\to\mathscr{H}$ be a μ-inverse strongly monotone mapping, and let $\mathscr{T}$ be a pseudo-contractive mapping without the Lipschitz assumption from $\mathscr{C}$ to itself. For $u_1\in\mathscr{C}$ arbitrarily, define a sequence $\{u_k\}$*

$$\begin{cases} v_k=\mathscr{P}_{\mathscr{C}}\left(u_k-\gamma_k\mathscr{A}u_k\right), \\ w_k=\mathscr{P}_{\mathscr{C}}\left((1-\delta_k)(v_k-\gamma_k\mathscr{A}v_k)\right), \\ u_{k+1}=(1-\eta_k)w_k+\eta_k\mathscr{T}w_k, \quad k\geq 1, \end{cases} \tag{4.1}$$

where $\{\gamma_k\}$, $\{\delta_k\}$, and $\{\eta_k\}$ are three sequences in $(0,1)$ satisfying $\inf_k\eta_k(1-\eta_k)>0$, $\inf_k\gamma_k(1-\delta_k)>0$, $\sum_{k=0}^{\infty}\delta_k=\infty$, and $\lim_{k\to\infty}\delta_k=0$, $\{\gamma_k\}$ is a sequence in $(0,2\mu)$. Then $\{u_k\}$ defined by (4.1) converges strongly to $\mathscr{P}_{\mathscr{S}}(0)$.

Proof. Fixing $u^*\in\mathscr{S}$, one has $u^*\in VIP(\mathscr{C},\mathscr{A})$, and then $u^*=\mathscr{P}_{\mathscr{C}}(u^*-\gamma\mathscr{A}u^*)$ from (1.2) for all $\gamma>0$. From (4.1), the definition of $\{\gamma_k\}$ and the nonexpansivity of $\mathscr{P}_{\mathscr{C}}$, proven in Theorem 3.1, we find

$$\|v_k-u^*\|^2\leq\|u_k-u^*\|^2-\gamma_k(2\mu-\gamma_k)\|\mathscr{A}u_k-\mathscr{A}u^*\|^2\leq\|u_k-u^*\|^2. \tag{4.2}$$

Specially, choosing $\gamma=\gamma_k(1-\delta_k)$, we have

$$u^*=\mathscr{P}_{\mathscr{C}}\left(u^*-\gamma_k(1-\delta_k)\mathscr{A}u^*\right)=\mathscr{P}_{\mathscr{C}}\left(\delta_k u^*+(1-\delta_k)(u^*-\gamma_k\mathscr{A}u^*)\right),\ k\geq 1.$$

Since both projector $\mathscr{P}_{\mathscr{C}}$ and $I-\gamma_k\mathscr{A}$ are nonexpansive, we attain

$$\begin{aligned} \|w_k-u^*\|^2\leq&\|(1-\delta_k)(v_k-\gamma_k\mathscr{A}v_k)-(\delta_k u^*+(1-\delta_k)(u^*-\gamma_k\mathscr{A}u^*))\|^2 \\ \leq&\delta_k\|u^*\|^2+(1-\delta_k)\|(I-\gamma_k\mathscr{A})v_k-(I-\gamma_k\mathscr{A})u^*\|^2 \\ \leq&\delta_k\|u^*\|^2+(1-\delta_k)\|v_k-u^*\|^2 \\ \leq&\delta_k\|u^*\|^2+(1-\delta_k)\|u_k-u^*\|^2. \end{aligned} \tag{4.3}$$

By (2.1) and (4.1), we have

$$\begin{aligned} &\|u_{k+1}-u^*\|^2 \\ &=(1-\eta_k)\|w_k-u^*\|^2+\eta_k\|\mathscr{T}w_k-u^*\|^2-\eta_k(1-\eta_k)\|w_k-\mathscr{T}w_k\|^2 \\ &=(1-\eta_k)\|w_k-u^*\|^2+\eta_k\langle\mathscr{T}w_k-w_k,\mathscr{T}w_k-u^*\rangle \\ &\quad+\eta_k\langle w_k-u^*,\mathscr{T}w_k-u^*\rangle-\eta_k(1-\eta_k)\|w_k-\mathscr{T}w_k\|^2 \\ &\leq\|w_k-u^*\|^2-\eta_k(1-\eta_k)\|w_k-\mathscr{T}w_k\|^2. \end{aligned} \tag{4.4}$$

From (4.3) and condition $\inf_k\eta_k(1-\eta_k)>0$, we attain

$$\|u_{k+1}-u^*\|^2\leq\|w_k-u^*\|^2\leq\delta_k\|u^*\|^2+(1-\delta_k)\|u_k-u^*\|^2\leq\max\left\{\|u^*\|^2,\|u_1-u^*\|^2\right\}.$$

Therefore, $\{u_k\}$ is bounded, so are $\{v_k\}$ and $\{w_k\}$. Backtracking to (4.2), (4.3) and (4.4), we deduce

$$\begin{aligned} \|u_{k+1}-u^*\|^2\leq&\delta_k\|u^*\|^2+(1-\delta_k)\|u_k-u^*\|^2-(1-\delta_k)\gamma_k(2\mu-\gamma_k)\|\mathscr{A}u_k-\mathscr{A}u^*\|^2 \\ &-\eta_k(1-\eta_k)\|w_k-\mathscr{T}w_k\|^2. \end{aligned}$$

Thus,

$$\gamma_k(1-\delta_k)(2\mu-\gamma_k)\|\mathscr{A}u_k-\mathscr{A}u^*\|^2+\eta_k(1-\eta_k)\|w_k-\mathscr{T}w_k\|^2$$
$$\leq \|u_k-u^*\|^2-\|u_{k+1}-u^*\|^2+\delta_k\|u^*\|^2.$$

Next, the monotonicity analysis of the sequence $\{\|u_k-u^*\|\}$ is given below.

Case 1. One assumes that there is a positive integer m that makes $\{\|u_k-u^*\|\}$ monotonically decreasing for all $k\geq m$. Thus, the existence of the limit of sequence $\{\|u_k-u^*\|\}$ is confirmed. So

$$\lim_{k\to\infty}\|\mathscr{A}u_k-\mathscr{A}u^*\|=\lim_{k\to\infty}\|w_k-Tw_k\|=0. \tag{4.5}$$

As proven in Theorem 3.1, by Proposition 2.1(3), we conclude from (4.1) that

$$\|v_k-u^*\|^2\leq\frac{1}{2}\left(\|u_k-u^*\|^2+\|v_k-u^*\|^2-\|u_k-v_k\|^2+2\gamma_k\|u_k-v_k\|\|\mathscr{A}u_k-\mathscr{A}u^*\|\right),$$

and

$$\begin{aligned}\|w_k-u^*\|^2\leq&\langle(1-\delta_k)(v_k-\gamma_k\mathscr{A}v_k)-(u^*-\gamma_k\mathscr{A}u^*),w_k-u^*\rangle\\
=&\frac{1}{2}\Big(\|(v_k-\gamma_k\mathscr{A}v_k)-(u^*-\gamma_k\mathscr{A}u^*)-\delta_k(v_k-\gamma_k\mathscr{A}v_k)\|^2+\|w_k-u^*\|^2\\
&-\|(v_k-\gamma_k\mathscr{A}v_k)-(u^*-\gamma_k\mathscr{A}u^*)-(w_k-u^*)-\delta_k(I-\gamma_k\mathscr{A})v_k\|^2\Big)\\
\leq&\frac{1}{2}\Big(\|(v_k-\gamma_k\mathscr{A}v_k)-(u^*-\gamma_k\mathscr{A}u^*)\|^2+\delta_k\mathscr{M}+\|w_k-u^*\|^2\\
&-\|(v_k-w_k)-(\gamma_k\mathscr{A}v_k-\gamma_k\mathscr{A}u^*)-\delta_k(I-\gamma_k\mathscr{A})v_k\|^2\Big)\\
\leq&\frac{1}{2}\left(\|v_k-u^*\|^2+\delta_k\mathscr{M}+\|w_k-u^*\|^2-\|v_k-w_k\|^2\right.\\
&\left.+2\gamma_k\|v_k-w_k\|\|\mathscr{A}v_k-\mathscr{A}u^*\|+2\delta_k\|v_k-w_k\|\|(I-\gamma_k\mathscr{A})v_k\|\right),\end{aligned}$$

where $\mathscr{M}>0$ is a bounded number. It follows that

$$\|v_k-u^*\|^2\leq\|u_k-u^*\|^2-\|u_k-v_k\|^2+2\gamma_k\|u_k-v_k\|\|\mathscr{A}u_k-\mathscr{A}u^*\|,$$

and

$$\begin{aligned}\|w_k-u^*\|^2\leq&\|v_k-u^*\|^2+\delta_kM-\|v_k-w_k\|^2\\
&+2\gamma_k\|v_k-w_k\|\|\mathscr{A}v_k-\mathscr{A}u^*\|+2\delta_k\|v_k-w_k\|\|(I-\gamma_k\mathscr{A})v_k\|\\
\leq&\|u_k-u^*\|^2-\|u_k-v_k\|^2-\|v_k-w_k\|^2+2\gamma_k\|u_k-v_k\|\|\mathscr{A}u_k-\mathscr{A}u^*\|\\
&+\delta_kM+2\gamma_k\|v_k-w_k\|\|\mathscr{A}v_k-\mathscr{A}u^*\|+2\delta_k\|v_k-w_k\|\|(I-\gamma_k\mathscr{A})v_k\|.\end{aligned}$$

Backtracking to (4.4), we deduce

$$\begin{aligned}\|u_{k+1}-u^*\|^2\leq&\|u_k-u^*\|^2-\|u_k-v_k\|^2-\|v_k-w_k\|^2+2\gamma_k\|u_k-v_k\|\|\mathscr{A}u_k-\mathscr{A}u^*\|\\
&+\delta_kM+2\gamma_k\|v_k-w_k\|\|\mathscr{A}v_k-\mathscr{A}u^*\|+2\delta_k\|v_k-w_k\|\|(I-\gamma_k\mathscr{A})v_k\|.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|u_k-v_k\|^2+\|v_k-w_k\|^2\\
&\leq\|u_k-u^*\|^2-\|u_{k+1}-u^*\|^2+2\gamma_k\|u_k-v_k\|\|\mathscr{A}u_k-\mathscr{A}u^*\|\\
&\quad+\delta_kM+2\gamma_k\|v_k-w_k\|\|\mathscr{A}v_k-\mathscr{A}u^*\|+2\delta_k\|v_k-w_k\|\|(I-\gamma_k\mathscr{A})v_k\|.\end{aligned}$$

From the restrictions, we confirm

$$\lim_{k\to\infty}\|u_k-v_k\|=\lim_{k\to\infty}\|v_k-w_k\|=0. \tag{4.6}$$

Thus, we can choose a subsequence $\{w_{k_i}\}$ of $\{w_k\}$ converges weakly to w for the boundedness of $\{w_k\}$. By (4.5) and Lemma 2.2, we observe that $w\in Fix(\mathscr{T})$.

On the other hand, we define a maximally monotone operator $\mathscr{V}x=\begin{cases}\mathscr{A}x+N_{\mathscr{C}}x, & x\in\mathscr{C};\\ \emptyset, & x\notin\mathscr{C}.\end{cases}$ Putting $(x,y)\in G(\mathscr{V})$, we obtain that $\langle x-w_k,y-\mathscr{A}x\rangle\geq 0$ for $y-\mathscr{A}x\in N_{\mathscr{C}}x$ and $w_k\in\mathscr{C}$. From $w_k\in\mathscr{C}$, we also have

$$\langle x-w_k,w_k-(1-\delta_k)(v_k-\gamma_k\mathscr{A}v_k)\rangle\geq 0,$$

that is,

$$\left\langle x-w_k,\frac{w_k-v_k}{\gamma_k}+\mathscr{A}v_k+\frac{\delta_k}{\gamma_k}(v_k-\gamma_k\mathscr{A}v_k)\right\rangle\geq 0.$$

So,

$$\begin{aligned}
&\langle x-w_{k_i},y\rangle\\
\geq&\langle x-w_{k_i},\mathscr{A}x\rangle\\
\geq&\langle x-w_{k_i},\mathscr{A}x\rangle-\left\langle x-w_{k_i},\frac{w_{k_i}-v_{k_i}}{\gamma_{k_i}}+\mathscr{A}v_{k_i}+\frac{\delta_{k_i}}{\gamma_{k_i}}(v_{k_i}-\gamma_{k_i}\mathscr{A}v_{k_i})\right\rangle\\
=&\langle x-w_{k_i},\mathscr{A}x-\mathscr{A}w_{k_i}\rangle+\langle x-w_{k_i},\mathscr{A}w_{k_i}-\mathscr{A}v_{k_i}\rangle-\left\langle x-w_{k_i},\frac{w_{k_i}-v_{k_i}}{\gamma_{k_i}}+\frac{\delta_{k_i}}{\gamma_{k_i}}(v_{k_i}-\gamma_{k_i}\mathscr{A}v_{k_i})\right\rangle\\
\geq&\langle x-w_{k_i},\mathscr{A}w_{k_i}-\mathscr{A}v_{k_i}\rangle-\left\langle x-w_{k_i},\frac{w_{k_i}-v_{k_i}}{\gamma_{k_i}}+\frac{\delta_{k_i}}{\gamma_{k_i}}(v_{k_i}-\gamma_{k_i}\mathscr{A}v_{k_i})\right\rangle.
\end{aligned}$$

From (4.6), we gain $\langle x-w,y\rangle\geq 0$. Since $\mathscr{V}$ is maximal monotone, we obtain $w\in\mathscr{V}^{-1}(0)$ and then $w\in VIP(\mathscr{C},\mathscr{A})$. Thus $w\in\mathscr{S}$ and

$$\begin{aligned}
&\limsup_{k\to\infty}\langle\mathscr{P}_{\mathscr{S}}(0),\mathscr{P}_{\mathscr{S}}(0)-w_k\rangle\\
&=\limsup_{i\to\infty}\langle\mathscr{P}_{\mathscr{S}}(0),\mathscr{P}_{\mathscr{S}}(0)-w_{k_i}\rangle\\
&=\langle\mathscr{P}_{\mathscr{S}}(0),\mathscr{P}_{\mathscr{S}}(0)-w\rangle\leq 0.
\end{aligned}$$

Finally, by the nonexpansivity of $I-\gamma_k\mathscr{A}$, one sees that

$$\begin{aligned}
&\|w_k-\mathscr{P}_{\mathscr{S}}(0)\|^2\\
=&\|\mathscr{P}_{\mathscr{C}}((1-\delta_k)(v_k-\gamma_k\mathscr{A}v_k))-\mathscr{P}_{\mathscr{C}}(\delta_k\mathscr{P}_{\mathscr{S}}(0)+(1-\delta_k)(\mathscr{P}_{\mathscr{S}}(0)-\gamma_k\mathscr{A}\mathscr{P}_{\mathscr{S}}(0)))\|^2\\
\leq&\langle\delta_k(-\mathscr{P}_{\mathscr{S}}(0))+(1-\delta_k)\left((v_k-\gamma_k\mathscr{A}v_k)-(\mathscr{P}_{\mathscr{S}}(0)-\gamma_k\mathscr{A}\mathscr{P}_{\mathscr{S}}(0))\right),w_k-\mathscr{P}_{\mathscr{S}}(0)\rangle\\
\leq&\delta_k\langle\mathscr{P}_{\mathscr{S}}(0),\mathscr{P}_{\mathscr{S}}(0)-w_k\rangle+(1-\delta_k)\|v_k-\mathscr{P}_{\mathscr{S}}(0)\|\|w_k-\mathscr{P}_{\mathscr{S}}(0)\|\\
\leq&\delta_k\langle\mathscr{P}_{\mathscr{S}}(0),\mathscr{P}_{\mathscr{S}}(0)-w_k\rangle+\frac{1-\delta_k}{2}\left(\|v_k-\mathscr{P}_{\mathscr{S}}(0)\|^2+\|w_k-\mathscr{P}_{\mathscr{S}}(0)\|^2\right),
\end{aligned}$$

which together with (4.4) results that

$$\|u_{k+1}-\mathscr{P}_{\mathscr{S}}(0)\|^2\leq(1-\delta_k)\|u_k-\mathscr{P}_{\mathscr{S}}(0)\|^2+2\delta_k\langle\mathscr{P}_{\mathscr{S}}(0),\mathscr{P}_{\mathscr{S}}(0)-w_k\rangle. \tag{4.7}$$

Since inequality (4.7) satisfies the requirements of Lemma 2.3, we can deduce that $\{u_n\}$ converges strongly to $\mathscr{P}_{\mathscr{S}}(0)$.

Case 2. One supposes that there is an integer n such that

$$\|u_n - \mathscr{P}_{\mathscr{S}}(0)\| \leq \|u_{n+1} - \mathscr{P}_{\mathscr{S}}(0)\|.$$

Setting $t_n = \{\|u_n - \mathscr{P}_{\mathscr{S}}(0)\|\}$, by the assumption above, we confirm $t_n \leq t_{n+1}$. We now define a sequence $\{\sigma(k)\}$ by $\sigma(k) = \max\{l \geq 1 : n \leq l \leq k, t_l \leq t_{l+1}\}$, $k \geq n$. We see that $\{\sigma(k)\}$ is a nondecreasing sequence satisfying $\lim_{k\to\infty} \sigma(k) = \infty$ and $t_{\sigma(k)} \leq t_{\sigma(k)+1}$, $k \geq n$. As discussed in Case 1, we derive that

$$\lim_{k\to\infty} \|u_{\sigma(k)} - v_{\sigma(k)}\| = \lim_{k\to\infty} \|v_{\sigma(k)} - w_{\sigma(k)}\| = \lim_{k\to\infty} \|w_{\sigma(k)} - \mathscr{T} w_{\sigma(k)}\| = 0,$$

and every weak cluster point of sequence $\{w_{\sigma(k)}\}$ is the solution of $VIP(\mathscr{C}, \mathscr{A})$. So, the set of weak cluster points of the sequence $\{w_{\sigma(k)}\}$ is a subset of $\mathscr{S}$. Thus,

$$\limsup_{k\to\infty} \left\langle \mathscr{P}_{\mathscr{S}}(0), \mathscr{P}_{\mathscr{S}}(0) - w_{\sigma(k)} \right\rangle \leq 0.$$

By $t_{\sigma(k)} \leq t_{\sigma(k)+1}$ and (4.7), we obtain

$$t^2_{\sigma(k)} \leq t^2_{\sigma(k)+1} \leq (1-\delta_{\sigma(k)})t^2_{\sigma(k)} + 2\delta_k \langle \mathscr{P}_{\mathscr{S}}(0), \mathscr{P}_{\mathscr{S}}(0) - w_{\sigma(k)} \rangle. \tag{4.8}$$

It turns out to be $t^2_{\sigma(k)} \leq 2\langle \mathscr{P}_{\mathscr{S}}(0), \mathscr{P}_{\mathscr{S}}(0) - w_{\sigma(k)} \rangle$. So, $\limsup_{k\to\infty} t_{\sigma(k)} \leq 0$, and hence $\lim_{k\to\infty} t_{\sigma(k)} = 0$. From (4.8), we conclude $\limsup_{k\to\infty} t^2_{\sigma(k)+1} \leq \limsup_{k\to\infty} t^2_{\sigma(k)}$, and hence $\lim_{n\to\infty} t_{\sigma(k)+1} = 0$. By Lemma 2.4, we have $0 \leq t_k \leq \max\{t_{\sigma(k)}, t_{\sigma(k)+1}\}$. Therefore, $\{t_k\}$ converges strongly to 0. That is, $\{u_k\}$ converges strongly to $\mathscr{P}_{\mathscr{S}}(0)$. This completes the proof. □

5. Numerical Example

In this section, we give a numerical example to illustrate the theoretical results.

Let $\mathscr{H} = \mathbb{R}$ and $\mathscr{C} = [0, +\infty)$. Define a self-mapping function $\mathscr{T} u = u - 1 + \frac{1}{u+1}$ for all $u \in C$. Clearly, $Fix(\mathscr{T}) = \{0\}$ and

$$\begin{aligned}\langle \mathscr{T} u - \mathscr{T} v, u - v\rangle &= \left\langle u - 1 + \frac{1}{u+1} - v + 1 - \frac{1}{v+1}, u - v \right\rangle \\ &\leq \left(1 - \frac{1}{(u+1)(v+1)}\right) |u-v|^2 \\ &\leq |u-v|^2,\end{aligned}$$

so, $\mathscr{T}$ is a pseudo-contractive mapping. Furthermore, $\mathscr{T}$ meets condition (2.1), that is,

$$\langle \mathscr{T} u - u, \mathscr{T} u - 0\rangle \leq -\frac{u^3}{(u+1)^2} \leq 0, \quad \forall u \in \mathscr{C}.$$

Let $Au = \frac{1}{2}u$ for all $u \in \mathscr{C}$. Let $\gamma_k = \alpha_k = 1$, $\delta_k = \frac{2}{3}$, and $\beta_k = \eta_k = \frac{1}{n}$. We rewrite (3.1) and (4.1) as follows, respectively,

$$\begin{cases} v_k = \mathscr{P}_C\left(\frac{1}{2}u_k\right), \\ u_{k+1} = v_k - \frac{1}{n} + \frac{1}{nv_k+n}, \ k \geq 1, \end{cases} \tag{5.1}$$

and

$$\begin{cases} v_k = \mathscr{P}_C\left(\frac{1}{2}u_k\right), \\ w_k = \mathscr{P}_C\left(\frac{1}{6}v_k\right), \\ u_{k+1} = w_k - \frac{1}{n} + \frac{1}{nw_k+n}, \ k \geq 1. \end{cases} \tag{5.2}$$

$\mathscr{A}$, $\mathscr{T}$, α_k, β_k, γ_k, δ_k and η_k satisfy the conditions in Theorem 3.1 and Theorem 4.1, respectively. Since $\{u_k\}$ is defined by (5.1) and (5.2), respectively, from Theorem 3.1 and 4.1, we deduce that $\{u_k\}$ converges strongly to 0. Taking $u_1 = 8$, the Figure 1 illustrates Theorem 3.1 and Theorem 4.1, respectively.

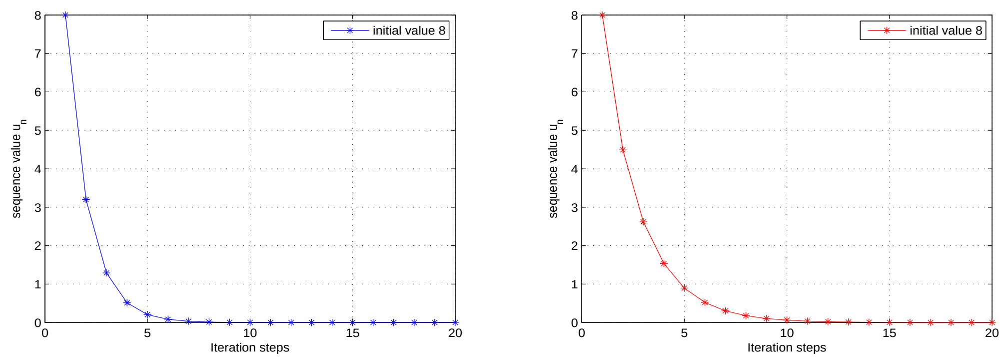

FIGURE 1. The weak (blue) and strong (red) convergence of $\{u_k\}$ with initial value 8, respectively.

Now, let us see the theoretical result in $\mathbb{R}^3$ by Figure 2.

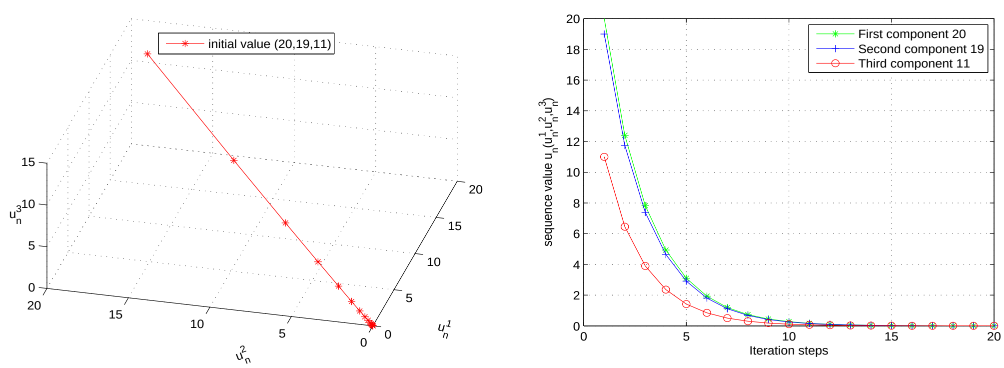

FIGURE 2. The strong convergence of $\{u_k\}$ with initial value (20,19,11).

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