J. Nonlinear Funct. Anal. 2023 (2023) 5 https://doi.org/10.23952/jnfa.2023.5

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

EXISTENCE OF MILD SOLUTIONS OF FRACTIONAL MEASURE EVOLUTION INCLUSIONS

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Abstract. In this paper, we consider a class of new fractional measure evolution inclusion. By using fixed point theorems and noncompactness measure methods, we obtain some sufficient conditions to guarantee the existence of mild solutions and the compactness of solution sets in the space of regulated functions endowed with weak and strong topologies, respectively. Finally, an example is given to illustrate the feasibility of our results.

Keywords. Fixed point theorem; Fractional derivative; Evolution inclusions; Mild solution; Noncompactness measure.

1. Introduction

When studying a large number of evolution processes in real life, like in physics, engineering, biology, or chemistry, one has to face the occurrence of discontinuities in the state, which can be seen as impulses. One way to mathematically describe such systems is offered by the theory of measure differential equations (see, e.g., [12, 16]). On the other hand, when a control is involved, or more complicated phenomena, it is more convenient to consider multivalued functions, i.e., the differential inclusions driven by different measures, also known as the measure differential inclusions. Measure differential inclusions, which encompass as special cases differential and difference equations and impulsive and hybrid problems, are the models of realistic problems arising from economics, optimal control, stochastic analysis. These kinds of problems have sparked much attention in recent years, as demonstrated in, for example, [8, 24].

Fractional calculus has been used extensively in the study of linear and nonlinear fractional differential equations (FDEs) arising from real-world challenges. It has been used to make some problems more approachable, such as the modeling of nonlinear phenomena, optimal control of complex systems, and other scientific research. An important characteristic of a fractional-order differential operator, in contrast to its integer-order counterpart, is its nonlocal

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Received January 27, 2022; Accepted January 7, 2023.

nature. This feature of fractional-order is regarded as one of the key factors in the mathematical modeling of several real world processes gives rise to more realistic models as these operators are capable of describing memory and hereditary properties. There have been many studies of fractional measure differential equations. Most of these equations differ from traditional impulsive differential equations in that infinitely many points of discontinuity in a finite time interval, allowing them to explain non-classical phenomena. Recently, scholars often focus their efforts on the existence and other properties of the solution of many classes of measure differential systems.

In [13], Gu and Sun studied the nonlocal controllability of the following fractional measure evolution equation by using noncompact measure method and fixed point theory:

$$\begin{cases} {}^C D^{\alpha}_{0+} x(t) = Ax(t)dt + (f(t,x(t),Bu(t))dg(t), \quad t \in (0,b], \\ x(0) = x_0 + p(x), \end{cases}$$

where ${}^C D^{\alpha}_{0+}$ is the Caputo fractional derivative, $0 < \alpha < 1$, A is a closed densely defined linear operator, $g : [0,b] \to R$ is a left non-decreasing function, and the control function $u(\cdot)$ takes values in a control set.

In [4], Cichoń and Satco studied the existence of mild solution of the following semilinear measure evolution inclusion by using fixed point theory:

$$\begin{cases} du(t) \in Au(t)dt + F(t,u(t))dg(t), \quad t \in [0,1], \\ u(0) = u_0, \end{cases}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t) : t \geq 0\}$ of contractions on a Banach space X, which is uniformly continuous on $(0,\infty)$ and $g : [0,1] \to R$ is a right non-decreasing function, and $F : [0,1] \times X \to p(X)$, where $p(X)$ is subsets of X.

In continuation to the previously works and in order to expand the compactness of solution sets outcomes to more measure driven problems, we investigate the existence of mild solutions and compactness of solution sets of the following fractional measure evolution inclusion:

$$\begin{cases} {}^C D^{\alpha}_{0+} u(t) \in Au(t)dt + F(t,u(t))dg(t), \quad t \in [0,1], \\ u(0) = u_0, \end{cases} \tag{1.1}$$

where $0 < \alpha < 1$, $A : D(A) \subset X \to X$ is a closed densely defined linear operator, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators (i.e., C_0-semigroup) $\{T(t)\}_{t\geq 0}$ in Banach space X, $g : [0,1] \to R$ is a left continuous and non-decreasing function, and $F : [0,1] \times X \to P(X)$ satisfying some assumptions, where $P(X)$ is subsets of X, $u_0 \in X$. If g is an absolutely continuous function, then (1) becomes a fractional evolution inclusion; if g is the sum of an absolutely continuous function with a step function, then (1) becomes an impulsive fractional evolution inclusion. Therefore, it is interesting and significant to investigate fractional measure evolution inclusion (1), i.e., the inclusion in the main parts of the considered problems is the generator of a equation.

To the best of our knowledge, we have not seen a relevant paper on the existence of mild solutions and the compactness of solution sets for fractional measure differential evolution inclusions of this type. In this paper, the existence results are obtained by the non-compactness measure and fixed point theorem. Moreover, we enlarge the framework where problem (1.1) can be studied from the point of view of existence theory by considering the multivalued case. A first existence is obtained by the non-compactness measure under the appropriate conditions for multifunction F. The second existence result is obtained by working with weak topologies, under an assumption of domination by an weakly compact convex-valued multifunction. The last result is obtained by considering the strong topology, the conditions imposed on the multifunction F being similar to the classical case where $g(t) = t$.

The rest of this paper is organized as follows. In Section 2, some notations and preparation are given. In Section 3, a suitable concept on mild solution for our problem is introduced and some existence results of the mild solution of system (1.1) are discussed. In Section 4, an example is provided to clarify the applicability of our results. In Section 5, the last section, a concluding remark is presented.

2. Preliminaries

In this section, we introduce some notations, definitions, and recall some basic known results which will are used throughout this paper. First, we recall some properties of the measure of noncompactness, and we denote by $\alpha(\cdot)$ the Kuratowski measure of noncompactness of the bounded sets.

Lemma 2.1. [3] *For all bounded subsets B, B_1, and B_2 of X, the following properties hold:*

(1) B is precompact if and only if $\alpha(B) = 0$;

(2) $\alpha(B)$ =$\alpha(\bar{B})$ = $\alpha(convB)$, where $\bar{B}$ and $conv(B)$ mean the closure and convex hull of B, respectively;

(3) $\alpha(B_1) \leq \alpha(B_2)$ if $B_1 \subset B_2$;

(4) $\alpha(B_1 \cup B_2) \leq \max\{\alpha(B_1), \alpha(B_2)\}$;

(5) $\alpha(lB) = |l|\alpha(B)$ for any $l \in R$;

(6) $\alpha(B_1 + B_2) \leq \alpha(B_1) + \alpha(B_2)$, where $B_1 + B_2 = \{x + y; x \in B_1, y \in B_2\}$;

(7) The map $Q : D(Q) \subseteq X \to Y$ is said to be an α-contraction if there exists a positive constant $k < 1$ such that $\alpha(QB) < k\alpha(B)$ for every bounded closed subset $B \subset D$.

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : P(X) \times P(X) \to R \bigcup \{\infty\}$ defined by

$$H_d = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a; b)$ and $d(a, B) = \inf_{b\in B} d(a; b)$. $(P_{cl}(X), H_d)$ is generalized metric space; see [15].

Lemma 2.2. [6] *A multivalued operator $N : X \to P_{cl}(X)$ is called γ-Lipschitz if and only if there exists $\gamma > 0$ such that $H_d(N(x), N(y)) \leq \gamma d(x, y)$ for each $x, y \in X$ and is contraction if and only if it is γ-Lipschitz with $\gamma < 1$, where $P_{cl}(X) = \{Y \in P(X) : Y \text{ is closed}\}$.*

Recall that a function $f : [0, 1] \to X$ satisfies the limits $\lim_{s\to t^-} f(s) = f(t^-)$ and $\lim_{s\to t^+} f(s) = f(t^+), t \in [0, 1]$ is called the regulated function on $[0, 1]$; see [21]. It was proved that the set of discontinuities of a regulated function is at most countable that regulated functions are bounded and the space $G([0, 1], X)$ of regulated functions is a Banach space when endowed with the norm $\|f\| = \sup_{t\in[0,1]} \|f(t)\|$. Recall that the Riemann-Liouville fractional integral of order $\alpha > 0$ for a function f is defined ([19]) as

$$I^{\alpha}_{0+} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t - s)^{1-\alpha}} ds, \alpha > 0,$$

where Γ is the gamma function. The Caputo derivative of order $\alpha > 0$ for a function f is defined ([19]) as

$$^{C}D_{0+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{(n-\alpha-1)}f^n(s)ds, n-1<\alpha<n, n=[\alpha]+1.$$

Next, we recall the definition of Henstock-Lebesgue-Stieljes integral. Note, that Henstock-Lebesgue-Stieltjes has become a tool for studying dynamical systems. We here point out the Henstock-Lebesgue-Stieltjes intergrability implies the Lebesgue-Stieltjes intergrability. For more details about the Henstock-Lebesgue-Stieltjes, we refer to [11] . Recall from [21] that $f : [0,1] \to X$ is called Henstock-Lebesgue-Stieltjes integrable over $[0,1]$ if there is a function denoted by $(HLS)\int_0^{\cdot} : [0,1] \to X$ such that, for given $\varepsilon > 0$, there exists a gauge δ_ε on [0, 1] with

$$\sum_{i=1}^{n}\left\| f(\xi_i)\big(g(t_i)-g(t_{i-1})\big) - \left((HLS)\int_0^{t_i} f(s)dg(s) - (HLS)\int_0^{t_{i-1}} f(s)dg(s)\right)\right\| < \varepsilon$$

for every ε-fine partition $(\xi_i,[t_{i-1},t_i]) : i=1,2,...,n$ of $[0,1]$. Let $HLS_g^p([0,1];X)(p>1)$ be a space of all p-ordered Henstock-Lebesgue-Stieltjes integral regulated functions from $[0,1]$ to X with respect to g with the norm $\|.\|_{HLS_g^p}$ defined by

$$\|f\|_{HLS_g^p} = \big((HLS)\int_0^1 \|f(s)\|^p dg(s)\big)^{\frac{1}{p}}.$$

Lemma 2.3. [13] *Consider the functions $M(t) \in HLS_g^p([0,1];X)(p>1)$ and $g : [0,1] \to R$ satisfying that g is regulated. Then the function*

$$j(t) = (HLS)\int_0^t (t-s)^{\alpha-1}M(s)dg(s), t \in [0,1]$$

is regulated and satisfies

$$j(t)-j(t^-) \le \left(\int_{t^-}^t [(t-s)^{\alpha-1}dg(s)]^{\frac{1}{q}} M(t)(\triangle^- g(t))^{\frac{1}{p}}\right), t\in[0,1],$$

and

$$j(t^+)-j(t) \le \left(\int_t^{t^+} [(t^+-s)^{\alpha-1}dg(s)]^{\frac{1}{q}} M(t)(\triangle^+ g(t))^{\frac{1}{p}}\right), t\in[0,1],$$

where $q>1, \frac{1}{p}+\frac{1}{q}=1, \triangle^+g(t)=g(t^+)-g(t), \triangle^-g(t)=g(t)-g(t^-)$, $g(t^+)$ and $g(t^-)$ denote the right and left limits of function g at point t.

Recall that a set $\mathscr{A} \in G([0,1],X)$ is said to be equi-regulated ([11]) if, for every $\varepsilon > 0$, $u \in \mathscr{A}$, and $t_0 \in [0,1]$, there exists $\delta > 0$ such that (1) for any $t_0-\delta < t' < t_0$, $\|u(t')-u(t_0^-)\| \le \varepsilon$; (2) for any $t_0-\delta < t'' < t_0+\delta$, $\|u(t'')-u(t_0^+)\| \le \varepsilon$. A set K of regulated functions is relatively compact in the space $G([0,1],X)$ ([10]) if and only if it is equi-regulated and for every $t \in [0,1], \{f(t), t\in[0,1]\}$ is relatively compact in X.

Proposition 2.4. [10] *A sequence $(u_n)_n \subset G([0,1],X)$ is weakly convergent to a function $u \in G([0,1],X)$ if and only if is (norm) bounded and for any $t \in [0,1]$ the sequence $(u_n)_n$ is weakly convergent to $u(t)$ in X.*

Proposition 2.5. [7] *Let X be a Banach space, and let μ be a finite measure. Then a bounded, uniformly integrable subset $\mathscr{A}$ of $L^1(\mu,X)$ such that $\mathscr{A}(t)$, a.e.t $\in [0,1]$ is relatively weakly compact in X. Then $\mathscr{A}$ is weakly relatively compact in $L^1(\mu,X)$.*

Lemma 2.6. [9] *(Kakutani-Fan-Glicksberg) Let K be a nonempty, compact, and convex subset of a locally convex Hausdorff space, and let the correspondence $N : K \to 2^K$ have closed graph and nonempty convex values. Then the set of fixed points of N is compact and nonempty.*

Lemma 2.7. [23] *(Eberlein-Smulian) Let K be a subset of a Banach space X. Then for the weak topology of X the following are equivalent:*

(1) The subset K is conditionally compact;

(2) The subset K is conditionally sequentially compact;

(3) The subset K is conditionally countably compact.

3. Main Results

In this section, by using fixed point theorems and noncompactness measure methods, we obtain sufficient conditions to ensure the existence of the mild solution of system (1.1). Throughout this paper, we assume that

(H1) $T(t)$ be a C_0-semigroup of contractions, uniformly continuous on $(0,\infty)$,

(H2) There exist a constant $q_1 \in (0,\alpha)$ and a compact convex set $S \subset X$ and $M(t) \in HLS_g^{\frac{1}{q_1}}([0,1], R^+)$ such that $F(t,x) \subset M(t)S$ for a.e. $t \in [0,1]$, $x \in X$.

Referring to the definition of mild solutions, given in see [13], we define the mild solution for fractional measure evolution inclusion (1.1) in the space of regulated function $G([0,1],X)$ as follows. Given $u \in G([0,1],X)$, we denote $Sel_F(u) = \Big\{ f \in L^{\frac{1}{q_1}}([0,1],X) : f(t) \in F(t,u(t))$, for a.e. $t \in [0,1] \Big\}$.

Definition 3.1. [13] A regulated function $u : [0,1] \to X$ is called a mild solution on [0,1] of problem (1.1) if there exists $f \in Sel_F(u)$ and satisfies the following integral equation: $u(t) = T_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s)dg(s)$, for $t \in [0,1]$, where $T_\alpha = \int_0^\infty \eta_\alpha(\theta)T(t^\alpha\theta)d\theta, S_\alpha = \alpha\int_0^\infty \theta\eta_\alpha(\theta)T(t^\alpha\theta)d\theta$, and η_α is probablity density function defined on $(0,\infty)$, that is, $\eta_\alpha(t) \geq 0, \theta \in (0,\infty)$ and $\int_0^\infty \eta_\alpha(\theta)d\theta = 1$.

Lemma 3.2. [17] *T_α and S_α have the following properties:*

(1) for every fixed $t \geq 0$, operators T_α and S_α are linear and bounded, i.e., for each $x \in X$, there is a positive constant M_1 such that

$$\|T_\alpha(t)x\| \leq M_1\|x\|, \|S_\alpha(t)x\| \leq \frac{M_1}{\Gamma(\alpha)}\|x\|;$$

(2) T_α and S_α are strongly continuous operators;

(3) for every $x \in X$, $t \to T_\alpha(t)x$ and $t \to S_\alpha(t)x$ are continuous functions from $[0,\infty) \to X$.

Theorem 3.3. *If the assumptions (H1) and the following conditions:*

(H3) $F : [0,1]\times X \to P_{kc}(X)$ is such that $F(\cdot,x)$ is measurable for every $x \in X$,

(H4) $H_d\big(F(t,x),F(t,\bar{x})\big) \leq \gamma(t)\big|x-\bar{x}\big|$ for almost all $t \in [0,1]$ and $x,\bar{x} \in X$, with $\gamma(t) \in HLS_g^{\frac{1}{q_1}}([0,1],R^+)$ and $d\big(0,F(t,0)\big) \leq \gamma(t)$ for almost all $t \in [0,1]$,

(H5) there exists a constant $r > 0$ such that $M_r(t) \in L^{\frac{1}{q_1}}([0,1],R^+)$, and for any $x \in X$, $\|x\| \leq r$: $F(t,x) \subset M_r(t)S$ for a.e. $t \in [0,1]$ are satisfied.

Then system (1.1) *has a mild solution provided that* $\tilde{a}\cdot\tilde{b}<1$, *where*

$$\tilde{a}=\sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t((t-s)^{\alpha-1})^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}$$

and

$$\tilde{b}=\max\left\{\|M_r\|_{HLS_g^{\frac{1}{q_1}}},\|\gamma\|_{HLS_g^{\frac{1}{q_1}}}\right\}.$$

Proof. The proof process is divided into three steps. First, define the operator N by

$$N(u)=\left\{h(t)\in G([0,1],X):h(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s),\right.$$
$$\left. f\ is\ L^{\frac{1}{q_1}}(dg,X)-integrable\ selection\ of\ F(t,u(t))\right\}.$$

Step I. We prove that there exists a big enough positive constant $r>0$ such that N maps B_r into itself. Since S is weakly compact, it is bounded in norm and let us say by some r. If this is not the case, there is a function u_r satisfying $\|h(t)\|>r$ for some $t\in[0,1]$. According to the assumptions, we have

$$\begin{aligned}
r<\|h(t)\|= &\left\|T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)\right\|\\
\le &\ M_1\|u_0\|+r\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t((t-s)^{\alpha-1})^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\left(\int_0^t(M_r(s))^{\frac{1}{q_1}}dg(s)\right)^{q_1}\\
\le &\ M_1\|u_0\|+\sup_{t\in[0,1]}r\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t((t-s)^{\alpha-1})^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|M_r\|_{HLS_g^{\frac{1}{q_1}}},
\end{aligned}$$

Then we can divide both sides of this inequality by r and take the limit $r\to\infty$ in both sides to obtain

$$1\le\sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t((t-s)^{\alpha-1})^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|M_r\|_{HLS_g^{\frac{1}{q_1}}},$$

which is a contradiction. Hence, we can find r such that $N(B_r)\subset B_r$.

Step II. We prove that the operator $N:B_r\to B_r$ is Lipschitz continuous.

According to Lemma 2.2, we have to show that $N(u)\in P_{cl}(G([0,1],X))$ for each $u\in G([0,1],X)$. Let $(h_n)_n\in N(u)$ such that $h_n\to h$ as $n\to\infty$. Then $h_n\in G([0,1],X)$ and there exists $v_n\in F(t,u(t))$ such that, for each $t\in[0,1]$,

$$h_n(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)v_n(s)dg(s).$$

As F has compact values, we obtain from the dominated convergence theorem a subsequence converges to v in $L([0,1],X)$. Thus, $v\in L([0,1],X)$ and for each $t\in[0,1]$, we have

$$h_n(t)\to h(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)v(s)dg(s).$$

Hence, $h\in N(u)$. Let $u,\bar{u}\in G([0,1],X)$ and $h_1\in N(u)$. Then there exists $v_1\in F(t,u(t))$ such that, for each $t\in[0,1]$,

$$h_1(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)v_1(s)dg(s).$$

By (H4), we have $H_d(F(t,u),F(t,\bar{u})) \le \gamma(t)|u-\bar{u}|$. So, there exists $w \in F(t,\bar{u}(t))$ such that

$$|v_1(t)-w(t)| \le \gamma(t)|u(t)-\bar{u}(t)|,\ t\in[0,1].$$

Define $U:[0,1]\to P(X)$ by

$$U(t)=\{w\in X: |v_1(t)-w(t)|\le \gamma(t)|u(t)-\bar{u}(t)|\}.$$

Since the multivalued operator $U(t)\bigcap F(t,\bar{u}(t))$ is measurable (see [5]), there exists a function $v_2(t)$, which is a measurable selection for U. So, $v_2(t)\in F(t,\bar{u}(t))$, and for each $t\in[0,1]$, we have $|v_1(t)-v_2(t)|\le\gamma(t)|u(t)-\bar{u}(t)|$. For each $t\in[0,1]$, let us define

$$h_2(t)=T_\alpha(t)u_0+\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)v_2(s)dg(s).$$

Thus,

$$\begin{aligned} |h_1(t)-h_2(t)| &= \left|\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)(v_1(s)-v_2(s))dg(s)\right| \\ &\le \left(\frac{M_1}{\Gamma(\alpha)}\int_0^t \|(t-s)^{\alpha-1}\gamma(s)\|dg(s)\right)\|u-\bar{u}\| \\ &\le \sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|\gamma\|_{HLS_g^{\frac{1}{q_1}}}\|u-\bar{u}\|. \end{aligned} \tag{3.1}$$

Step III. We prove that $N: B_r\to B_r$ is contraction map.
For any bounded set $D\subset B_r$, By Lemma 2.1 and (3.1), we know

$$\alpha\big(N(D)\big)\le \sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|\gamma\|_{HLS_g^{\frac{1}{q_1}}}\alpha(D).$$

Therefore, $N: B_r\to B_r$ is contraction map with

$$k=\sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t \left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|\gamma\|_{HLS_g^{\frac{1}{q_1}}}.$$

It implies that N has at least one fixed point $u\in B_r$, which is a mild solution to system (1.1). □

Let X be a separable Banach space, and let (X,w) be the space endowed with its weak topology. Let $P_{kc}(X)$ and $P_{wkc}(X)$ denote the families of all nonempty of X which are compact convex and weakly compact convex, respectively. Denote by $L(X)$ the space of bounded linear operators from X to X. In particular, the operator norm $\|\cdot\|$ is defined on $L(X)$ by $\|T\|=\sup_{t\le1}\|T(t)\|_{L(X)}$.

Next, we investigate the existence of mild solutions, as well as the compactness of the solution set.

Theorem 3.4. *Assume that both (H1) and (H2) are satisfied. In addition, assume that the following condition holds:*

(H6) $F:[0,1]\times X\to P_{wkc}(X)$, $F(\cdot,x)$ is sequentially upper semi-continuous from $(X,w)\to(X,w)$ for dg-a.e. $t\in[0,1]$ and $F(\cdot,x)$ is measurable for every $x\in X$.

Then the solution set of system (1.1) is non-empty and weakly compact.

Proof. The proof process is divided into three steps. We denote the set

$$K=\Big\{u\in G([0,1],X), u(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s),$$
$$f \textit{ is measurable}, f(t)\in M(t)S, t\in[0,1]\Big\}.$$

Step I. We prove that K is bounded. For every $u\in K$, there exists f measurable with $f(t)\in M(t)S$ for all $t\in[0,1]$ and such that

$$u(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)$$

for all $t\in[0,1]$. Then

$$\begin{aligned}\|u\|\le\ & M_1\|u_0\|+r\frac{M_1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}M(s)dg(s)\\ \le\ & M_1\|u_0\|+r\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t\left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\left(\int_0^t(M(s))^{\frac{1}{q_1}}dg(s)\right)^{q_1}\\ \le\ & M_1\|u_0\|+r\sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t\left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|M\|_{HLS_g^{\frac{1}{q_1}}}.\end{aligned}$$

Step II. We show the relative weak compactness of K. First, applying Lemma 2.7, it is sufficient to show that, for every sequence

$$\left(T_\alpha(\cdot)u_0+\int_0^{\cdot}(\cdot-s)^{\alpha-1}S_\alpha(\cdot-s)f_n(s)dg(s)\right)_n$$

of K, one can extract a subsequence, which is weakly convergent in $G([0,1],X)$. By Proposition 2.4, it is sufficient to prove that this subsequence is weakly pointwisely convergent. Since $(f_n)_n$ is in $M(t)S$ and satisfies the assumptions of Proposition 2.5, there exists a subsequence $(f_{n_k})_k$, which is $\sigma\big(L^{\frac{1}{q_1}}(dg,X),L^\infty(dg,X^*_{w^*})\big)$-weakly convergent to some integrable (with respect to g) function f, and it can be seen that, for a fixed $t\in[0,1]$, the operator

$$f(\cdot)\in L^{\frac{1}{q_1}}\big([0,1],dg,X\big)\mapsto(t-\cdot)^{\alpha-1}S_\alpha(t-\cdot)f(\cdot)\in L^1\big([0,t],dg,X\big)$$

is linear and continuous due to

$$\begin{aligned}&\left\|(t-\cdot)^{\alpha-1}S_\alpha(t-\cdot)f(\cdot)\right\|_{L^1}\\ &=\left\|\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)\right\|_{L^1}\\ &\le r\sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t\left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\left(\int_0^1\left(f(s)\right)^{\frac{1}{q_1}}dg(s)\right)^{q_1}\\ &\le r\sup_{t\in[0,1]}\frac{M_1}{\Gamma(\alpha)}\left(\int_0^t\left((t-s)^{\alpha-1}\right)^{\frac{q_1-1}{q_1}}dg(s)\right)^{\frac{q_1}{q_1-1}}\|f\|_{HLS_g^{\frac{1}{q_1}}}.\end{aligned}$$

Observe that we have a semigroup of contractions. Since it is linear, it follows that it is also weakly-weakly (sequentially) continuous. It further follows that, for every $t\in[0,1]$, $\big((t-\cdot)^{\alpha-1}S_\alpha(t-\cdot)f_{n_k}(\cdot)\big)_k$ weakly converges to $(t-\cdot)^{\alpha-1}S_\alpha(t-\cdot)f(\cdot)$. For any continuous linear functional on $L^p(\mu,X)$, a finite measure μ can be represented by an element of $L^\infty(\mu,X^*_{w^*})$, where $X^*_{w^*}$ denotes the space of all weak equivalence classes of X^*-valued, X-weakly measurable functions $h:[0,1]\to X^*$ such that they are essentially bounded. Therefore, taking in particular functions of the form $h(s)=x^*\chi_{[0,t]}(s)$ (for some $x^*\in X^*$) one sees that

$$\begin{aligned}
&\int_0^t \langle x^*,(t-s)^{\alpha-1}S_\alpha(t-s)f_{n_k}(s)\rangle dg(s)\\
&=\int_0^1 h(s)(t-s)^{\alpha-1}S_\alpha(t-s)f_{n_k}(s)dg(s) \quad\to\quad \int_0^1 h(s)(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)\\
&\qquad\qquad = \int_0^t \langle x^*,(t-s)^{\alpha-1}S_\alpha(t-s)f(s)\rangle dg(s).
\end{aligned}$$

So, the sequence $\left(\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f_{n_k}(s)dg(s)\right)_k$ is weakly convergent to $\int_0^t(t-s)^{\alpha-1}$ $S_\alpha(t-s)f(s)dg(s)$, Thus, we obtained a weakly convergent subsequence.

Secondly, we show that it is weakly sequentially closed. For this purpose, let $(u_n)_n$ be a sequence of elements of K which converges weakly to a regulated function u and prove that u is also an element of K. There exists a sequence of measurable functions $(f_n(\cdot))_n$ such that $f_n(t)\in M(t)S$ for every $t\in[0,1]$ and for each n:

$$u_n(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f_n(s)dg(s), t\in[0,1].$$

Since $(f_n)_n$ satisfies the assumptions of Lemma 2.3, there exists a subsequence $\left(f_{n_k}\right)_k$ which is $\sigma\left(L^{\frac{1}{q_1}}(dg,X),L^\infty(dg,X^*_{w^*})\right)$-weakly convergent to some integrable (with respect to g) function f. It follows that for every $t\in[0,1]$, $\left((t-\cdot)^{\alpha-1}S_\alpha(t-\cdot)f_{n_k}(\cdot)\right)_k$ weakly converges to $(t-\cdot)^{\alpha-1}S(t-\cdot)f(\cdot)$. Therefore, the sequence $\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f_{n_k}(s)dg(s)$ is weakly convergent to $\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)$. Hence,

$$u(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)$$

and the sequential closeness (thus, the weak sequential compactness) of K is proved. By Lemma 2.7, we obtain the weak compactness of the set.

Step III. Define on the convex weakly compact set K the multivalued operator $N:K\subset G([0,1],X)\to 2^{G([0,1],X)}$ by

$$N(u)=\Big\{v\in G([0,1],X):v(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s),$$
$$f \text{ is } L^{\frac{1}{q_1}}(dg,X)-\text{integrable selection of } F(t,u(t))\Big\}.$$

We demonstrate that it satisfies Lemma 2.6 when the considered Hausdorff topology on $G([0,1],X)$ is its weak topology. From the definition of K and our hypotheses, $N(K)\subset K$. Evidently, the values of N are nonempty (and convex) since any measurable selection f of $F(\cdot,u(\cdot))$ (which exists, by [16]) is $L^{\frac{1}{q_1}}$ integrable with respect to g and the regulated function

$$v(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s), t\in[0,1]$$

is an element of $N(u)$.

Next, we prove that N has weakly closed graph. By Lemma 2.7, it is sufficient to prove that N has weakly sequentially closed graph. Consider a sequence $(u_n, v_n)_n \in Graph(N)$ weakly convergent to the pair of regulated functions (u, v) and prove that $(u, v) \in Graph(N)$, equivalently that $v \in N(u)$. According to the definition, there exists a sequence $(f_n)_n$ of $F(\cdot, u_n(\cdot))$ is $L^{\frac{1}{q_1}}$ integrable with respect to g such that for every $n \in N$:

$$v_n(t) = T_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) dg(s), \forall t \in [0,1].$$

As before, we can extract a subsequence $(f_{n_k})_k$, which is $\sigma\big(L^{\frac{1}{q_1}}(dg, X), L^\infty(dg, X^*_{w^*})\big)$-weakly convergent to some measurable function f and $\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_{n_k}(s) dg(s)$ is weakly convergent to $\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) dg(s)$. Hence,

$$v(t) = T_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) dg(s), \ \forall t \in [0,1].$$

In addition, by the upper semi-continuity hypothesis, we can obtain that, for every neighborhood V of 0 and dg-a.e. $t \in [0,1]$, there exists $n_{t,V} \in N$ such that $F(t, u_n(t)) \subset F(t, u(t)) + V$, $\forall n \geq n_{t,V}$. By Mazur's Lemma, one can find a sequence $(g_k)_k$ of convex combinations of $\{f_{n_m}, m \geq k\}$, which is strongly $L^{\frac{1}{q_1}}$-convergent, and then, on a subsequence, dg-a.e. convergent to f. It follows that f is a selection of $F(\cdot, u(\cdot))$ and thus the graph of N is weakly sequentially closed. By Lemma 2.6, we obtain that N has at least one fixed point, which is a mild solution to system (1.1) in $[0,1]$ and the solution set is weak compact, which completes the proof. □

Next, we prove in strong topological sense.

Theorem 3.5. *Let both the assumptions (H1) and (H2) be satisfied. In addition, assume that the following condition:*

(H7) $F : [0,1] \times X \to P_{kc}(X)$, $F(t, \cdot)$ is upper semi-continuous for dg-a.e. $t \in [0,1]$ and $F(\cdot, x)$ is measurable for every $x \in X$ holds.

Then the solution set of system (1.1) is non-empty and compact.

Proof. The proof is divide into two steps.

Step I. Let

$$K = \Big\{ u \in G([0,1], X), u(t) = T_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) dg(s),$$
$$f \text{ is measurable}, \ f(t) \in M(t)S, t \in [0,1] \Big\}.$$

We prove the set is compact and convex in the space of regulated functions, by using Proposition 2.4. First, we have to show that, for each $t \in [0,1]$, $K(t) \in X$ is relatively compact. Indeed, because of the linear continuity of $(t-s)^{\alpha-1} S_\alpha(t-s)$ for all $s \in [0,t]$,

$$(t-s)^{\alpha-1} S_\alpha(t-s) f(s) \in (t-s)^{\alpha-1} S_\alpha(t-s) M(s) S,$$

which is relatively compact. In addition, we know that the multivalued integral of an integrably bounded multifunction with compact convex values is compact and convex. So $K(t) \subset X$ is relatively compact. Next, let us see that K is equi-regulated. For any $t \in [0,1]$, we have

$$\begin{aligned}\|u(t)-u(t_0^+)\| &= \Big\|T_\alpha(t)u_0+\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)-T_\alpha(t_0^+)u_0 \\ &\quad +\int_0^{t_0^+}(t_0^+-s)^{\alpha-1}S_\alpha(t_0^+-s)f(s)dg(s)\Big\| \\ &\le \big\|(T_\alpha(t)-T_\alpha(t_0^+))u_0\big\| \\ &\quad +\int_0^{t_0^+}\big\|((t-s)^{\alpha-1}S_\alpha(t-s)-(t_0^+-s)^{\alpha-1}S_\alpha(t_0^+-s))f(s)\big\|dg(s) \\ &\quad +\int_{t_0^+}^{t_0}\big\|(t-s)^{\alpha-1}S_\alpha(t-s)f(s)\big\|dg(s) \\ &= A_1+A_2+A_3,\end{aligned}$$

where

$$A_1=\big\|\big(T_\alpha(t)-T_\alpha(t_0^+)\big)u_0\big\|,$$

$$A_2=\int_0^{t_0^+}\big\|\big((t-s)^{\alpha-1}S_\alpha(t-s)-(t_0^+-s)^{\alpha-1}S_\alpha(t_0^+-s)\big)f(s)\big\|dg(s),$$

and

$$A_3=\int_{t_0^+}^{t_0}\big\|(t-s)^{\alpha-1}S_\alpha(t-s)f(s)\big\|dg(s).$$

From the combination of compactness of $T(t)$ and its strongly continuity of $T(t)$ in the uniform operator topology, and the dominated convergence theorem, we can derive that A_1 and A_2 tend to zero independently of x as $t\to t_0^+$. Let $j(t)=\int_0^t k(s)dg(s)$, where $k(s)=(t-s)^{\alpha-1}M(s)$. From Lemma 2.6, $j(t):[0,1]\to X$ is a regulated function. Then

$$\begin{aligned}A_3 &\le r\frac{M_1}{\Gamma(\alpha)}\int_{t_0^+}^{t_0}\big\|(t-s)^{\alpha-1}M(s)\big\|dg(s) \\ &\le r\frac{M_1}{\Gamma(\alpha)}\Big(\big\|j(t)-j(t_0^+)\big\|+\int_0^{t_0^+}\big\|\big((t-s)^{\alpha-1}-(t_0^+-s)^{\alpha-1}\big)f(s)\big\|dg(s)\Big).\end{aligned}$$

So, as $t\to t_0^+$, we have $A_3\to 0$. With the same analysis as above, we can also derive that $\|u(t_0^-)-u(t)\|\to 0$ as $t_0^-\to t$ for every $t\in[0,1]$. So K is equi-regulated.

Finally, we prove that K is closed. Let

$$u_n(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f_n(s)dg(s), t\in[0,1]$$

be a sequence of elements of K which converges to a regulated function u and show that u is also an element of K. As before, we can extract a subsequence $(f_{n_k})_k$ which is $\sigma\big(L^{\frac{1}{q_1}}(dg,X),L^\infty(dg,X^*_{w^*})\big)$-weakly convergent to some measurable function f. It follows, as in the proof of Theorem 3.4, that, for $t\in[0,1]$, $\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f_{n_k}(s)dg(s)$ is weakly convergent to $\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)$. Therefore,

$$u(t)=T_\alpha(t)u_0+\int_0^t(t-s)^{\alpha-1}S_\alpha(t-s)f(s)dg(s)$$

and the closeness of K is proved.

Step II. Define on the convex compact set K the multivalued operator $N : K \subset G([0,1],X) \to 2^{G([0,1],X)}$ by

$$N(u) = \{v \in G([0,1],X) : v(t) = T_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) dg(s),$$
$$f \text{ is } L^{\frac{1}{q_1}}(dg,X) - \text{integrable selection of} F(t,u(t))\}.$$

We prove that it satisfies Lemma 2.6 when $G([0,1],X)$ is considered endowed with its strong topology. Obviously, $N(K) \subset K$, and the values of N are nonempty (and convex).

Next we prove that N has closed graph. Let $(u_n, v_n)_n \in Graph(N)$ converge to the pair of regulated functions (u,v) and prove that $(u,v) \in Graph(N)$, equivalently that $v \in N(u)$. There exists a sequence $(f_n)_n$ of $F(\cdot, u_n(\cdot))$, which is $L^{\frac{1}{q_1}}$-integrable with respect to g such that, for every $n \in N$,

$$v_n(t) = T_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_n(s) dg(s), \forall t \in [0,1].$$

As before, one can extract a subsequence $(f_{n_k})_k$ which is $\sigma\big(L^{\frac{1}{q_1}}(dg,X), L^\infty(dg,X^*_{w^*})\big)$-weakly convergent to some measurable function f and $\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f_{n_k}(s) dg(s)$ is weakly convergent to $\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) dg(s)$. Therefore,

$$v(t) = T_\alpha(t)u_0 + \textstyle\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) f(s) dg(s), \text{ for all } t \in [0,1].$$

From (H6), one deduces that, for every neighborhood V of 0 and dg-a.e. $t \in [0,1]$, there exists $n_{t,V} \in N$ such that $F(t, u_n(t)) \subset F(t,u(t)) + V$ for all $n \geq n_{t,V}$. By Mazur's Lemma, one can find a sequence $(g_k)_k$ of convex combinations of $\{f_{n_m}, m \geq k\}$, which is strongly $L^{\frac{1}{q_1}}$-convergent, therefore, on a subsequence, dg-a.e. convergent to f. It follows that f is a selection of $F(\cdot,u(\cdot))$ and thus the graph of N is closed. By Lemma 2.6, operator N has fixed points, so system (1.1) has mild solutions and the solution set is compact, which completes the proof. □

Remark 3.6. If g is absolutely continuous, then system (1.1) becomes the following

$$\begin{cases} {}^C D^\alpha u(t) \in Au(t)dt + F(t,u(t)), \quad t \in [0,1], \\ u(0) = u_0. \end{cases}$$

If (H1), (H2), and (H6) are satisfied, this system has mild solutions $[0,1]$. So our results cover [14, 18, 22]. When A is an almost sectorial operator, Theorem 3.1 and Theorem 8.2 of [18] have almost the same assumptions. This demonstrates that the conditions that we present in this paper are wide.

If g is the sum of absolutely continuous functions with piecewise constant step function, then system (1.1) becomes the following

$$\begin{cases} {}^C D^\alpha u(t) \in Au(t)dt + F(t,u(t)), \quad t \in [0,1], \\ u(t_i^+) = u(t_i) + I_i(u(t_i)), i = 1,2,\cdots,m, \\ u(0) = u_0, \end{cases} \tag{3.2}$$

which was studied by Alsarori and Ghadle [1]. When (H1), (H2), and (H6) are satisfied, system (3.2) has mild solutions on $[0,1]$.

Our results also extend and unify the differential inclusions with impulses [1] and the difference inclusions [2]. There is an interesting case that the solutions of the impulsive differential inclusion should be discontinuous (see [1]), however we find that any mild solution to system (1.1) is an impulsive mild solution to system (3.2) since we can put $F(t,u) = G(t,u)$ for $t \neq t_i$ and $F(t,u) = I_i(u(t))$ whenever $t = t_i$. F, G are essentially the same in continuity. In addition, let $g(t) = t + \sum_i \chi_{(t_i)}$. It is clear that g is non-decreasing, but discontinuous. Thus our main results also extend some existence results in [1, 2]. Moreover, there is no need to consider whether the number of jumps are finite in this case. Thus our method is efficient to study the Zeno phenomenon for hybrid systems (with infinite number of jumps) than before.

4. Example

As an application of our results, we consider the following fractional differential inclusion of the form

$$\begin{cases} {}^C D^{\alpha}_{0+} u(t,z) \in \frac{\partial^2}{\partial z^2} u(t,z) + G(t, u(t,z)) dg(t), \quad t, z \in [0,1], \\ u(0)(z) = u_0(z), \end{cases} \tag{4.1}$$

where ${}^C D^{\alpha}_{0+}$ is the Caputo fractional partial derivative of order α, $\alpha \in (0,1)$, and $G : [0,1] \times X \to 2^X \backslash \{\emptyset\}$. In order to rewrite (4.1) in the abstract form, we consider the linear operator A in Banach space X defined by $Aw = \frac{\partial w^2}{\partial z^2}$, $w \in D(A)$, where

$$D(A) = \Big\{ w \in X : w, \frac{\partial w}{\partial z} \text{ are absolutely continuous}, \frac{\partial^2 w}{\partial z^2} \in X, w(t,0) = w(t,1) = 0 \Big\}.$$

Then $Aw = \sum_{n=1}^{\infty} n^2 (w, w_n) w_n$, $w \in X$, where $W_n = \sqrt{\frac{2}{\pi}} \sin(ns)$, $0 \leq s \leq 1$, $n = 1, 2, \cdots$, is the orthogonal set of eigenfunctions of A, and (w, w_n) is the L^2 inner product. It follows from [20] that operator A generates a strongly continuous semigroup $T(t), t \geq 0$ defined by

$$T(t)w := \sum_{n=1}^{\infty} e^{-n^2} \langle w, w_n \rangle w_n.$$

Define $u(t)(z) = u(t,z), F(t, u(t))(z) = G(t, u(t,z)), t, z \in [0,1]$, Then system (4.1) can be transformed into the abstract form of system (1.1). Take

$$g(t) = \begin{cases} \frac{1}{2}, t \in [0, 1 - \frac{1}{2}], \\ \cdots, \\ 1 - \frac{1}{n-1}, t \in (1 - \frac{1}{n-1}, 1 - \frac{1}{n}], \\ \cdots, \\ 1, t = 1. \end{cases}$$

It is evident that $g : [0,1] \to R$ is a left continuous and nondecreasing function on $[0,1]$. We assume that the assumptions (H2), (H3), and (H4) are satisfied. From Theorem 3.3, system (4.1) has at least one mild solution on $[0,1]$.

5. Conclusion

In this paper, sufficient conditions for a class of fractional measure evolution inclusion were considered. By adopting the noncompactness measure method and fixed point theorem, we obtained the existence of mild solutions. An example was presented to support our main results.

Funding
This work was supported by National Natural Science Foundation of China (11961069), Outstanding Young Science and technology Training program of Xinjiang (2019Q022), Natural Science Foundation of Xinjiang (2019D01A71), and Scientific Research Programs of Colleges in Xinjiang (XJEDU2018Y033).

Acknowledgments
The authors would like to thank the editor and anonymous reviewers for their helpful comments and suggestions which helped to improve the quality of our present paper.

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