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NASH EQUILIBRIA FOR COMPONENTWISE VARIATIONAL SYSTEMS

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Abstract. In this paper, we generalize an existing result regarding the existence of a Nash equilibrium for a system of fixed point equations. The problem is considered in a more general form and the initial conditions are also improved, without changing the final conclusion. This is achieved by combining the idea of a solution operator with monotone operator techniques and classical fixed point principles. An application to a coupled system with Dirichlet boundary conditions involving the p-Laplacian is provided.

Keywords. Dirichlet boundary condition; Monotone operator; Nash equilibrium.

1. Introduction

Numerous equations can be reduced to a fixed point equation $N(u) = u$, where N is an operator. The equation is said to admit a variational structure if there exists a Fréchet differentiable functional E (called the ”energy functional”) such that any solution u is also a solution to the critical point equation $E'(u) = 0$, and vice versa.

In this paper, we are concerned with an existence result for a system of type

$$\begin{cases} N_1(u,v) = J_1(u), \\ N_2(u,v) = J_2(v), \end{cases} \tag{1.1}$$

where each of the equations admits a variational structure, i.e., there are two functionals E_1, E_2 such that the system is equivalent with

$$\begin{cases} E_{11}(u,v) = 0, \\ E_{22}(u,v) = 0, \end{cases}$$

where E_{11} is the partial Fréchet derivative of E_1 with respect to the first variable, and E_{22} is the partial Fréchet derivative of E_2 with respect to the second variable. Here J_1 and J_2 are two duality mappings, and N_1 and N_2 two continuous operators.

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In what follows, we study more than an existence result for (1.1), that is, under what conditions we have a solution (u^*, v^*), which is also a Nash equilibrium for the corresponding energy functionals, i.e.,

$$\begin{cases} E_1(u^*, v^*) = \inf E_1(\cdot, v^*), \\ E_2(u^*, v^*) = \inf E_2(u^*, \cdot). \end{cases}$$

We aim to generalize the result from [10], where a similar system was considered on Hilbert spaces. This was achieved imposing a Perov contraction condition and making use of Ekeland variational principle. Our contribution in the present paper aims to improve both the functional framework (real, separable and uniformly convex Banach spaces) as well as the initial conditions (less restrictive ones), while retaining the same conclusion. The result is obtained combining the idea of a solution operator, inspired from [1], with monotone operators techniques (Minty-Browder Theorem) and a fixed point principle (Leray-Schauder Fixed Point Theorem).

2. Preliminaries

Let X be a real, separable, and uniformly convex Banach space. Let X^* be its dual, and let $\langle \cdot, \cdot \rangle$ stand for the dual pairing between X^* and X. We denote by J the duality mapping corresponding to the gauge function $\varphi(t) := t^{p-1}$, where $p > 1$, i.e.,

$$Jx := \{x^* \in X^* : \langle x^*, x \rangle = |x|^p, |x^*|_{X^*} = |x|^{p-1}\}. \tag{2.1}$$

Below some important properties of duality mapping J are stated. For proofs and further details, we refer to Dinca, Jebelean, and Mawhin [6].

Lemma 2.1. *The duality mapping J, defined by (2.1), has the following properties:*

- *i) J is single valued;*
- *ii) J is strictly monotone, i.e., $\langle Jx - Jy, x - y \rangle > 0$ for all $x \neq y$.*
- *iii) J satisfies the $(S)_+$ condition, i.e., if $x_n \to x$ weakly and $\limsup_{n\to\infty} \langle Jx_n, x_n - x \rangle \leq 0$, then $x_n \to x$ strongly.*
- *iv) J is demicontinuous, i.e., if $x_n \to x$ strongly, then $Jx_n \to Jx$ weakly.*
- *v) J is bijective from X to X^*.*

A square matrix of non-negative numbers $A = [a_{i,j}]_{1\leq i,j\leq n} \in \mathbb{M}_{n,n}(\mathbb{R}_+)$ is said to be *convergent to zero* if $A^k \to O_n$ as $k \to \infty$, where O_n is the zero matrix. In case $n = 2$, we have the following equivalent characterization (see [2]).

Lemma 2.2. *Let $A = [a_{i,j}]_{1\leq i,j\leq 2}$ be a square matrix of non-negative real numbers. Then A is convergent to zero if and only if $a_{11}, a_{22} < 1$ and $a_{11} + a_{22} < 1 + a_{11}a_{22} - a_{12}a_{21}$.*

For the convenience of the reader, we present a list of theoretical results used throughout this paper. Because they are some classics in the theory of nonlinear analysis, we omit the proofs. However, further details can be found in the indicated sources.

Theorem 2.3 (Minty-Browder [4, Theorem 9.14])**.** *Let X be a real, reflexive, and separable Banach space. Assume that $T : X \to X^*$ is a bounded, demicontinuous, coercive, and monotone operator. Then, for any given $v \in X^*$, there exist a unique $u \in X$ such that $T(u) = v$.*

Theorem 2.4 (Leray-Schauder [5, Theorem 2.11])**.** *Let X be a Banach space, and let $T : X \to X$ be a continuous compact mapping which satisfies the following condition: there exists $R > 0$*

such that the set $\cup_{\lambda\in[0,1]}\{x \in X : x = \lambda Tx\}$ *lies in a ball of radius R, centered in the origin. Then T admits at least one fixed point.*

Theorem 2.5 ([5, Lemma 1.1])**.** *Let X be a topological space, and let* x_n *a sequence from X with the following property: there exist* $x \in X$ *such that, from any subsequence of* x_n*, one can extract a further subsequence converging to x. Then the whole sequence* x_n *is convergent to x.*

We conclude this preliminary section with some known results related to the p-Laplacian. For proofs and further details, we refer to [6], [7], and [9]. Let Ω be a bounded domain from $\mathbb{R}^n$ with Lipschitz boundary. Consider the well known Sobolev space $W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) \mid u_{\partial\Omega} = 0\}$ endowed with norm

$$|u|_{1,p} := |\nabla u|_{L^p} = \left(\int_\Omega |\nabla u|^p\right)^{1/p}.$$

It is known that $\left(W_0^{1,p}(\Omega), |\cdot|_{1,p}\right)$ is a separable and uniformly convex real Banach space (see [6, Theorem 6]). The notation $W^{-1,p'}(\Omega)$ stands for the dual of $W^{1,p}(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Proposition 2.6 ([6, Proposition 6])**.** *Let* $f\colon \Omega\times\mathbb{R}\to\mathbb{R}$ *be a Carathéodory function which satisfies the following growth condition:* $|f(x,s)| \le a|s|^{p-1}+b(x)$*, where* $b \in L^{p'}$*. Then the Nemytskii's operator* $\left(N_f u\right)(x) = f(x,u(x))$*, is continuous and bounded from* $L^p(\Omega)$ *to* $L^{p'}(\Omega)$*.*

One has the following diagram (see [6, p. 355])

$$W_0^{1,p}(\Omega) \rightrightarrows L^p(\Omega) \xrightarrow{N_f} L^{p'}(\Omega) \hookrightarrow W^{-1,p}(\Omega),$$

the Poincare's inequalities: for any $u \in W_0^{1,p}(\Omega)$, $|u|_{L^p} \le C|u|_{1,p} = C|\nabla u|_{L^p}$, and, for any $f \in L^{p'}(\Omega)$, $|f|_{W^{-1,p'}} \le C|f|_{L^{p'}}$, where C is a constant depending only on Ω and n.

The following result establishes an equivalence between p-Laplacian and the duality mapping corresponding to the gauge function $\varphi(t)=t^{p-1}$ on $\left(W_0^{1,p}(\Omega), |\cdot|_{1,p}\right)$.

Theorem 2.7 ([6, Theorem 7])**.** *The operator* $-\Delta_p\colon W_0^{1,p}(\Omega)\to W^{-1,p'}(\Omega)$ *is the Fréchet derivative of functional* $\psi\colon W_0^{1,p}(\Omega)\to\mathbb{R}$ *defined as* $\psi(u)=\frac{1}{p}|u|_{1,p}^p$*. More exactly,*

$$\psi' = -\Delta_p = J_\varphi,$$

where J_φ *represents the duality mapping corresponding to the gauge function* $\varphi(t)=t^{p-1}$*.*

3. Main Results

In what follows, $(X_1, |\cdot|_1)$ and $(X_2, |\cdot|_2)$ are two separable and uniformly convex real Banach spaces. Further, X_1^* and X_2^* stand for the dual spaces of X_1 and X_2, and $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ for the dual pairing between X_1^* and X_1, and X_2^* and X_2.

We denote with J_1 and J_2 the duality mappings corresponding to gauge functions $\varphi_1(t) := t^{p-1}$ and $\varphi_2(t)=t^{q-1}$, respectively, where $p \ge q > 1$. Clearly J_1 and J_2 satisfy all the properties of duality mappings stated in Lemma 2.1.

It is assumed that system (1.1) admits a variational structure on each equation, i.e., there are two energy functionals $E_1, E_2\colon X_1\times X_2\to\mathbb{R}$ are such that

$$\begin{cases} E_{11}(u,v) = J_1(u) - N_1(u,v), \\ E_{22}(u,v) = J_2(v) - N_2(u,v), \end{cases}$$

where E_{ii} represents the Fréchet derivative of E_i with respect to the ith component. Thus any point $(u^*,v^*)\in X_1\times X_2$ which satisfies simultaneously $E_{11}(u^*,v^*)=0$ and $E_{22}(u^*,v^*)=0$ is a solution of (1.1).

We assume that the operators $N_i\colon X_1\times X_2\to X_i^*$ $(i\in\{1,2\})$ are continuous and satisfy the following monotony conditions: there are real numbers $a_{11},a_{22}\in[0,1)$ such that

$$\langle N_1(u,v)-N_1(\overline{u},v)\,,\,u-\overline{u}\rangle_1\le a_{11}\langle J_1(u)-J_1(\overline{u})\,,\,u-\overline{u}\rangle,\text{ for all }u,\overline{u}\in X_1\text{ and }v\in X_2\,,\tag{3.1}$$

$$\langle N_2(u,v)-N_2(u,\overline{v})\,,\,v-\overline{v}\rangle_2\le a_{22}\langle J_2(v)-J_2(\overline{v})\,,\,u-\overline{v}\rangle,\text{ for all }v,\overline{v}\in X_2\text{ and }u\in X_1.\tag{3.2}$$

Below, we present an auxiliary result (Theorem 3.1) which allows us to split the problem of finding a solution which is a Nash equilibrium solution to system (1.1) as two individual problems: any solution is a Nash equilibrium and there is at least one solution.

3.1. Nash equilibria property.

Theorem 3.1. *Under the previous assumptions, if system* (1.1) *admits a solution* $(u^*,v^*)\in X_1\times X_2$ *,then it is a Nash equilibrium for the energy functionals* (E_1,E_2)*, i.e.,*

$$E_1(u^*,v^*)=\inf_{X_1}E_1(\cdot,v^*),$$
$$E_2(u^*,v^*)=\inf_{X_2}E_2(u^*,\cdot)\,.$$

Proof. It is sufficient to demonstrate that, for all $u\in X_1$ and $v\in X_2$,

$$E_1(u^*,v^*)\le E_1(u^*+u,v^*)\quad\text{and}\quad E_2(u^*,v^*)\le E_2(u^*,v^*+v).$$

Let $u\in X_1$. Since $E_1(\cdot,v^*)$ is Fréchet differentiable and $N_1(u^*,v^*)=J_1(u^*)$, we obtain

$$\begin{aligned}
&E_1(u^*+u,v^*)-E_1(u^*,v^*)\\
&=\int_0^1\langle E_{11}(u^*+t\,u,v^*)\,,\,u\rangle_1dt\\
&=\int_0^1\langle J_1(u^*+t\,u)-N_1(u^*+t,v^*)\,,\,u\rangle_1dt\\
&=\int_0^1\langle J_1(u^*+t\,u)-J_1(u^*)\,,\,u\rangle_1-\langle N_1(u^*+t\,u,v^*)-N_1(u^*,v^*)\,,\,u\rangle_1dt.
\end{aligned}$$

Condition (3.1) yields

$$E_1(u^*+u,v^*)-E_1(u^*,v^*)\ge\int_0^1\frac{(1-a_{11})}{t}\langle J_1(u^*+tu)-J_1(u^*)\,,\,u^*+t\,u-u^*\rangle_1dt.$$

Furthermore, from the monotony of J_1, we deduce

$$E_1(u^*+u,v^*)-E_1(u^*,v^*)\ge0,\text{ for all }u\in X_1.$$

Using a similar reasoning, we find

$$E_2(u^*,v^*+v)-E_2(u^*,v^*)\ge0,\text{ for all }v\in X_2.$$

□

Now, we are ready to state our main existence result. Taking into account Theorem 3.1, if one can prove the existence of a solution to system (1.1), then it is a Nash equilibrium for the associated energy functionals E_1,E_2.

3.2. Existence result.

Theorem 3.2. *Under the previous mentioned settings, we additionally assume:*

- ***(h1)*** *the operator $J_2^{-1} \circ N_2 \colon X_1 \times X_2 \to X_2$ is compact.*
- ***(h2)*** *there are real numbers $a_{12}, a_{21} \in (0,1)$, and $M_1, M_2 \in \mathbb{R}_+$ such that*

$$|N_1(0,v)| \leq a_{12}|v|_1^{p-1} + M_1, \quad \text{for all } v \in X_2, \tag{3.3}$$

$$|N_2(u,0)| \leq a_{21}|u|_1^{q-1} + M_2, \quad \text{for all } u \in X_1, \tag{3.4}$$

and the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is convergent to zero.

Then there exists a solution $(u^*, v^*) \in X_1 \times X_2$ to system (1.1).

Proof. Our approach is based on [1], where a solution operator was constructed from the first equation. This is then used, together with the second equation, to build a further operator, which is later shown to admit a fixed point. Let $v \in X_2$ be arbitrarily chosen. First, note that from the monotony condition (3.3) the operator $J_1(\cdot) - N_1(\cdot, v)$ is monotone and coercive. Indeed, for any $u, \overline{u} \in X_1$, we have

$$\langle J_1(u) - N_1(u,v) - J_1(\overline{u}) + N_1(\overline{u},v)\,,\, u - \overline{u}\rangle_1 \geq (1 - a_{11})\langle J_1(u) - J_1(\overline{u})\,,\, u - \overline{u}\rangle_1 \geq 0,$$

and

$$\begin{aligned}
\frac{\langle J_1(u) - N_1(u,v)\,,\, u\rangle_1}{|u|_1} &= \frac{\langle J_1(u)\,,\, u\rangle_1 - \langle N_1(u,v) - N_1(0,v)\,,\, u\rangle_1}{|u|_1} - \frac{\langle N_1(0,v)\,,\, u\rangle_1}{|u|_1} \\
&\geq (1 - a_{11})\frac{\langle J_1(u), u\rangle_1}{|u|_1} - |N_1(0,v)| \\
&= (1 - a_{11})|u|_1^{p-1} - |N_1(0,v)| \to \infty \text{ as } |u|_1 \to \infty.
\end{aligned}$$

Moreover, since $N_1(\cdot, v)$ is continuous and J_1 is bounded and demicontinuous, $J_1(\cdot) - N_1(\cdot, v)$ is also bounded and demicontinuous. Now, in virtue of Theorem 2.3, there exist a unique element $S(v) \in X_1$ such that

$$J_1(S(v)) = N_1(S(v), v). \tag{3.5}$$

Thus, we defined by (3.5) the solution operator $S \colon X_2 \to X_1$. In what follows, we prove that S is continuous. Let v_n a sequence from X_2 convergent to some $v \in X_2$.

Boundedness of $S(v_n)$. From the monotony condition (3.1) and relation (3.5), we obtain

$$\begin{aligned}
|S(v_n)|_1^p &= \langle J_1\, S(v_n)\,,\, S(v_n)\rangle_1 \\
&= \langle N_1\,(S(v_n), v_n) - N_1(0, v_n)\,,\, S(v_n)\rangle_1 + \langle N_1(0, v_n)\,,\, S(v_n)\rangle_1 \\
&\leq a_{11}\langle J_1\, S(v_n)\,,\, S(v_n)\rangle_1 + |N_1(0, v_n)|\,|S(v_n)|_1 \\
&= a_{11}|S(v_n)|_1^p + |N_1(0, v_n)|\,|S(v_n)|_1.
\end{aligned}$$

Thus,

$$|S(v_n)|_1^{p-1} \leq \frac{1}{1 - a_{11}}|N_1(0, v_n)|,$$

which guarantees the boundedness of $S(v_n)$. Let $S(v_n)$ be a subsequence of $S(v_n)$ (for simplicity, we keep the same indices). Since $S(v_n)$ is bounded, there is a further subsequence (also denoted with $S(v_n)$) and an element $w\in X_1$ such that $S(v_n)$ converges weakly to w (see [3, Theorem 3.18]).

Strong convergence of $S(v_n)$ to w. Observe that

$$\begin{aligned}\langle J_1\,S(v_n)\,,\,S(v_n)-w\rangle_1 &= \langle N_1\left(S(v_n)\,,\,v_n\right)\,,\,S(v_n)-w\rangle_1\\ &= \langle N_1\left(S(v_n)\,,\,v_n\right)-N_1(w,v_n)\,,\,S(v_n)-w\rangle_1+\langle N_1(w,v_n)\,,\,S(v_n)-w\rangle_1\\ &\le a_{11}\langle J_1\,S(v_n)-J_1w\,,\,S(v_n)-w\rangle_1+\langle N_1(w,v_n)\,,\,S(v_n)-w\rangle_1.\end{aligned}$$

Consequently,

$$\begin{aligned}\langle J_1\,S(v_n)\,,\,S(v_n)-w\rangle_1 &\le \frac{1}{1-a_{11}}\Big(\langle J_1w\,,\,S(v_n)-w\rangle_1+\langle N_1(w,v_n)\,,\,S(v_n)-w\rangle_1\Big)\\ &\le \frac{1}{1-a_{11}}\langle J_1w-N_1(w,v)\,,\,S(v_n)-w\rangle_1\\ &\quad+\frac{|S(v_n)|_1+|w|_1}{1-a_{11}}|N_1(w,v_n)-N_1(w,v)|.\end{aligned}\tag{3.6}$$

Since $S(v_n)$ is bounded, $S(v_n)\to w$ weakly, and $N_1(w,v_n)\to N_1(w,v)$ strongly,

$$\begin{aligned}&\langle J_1w-N_1(w,v)\,,\,S(v_n)-w\rangle_1\to 0 \text{ and}\\ &|N_1(w,v_n)-N_1(w,v)|\,\left(|S(v_n)|_1+|w|_1\right)\to 0.\end{aligned}$$

Hence, passing to $\limsup$ in (3.6), we deduce $\limsup_{n\to\infty}\langle J_1\,S(v_n)\,,\,S(v_n)-w\rangle_1\le 0$, which guarantees the strong convergence of $S(v_n)$ to w, based on the $(S)_+$ property of duality mapping J_1.

The equality w=S(v). Since $N_1(\cdot,v)$ is continuous, one has $\lim_{n\to\infty}J_1\,S(v_n)=N_1(w,v)$. Then

$$\begin{aligned}\langle J_1w-J_1S(v)\,,\,w-S(v)\rangle_1 &= \langle N_1(w,v)-N_1(S(v),v)\,,\,w-S(v)\rangle_1+\langle J_1w-N_1(w,v)\,,\,w-S(v)\rangle_1\\ &\le a_{11}\langle J_1w-J_1\,S(v)\,,\,w-S(v)\rangle_1+\langle J_1w-J_1\,S(v_n)\,,\,w-S(v)\rangle_1\\ &\quad+\langle J_1\,S(v_n)-N_1(w,v)\,,\,w-S(v)\rangle_1,\end{aligned}$$

that is,

$$\begin{aligned}&\langle J_1w-J_1S(v)\,,\,w-S(v)\rangle_1\\ &\le\frac{1}{1-a_{11}}\Big(\langle J_1w-J_1\,S(v_n)\,,\,w-S(v)\rangle_1+\langle J_1\,S(v_n)-N_1(w,v)\,,\,w-S(v)\rangle_1\Big).\end{aligned}\tag{3.7}$$

Since J_1 is demicontinuous, one has $\langle J_1w-J_1\,S(v_n)\,,\,w-S(v)\rangle_1\to 0$. Thus, passing to limit in (3.7), we conclude that $\langle J_1w-J_1S(v)\,,\,w-S(v)\rangle_1\le 0$. Now, from the strict monotony of dual mapping J_1, we have $w=S(v)$ immediately.

Finally, we obtain the continuity of the operator S. Note that S also satisfies the growth condition (3.8). Indeed, from (3.3), we have

$$\begin{aligned}|S(v)|_1^p &= \langle J_1\,S(v)\,,\,S(v)\rangle_1\\ &= \langle N_1(S(v),v)-N_1(0,v)\,,\,S(v)\rangle_1+\langle N_1(0,v)\,,\,S(v)\rangle_1\\ &\le a_{11}\langle J_1\,S(v)\,,\,S(v)\rangle_1+|N_1(0,v)|\,|S(v)|_1\\ &= a_{11}|S(v)|_1^p+|N_1(0,v)|\,|S(v)|_1,\end{aligned}$$

that is,

$$|S(v)|_1 \leq \left(\frac{1}{1-a_{11}} |N_1(0,v)| \right)^{\frac{1}{p-1}}. \tag{3.8}$$

Now, we have proved that the solution operator is continuous and satisfies the growth condition (3.8). Next, with the aid of Leray-Schauder theorem (Theorem 2.4), we prove that the fixed point equation $v = J_2^{-1} N_2(S(v), v)$ admits a fixed point. First, note that $J_2^{-1} \circ N_2 \circ (S, I)$ is compact since $J_2^{-1} \circ N_2$ is compact and S, I are bounded and continuous operators. Further, we show that there exists $R > 0$ such that any fixed point of the operator $\lambda J_2^{-1} N_2(S(\cdot), \cdot)$ lies in the ball of radius R for any $\lambda \in (0, 1]$. Let $\lambda \leq 1$ and v a fixed point of the operator $\lambda J_2^{-1} N_2(S(\cdot), \cdot)$, i.e., $v = \lambda J_2^{-1} N_2(S(v), v)$. Since $J_2(\alpha v) = \alpha^{q-1} J_2(v)$ for any $\alpha \in \mathbb{R}_+$, we have

$$\begin{aligned}
|v|_2^q &= \lambda^q \langle J_2 \tfrac{1}{\lambda} v, \tfrac{1}{\lambda} v \rangle_2 \\
&= \lambda^{q-1} \langle N_2(S(v), v) - N_2(S(v), 0), v \rangle_2 + \lambda^{q-1} \langle N_2(S(v), 0), v \rangle_2 \\
&\leq a_{22} \lambda^{q-1} \langle J_2 v, v \rangle_2 + \lambda^{q-1} |v|_2 |N_2(S(v), 0)| \\
&\leq a_{22} \lambda^{q-1} |v|_2^q + |v|_2 |N_2(S(v), 0)|.
\end{aligned}$$

Now, from (3.8) and the growth conditions (3.3) and (3.4)), we obtain

$$\begin{aligned}
(1-a_{22}) |v|_2^{q-1} &\leq |N_2(S(v), 0)| \\
&\leq a_{21} |S(v)|_1^{q-1} + M_2 \\
&\leq a_{21} \left(\frac{a_{12}}{1-a_{11}} |v|^{p-1} + \frac{M_1}{1-a_{11}} \right)^{\frac{q-1}{p-1}} + M_2.
\end{aligned}$$

Since $\frac{q-1}{p-1} \leq 1$, one has

$$\begin{aligned}
(1-a_{22}) |v|_2^{q-1} &\leq a_{21} \left(\frac{a_{12}}{1-a_{11}} \right)^{\frac{q-1}{p-1}} |v|_2^{q-1} + M' \\
&\leq \frac{a_{21} a_{12}}{1-a_{11}} |v|_2^{q-1} + M',
\end{aligned}$$

which gives

$$|v|_2^{q-1} \leq \left(1 - a_{22} - \frac{a_{12} a_{21}}{1-a_{11}} \right)^{-1} M',$$

where $M' := \left(\frac{1}{1-a_{11}} M_1 \right)^{\frac{q-1}{p-1}} + M_2$. Since matrix A is convergent to zero, Lemma 2.2 yields that the solution v lies in the ball $B(0, R)$, where

$$R = \left(1 - a_{22} - \frac{a_{12} a_{21}}{1-a_{11}} \right)^{-\frac{1}{q-1}} (M')^{\frac{1}{q-1}}.$$

Now, the Leray-Schauder fixed point theorem guarantees the existence of a point v^* such that $J_2(v^*) = N_2(S(v^*), v^*)$. One can see that $(S(v^*), v^*)$ is a solution to system (1.1), which ends our proof. □

4. APPLICATIONS

Consider the (p,q)-Laplacian system with Dirichlet boundary conditions

$$\begin{cases} -\Delta_p\, u = f_1(\cdot,u,v), \\ -\Delta_q\, v = f_2(\cdot,u,v) \text{ on } \Omega, \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

where $p \geq q > 1$ and Ω is some bounded domain from $\mathbb{R}^n$ with Lipschitz boundary. We consider the Sobolev spaces $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively, endowed with the usual norms $|u|_{1,p} := |\nabla u|_{L^p}$ and $|u|_{1,q} := |\nabla u|_{L^q}$.

Note that, in the light of Theorem 2.7, the dual mapping J_1 is the p-Laplacian operators $-\Delta_p$ and J_2 the q-Laplacian operator $-\Delta_q$. We assume that the functions $f_1, f_2 \colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ are of Carathéodory type and satisfy the growth conditions

$$|f_1(x,r,s)| \leq C_1|r|^{p-1} + C_2|s|^{p-1} + a(x)\,, \tag{4.2}$$

$$|f_2(x,r,s)| \leq C_1|r|^{q-1} + C_2|r|^{q-1} + b(x)\,, \tag{4.3}$$

for all real numbers $r,s \in \mathbb{R}$, where $C_1, C_2 \in \mathbb{R}$, $a \in L^{p'}(\Omega)$ and $b \in L^{q'}(\Omega)$. Since f_1 and f_2 satisfy the growth conditions (4.2) and (4.3), the Nemytskii operators

$$N_{f_1}(u,v)(x) := f_1(x,u(x),v(x)) \text{ and } N_{f_2}(u,v)(x) := f_2(x,u(x),v(x))$$

are well defined from $L^p(\Omega)$ to $L^{p'}(\Omega)$, respectively $L^q(\Omega)$ to $L^{q'}(\Omega)$. Also, they are continuous and bounded. Due to the compact embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega)$ and $W_0^{1,q}(\Omega)$ in $L^q(\Omega)$, the operator

$$T = (-\Delta_q)^{-1} N_{f_2}(u,v) \colon W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) \to W_0^{1,q}(\Omega)$$

is compact (see Dinca and Jebelean [8]).

Note that each equation from (4.1) admits a variational structure, given by the energy functionals $E_1, E_2 \colon W^{1,p}(\Omega) \times W^{1,q}(\Omega) \to \mathbb{R}$,

$$E_1(u,v) := \frac{1}{p}|u|_{1,p}^p - \int_\Omega F_1(\cdot,u,v),\ E_2(u,v) := \frac{1}{q}|u|_{1,q}^q - \int_\Omega F_2(\cdot,u,v),$$

where

$$F_1(x,u(x),v(x)) := \int_0^{u(x)} f_1(x,s,w(x))ds,\ F_2(x,u(x),v(x)) := \int_0^{v(x)} f_2(x,u(x),s)ds.$$

Theorem 4.1. *Let the above conditions be fulfilled. In addition, assume that*

(H1) *there are non-negative real numbers $\overline{a}_{11}$ and $\overline{a}_{22}$ such that*

$$(r-\overline{r})(f_1(\cdot,r,s) - f_1(\cdot,\overline{r},s)) \leq \overline{a}_{11}|r-\overline{r}|^p, \tag{4.4}$$

$$(s-\overline{s})(f_2(\cdot,r,s) - f_2(\cdot,r,\overline{s})) \leq \overline{a}_{22}|s-\overline{s}|^q, \tag{4.5}$$

for all real numbers $r, \overline{r}, s$, and $\overline{s}$;

(H2) *there are non-negative real numbers $\overline{a}_{12}, \overline{a}_{21}, M_1$, and M_2 such that*

$$|f_1(\cdot,0,s)| \leq \overline{a}_{12}|s|^{p-1} + M_1, \tag{4.6}$$

$$|f_2(\cdot,r,0)| \leq \overline{a}_{21}|r|^{q-1} + M_2, \tag{4.7}$$

for all real numbers r and s;

(H3) *the matrix*

$$A := \begin{bmatrix} C^p \overline{a}_{11} & C^p \overline{a}_{12} \\ C^q \overline{a}_{21} & C^q \overline{a}_{22} \end{bmatrix}$$

is convergent to zero.

Then there exist $(u^*, v^*) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$*, a solution of the system* (4.1)*, which is a Nash equilibrium for the energy functionals* E_1, E_2*.*

For the proof, we need the following lemma.

Lemma 4.2 ([7, Proposition 8])**.** *Under the growth conditions* (4.6)-(4.7)*, the Nemytskii's operators* $(\overline{N}_{f_1} v)(x) := f_1(x, 0, v(x))$ *and* $(\overline{N}_{f_2} u)(x) := f_2(x, u(x), 0)$ *satisfy*

$$|\overline{N}_{f_1} v|_{L^{p'}} \le \overline{a}_{12} |v|_{L^p}^{p-1} + M_1'$$
$$|\overline{N}_{f_2} u|_{L^{q'}} \le \overline{a}_{21} |u|_{L^q}^{q-1} + M_2'.$$

Proof of the Theorem. We verify that all conditions of Theorem 3.2 are fulfilled.

Check (3.1) and (3.2). Let $u, \overline{u} \in W_0^{1,p}(\Omega)$ and $v \in W_0^{1,q}(\Omega)$. Then, from (4.4), we obtain

$$\begin{aligned} \langle f_1(\cdot,u,v) - f_1(\cdot,\overline{u},v)\,,\, u-\overline{u}\rangle_{W^{-1,p'}} &= \int_\Omega (u-\overline{u})\,(f_1(\cdot,u,v) - f_1(\cdot,\overline{u},v)) \\ &\le \overline{a}_{12} |u-\overline{u}|_{L^p}^p \\ &\le \overline{a}_{12} C^p \, |u-\overline{u}|_{1,p} \\ &= \overline{a}_{12} C^p \, \langle (-\Delta_p)u - (-\Delta_p)\overline{u}\,,\, u-\overline{u}\rangle_{W^{-1,p'}}. \end{aligned}$$

Similarly, (4.5) yields

$$\langle f_2(\cdot,u,v) - f_2(\cdot,u,\overline{v})\,,\, v-\overline{v}\rangle_{W^{-1,q'}} \le \overline{a}_{22} C^q \, \langle (-\Delta_q)v - (-\Delta_q)\overline{v}\,,\, v-\overline{v}\rangle_{W^{-1,q'}}.$$

*Check (**h1**).* The condition is trivially satisfied since $(-\Delta_q)^{-1} N_{f_2}(u,v)$ is compact.

*Check (**h2**).* Let $v \in W_0^{1,q}(\Omega)$. Then, Lemma 4.2 yields

$$\begin{aligned} |f_1(\cdot,0,v)|_{W^{-1,p'}} &\le C |f_1(\cdot,0,v)|_{L^{p'}} \\ &\le \overline{a}_{12} C^p \, |v|_{1,p}^{p-1} + C M_1'. \end{aligned}$$

Similarly, $|f_2(\cdot,u,0)|_{W^{-1,q'}} \le \overline{a}_{21} C^q \, |u|_{1,q}^{q-1} + C M_2'$.

Finally, note that all the assumptions from Theorem 3.2 are fulfilled, where

$$a_{11} = \overline{a}_{11} C^p\,;\, a_{22} = \overline{a}_{22} C^q$$
$$a_{12} = \overline{a}_{12} C^p\,;\, a_{21} = \overline{a}_{21} C^q.$$

Therefore, there exists $(u^*, v^*) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, a solution of system (4.1). Moreover, from Theorem 3.1, it is a Nash equilibrium for the energy functionals E_1, E_2. □

Example 4.3. Consider the Dirichlet problem

$$\begin{cases} -u'' = -u + \pi \sin(u) + \frac{\pi}{2} v \\ -v'' = u + \cos(v) & \text{on } (0,1) \\ u(0) = v(0) = u(1) = v(1) = 0. \end{cases} \tag{4.8}$$

We apply Theorem 4.1 with

$$\Omega = (0,1),\ p = q = 2,\ n = 1,\ C = \frac{1}{\pi}$$

$$f_1(x,r,s) = -r + \pi \sin(r) + \frac{\pi}{2}s,\ f_2(x,r,s) = r + \cos(s).$$

Note that growth conditions (4.2-4.3) are fulfilled with $C_1 = 1$, $C_2 = \frac{\pi}{2}$, $a(x) = \pi$, and $b(x) = 1$. Since $(-r+\bar{r})(r-\bar{r}) = -(r-\bar{r})^2 \le 0$, one has

$$(f_1(x,r,s) - f_1(x,\bar{r},s)(r-\bar{r}) \le \pi|r-\bar{r}|.$$

Similarly,

$$(f_2(x,r,s) - f_1(x,r,\bar{s})(s-\bar{s}) \le |s-\bar{s}|.$$

Hence, condition (H1) holds with $\bar{a}_{11} = \pi$ and $\bar{a}_{22} = 1$. Simple calculation demonstrates that (H2) also holds with $\bar{a}_{12} = \frac{\pi}{2}$, $\bar{a}_{21} = 1$, $M_1 = 0$, and $M_2 = 1$. One sees that the matrix

$$A = \begin{bmatrix} \frac{1}{\pi} & \frac{1}{2\pi} \\ \frac{1}{\pi^2} & \frac{1}{\pi^2} \end{bmatrix}$$

is convergent to zero. Therefore, system (4.8) has a solution $(u^*, v^*) \in W_0^{1,2}(0,1) \times W_0^{1,2}(0,1)$ which is a Nash equilibrium for the corresponding energy functionals.

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