J. Nonlinear Funct. Anal. 2023 (2023) 7 https://doi.org/10.23952/jnfa.2023.7

MATHRES

Journal of Nonlinear Functional Analysis

Available online at http://jnfa.mathres.org

VISCOSITY APPROXIMATION OF A MODIFIED INERTIAL SIMULTANEOUS ALGORITHM FOR A FINITE FAMILY OF DEMICONTRACTIVE MAPPINGS

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Abstract. In this paper, we propose a new modified inertial simultaneous algorithm of common fixed point problems for a finite family of demicontractive mappings and obtain some strong convergence results in real Hilbert spaces. Meanwhile, we also give a numerical example to demonstrate the efficiency of our proposed algorithm. Our results improve and extend some corresponding known results.

Keywords. Demicontractive mapping; Inertial algorithm; Numerical example; Split common fixed point problem; Viscosity approximation.

1. Introduction

In the study of the real problems in physics and control theory, the strong convergence plays a more important role, compared with the weak convergence. Thus, in order to obtain the strong convergence, numerous scholars justifiably devised a variety of iterative algorithms including the Mann algorithm; see, e.g., [2, 3, 8]. Moreover, it is known that the convergence rate of the Mann algorithm is slow. Recently, spotlight sheds on various fast algorithms, which are significant from the viewpoint of real applications; see, e.g., [1, 5, 15] and the references therein. In 1964, Polyak [13] first proposed an inertial type extrapolation as an acceleration process. Since then, based on inertial extrapolation techniques, authors have introduced various new iterative algorithms and obtained valuable results, such as, inertial forward-backward splitting algorithms [9], inertial Mann algorithms [10], and inertial extragradient algorithms [4].

In 2008, Mainge [10] introduced the following inertial Mann algorithm with the aid of the inertial extrapolation

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ x_{n+1} = \phi_n w_n + (1-\phi_n)T(w_n), \end{cases} \tag{1.1}$$

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Received October 14, 2022; Accepted January 16, 2023.

where T is a nonexpansive mapping (see below), and $\{\delta_n\}$ and $\{\phi_n\}$ are two real sequences. He proved that the iterative sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T under some mild assumptions in Hilbert spaces.

Recently, Tan, Zhou and Li [14] presented a modified inertial Mann algorithm:

$$\begin{cases} w_n = x_n + \delta_n(x_n - x_{n-1}), \\ y_n = \phi_n w_n + (1-\phi_n)T(w_n), \\ x_{n+1} = \nu_n u + (1-\nu_n)y_n, \end{cases} \tag{1.2}$$

where T is a nonexpansive mapping, and $\{\delta_n\}$ and $\{\phi_n\}$ are two real sequences. They stated that the iterative sequence $\{x_n\}$ defined by (1.2) converges strongly to $p = P_{F(T)}u$ in Hilbert spaces without the aid of compact assumptions.

In recent years, the following split common fixed point problem (SCFPP) has attracted wide attention in the community of nonlinear optimization, which is to find

$$x^* \in \cap_{i=1}^{p} F(U_i) \text{ and } Ax^* \in \cap_{j=1}^{s} F(T_j), \tag{1.3}$$

where $p,s \geq 1$ are integers, A: $H_1 \to H_2$ is a bounded linear operator, $\{U_i\}_{i=1}^{p} : H_1 \to H_1$ and $\{T_j\}_{j=1}^{s} : H_2 \to H_2$ are families of nonlinear operators, and $F(U_i)$ and $F(T_j)$ stand for the sets of all fixed points of U_i and T_j, respectively.

To solve the SCFPP, Tang, Peng and Liu [16] introduced the following simultaneous iterative algorithm, which is also called the parallel iterative algorithm:

$$\begin{cases} u_k = x_k + \gamma A^* \sum_{j=1}^{s} \eta_j (T_j - I)Ax_k, \\ x_{k+1} = (1-\alpha_k)u_k + \alpha_k \sum_{i=1}^{p} \omega_i U_i(u_k), \end{cases}$$

where γ is some positive real number, $\{\alpha_k\} \subset (0,1)$, $\{\omega_i\}_{i=1}^{p} \subset (0,1)$, and $\{\eta_j\}_{j=1}^{s} \subset (0,1)$ with $\sum_{i=1}^{p} \omega_i = 1$ and $\sum_{j=1}^{s} \eta_j = 1$. They obtained the weak convergence of this algorithm and solved the SCFPP (1.3) governed by demicontractive mappings $U_i (1 \leq i \leq p)$ and $T_j (1 \leq j \leq s)$.

Inspired and motivated by the above works, we introduce a new modified inertial simultaneous algorithm for a finite family of demicontractive mappings by using viscosity approximation in Hilbert spaces. In addition, under suitable conditions, we prove some strong convergence results. Finally, we give a numerical example to demonstrate the efficiency of our proposed algorithm.

2. Preliminaries

Throughout this paper, let R be the set of real numbers, and let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a closed, convex, and nonempty subset of H. We denote the weak and strong convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. From now on, $F(S)$ denotes the fixed-point set of a mapping S.

Recall that the metric (or nearest point) projection P_C from H onto C is defined as follows. For any given $x \in H$, there exists a unique vector in C, $P_C x$ such that $P_C x := \arg\min_{y \in C} \|x - y\|$. It is well-known [19] that P_C is a nonexpansive mapping and is characterized by

$$P_C x \in C, \langle x - P_C x, y - P_C x \rangle \leq 0, \forall\, y \in C.$$

For each $x, y \in H$, we also have some following known facts

(a_1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$;

(a_2) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall\, t \in R.$

Definition 2.1. Let C be a nonempty, convex, and closed subset of a real Hilbert space H. A mapping $S:C\to C$ is said to be
(i) contractive if there exists $\alpha\in(0,1)$ such that
$$\|Sx-Sy\|\le\alpha\|x-y\|,\ \forall\, x,y\in C;$$
(ii) nonexpansive if
$$\|Sx-Sy\|\le\|x-y\|,\ \forall\, x,y\in C;$$
(iii) quasi-nonexpansive if $F(S)\neq\emptyset$ and
$$\|Sx-q\|\le\|x-q\|,\ \forall\,(x,q)\in C\times F(S);$$
(iv) firmly nonexpansive if
$$\|Sx-Sy\|^2\le\|x-y\|^2-\|(I-S)x-(I-S)y\|^2,\ \forall\, x,y\in C;$$
(v) directed if
$$\|Sx-q\|^2\le\|x-q\|^2-\|x-Sx\|^2,\ \forall\,(x,q)\in C\times F(S);$$
(vi) μ-demicontractive if $F(S)\neq\emptyset$ and there exists a constant $\mu\in(-\infty,1)$ such that
$$\|Sx-q\|^2\le\|x-q\|^2+\mu\|x-Sx\|^2,\ \forall\,(x,q)\in C\times F(S).$$

Remark 2.2. [17] Note that every 0-demicontractive mapping is exactly quasi-nonexpansive. In particular, if $\mu\le0$, then every μ-demicontractive mapping becomes quasi-nonexpansive. Moreover, we say that it is quasi-strict pseudo-contractive [11] if $0\le\mu<1$. Therefore, it is sufficient to only take $\mu\in(0,1)$ in (vi) of Definition 2.1.

It is easy to know that every quasi-nonexpansive mapping is demicontractive. However, there exist some demicontractive mappings which are not quasi-nonexpansive. We can see this via following example.

Example 2.3. [18] Let $H=l_2$ and $S:l_2\to l_2$ be defined by $Sx=-kx$, for $\forall x\in l_2$, where $k>1$. Then S is a $\frac{k-1}{k+1}$-demicontractive mapping which is not quasi-nonexpansive.

Definition 2.4. Let C be a nonempty, convex, and closed subset of a real Hilbert space H. An operator $S:C\to C$ is said to be demiclosed at 0 if $\{x_n\}$ converges weakly to x, and $\{Sx_n\}$ converges strongly to 0 for any sequence $\{x_n\}$, then $Sx=0$.

For a quasi-nonexpansive mapping $S:C\to H$, is $I-S$ still demiclosed on C? The answer is negative even at 0, which can be checked via the following example.

Example 2.5. [17] Let $S:[0,1]\to[0,1]$ be a mapping defined by
$$Sx=\begin{cases}\frac{x}{5}, & x\in[0,\frac{1}{2}],\\ x\sin\pi x, & x\in(\frac{1}{2},1].\end{cases}$$
Then S is quasi-nonexpansive, but $I-S$ is not demiclosed at 0.

To prove our main results, we also need the following lemmas.

Lemma 2.6. [12] *Let S be μ-demicontractive self-mapping on H with $F(S)\neq\emptyset$ and set $S_\lambda=(1-\lambda)I+\lambda S$ for $\lambda\in[0,1]$. Then, S_λ is quasi-nonexpansive provided that $\lambda\in[0,1-u]$, and*
$$\|S_\lambda x-q\|^2\le\|x-q\|^2-\lambda(1-\mu-\lambda)\|x-Sx\|^2,\ (x,q)\in H\times F(S).$$

Lemma 2.7. [6] *Let C be a nonempty, convex, and closed subset of a real Hilbert space H, and let $S : C \to H$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Sx_n - x_n \to 0$ as $n \to \infty$. Then $x \in F(S)$.*

Lemma 2.8. [11] *Let C be a nonempty, convex, and closed subset of a real Hilbert space H. Let $S : C \to C$ be a self-mapping on C. If S is a μ-demicontractive mapping (which is also called a μ-quasi-strict-pseudo-contraction in [11]), then the fixed point set $F(S)$ is convex and closed.*

Lemma 2.9. [7] *Assume that $\{V_n\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} V_{n+1} \leq (1-\beta_n)V_n + \beta_n\delta_n, \\ V_{n+1} \leq V_n - \eta_n + \mu_n, \end{cases}$$

where $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\beta_n\}$ is a sequence in $(0,1)$, and $\{\delta_n\}$ and $\{\mu_n\}$ are two real sequences such that (i) $\sum_{n=1}^{\infty}\beta_n = \infty$; (ii) $\lim_{n\to\infty}\mu_n = 0$; (iii) $\lim_{n\to\infty}\mu_{n_k} = 0$ implies $\lim_{k\to\infty}\delta_{n_k} = 0$ for any subsequence $\{n_k\} \subset \{n\}$. Then $\lim\limits_{n\to\infty} V_n = 0$.

3. Main Results

Theorem 3.1. *Let H be a real Hilbert space. Let $U_i : H \to H\,(1 \leq i \leq s)$ be a τ_i-demicontractive mapping and $f : H \to H$ be a contraction with constant $\alpha \in \left(0, \frac{1}{\sqrt{2}}\right)$. Suppose that $I - U_i\,(1 \leq i \leq s)$ is demiclosed at 0 and $\bigcap_{i=1}^{s} F(U_i) \neq \emptyset$. Let $\left\{\lambda_n^{(i)}\right\}(0 \leq i \leq s)$ be a sequence in $[0,1]$ such that $\sum_{i=0}^{s}\lambda_n^{(i)} = 1$. Let $\{\beta_n\}$ be a sequence in $(0,1)$, and assume that the following conditions hold*

(D_1) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;
(D_2) $\lim_{n\to\infty}\frac{\rho_n}{\beta_n}\|x_n - x_{n-1}\| = 0$;
(D_3) $\liminf\limits_{n\to\infty}\lambda_n^{(0)} > \tau$;
(D_4) $\liminf\limits_{n\to\infty}\lambda_n^{(i)} > 0\,(1 \leq i \leq s)$,

where $\tau = \max_{1\leq i\leq s}\tau_i$. Let $x_0, x_1 \in H$ be two arbitrary initials. Define a sequence via the following algorithm:

$$\begin{cases} u_n = x_n + \rho_n(x_n - x_{n-1}), \\ y_n = \lambda_n^{(0)}u_n + \sum_{i=1}^{s}\lambda_n^{(i)}U_i(u_n), \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n)y_n. \end{cases} \tag{3.1}$$

Then the sequence $\{x_n\}$ defined by (3.1) converges strongly to $p = P_\Omega f(p)$, where $\Omega = \bigcap_{i=1}^{s} F(U_i)$.

Proof. First, we prove that $\{x_n\}$ is bounded. In fact, from Lemma 2.8, one sees that, for any $i \in \{1,2,\ldots,s\}$, $F(U_i)$ is close and convex, which indicates that Ω is closed and convex. Since P_Ω is nonexpansive, and f is contractive, we, according to the Banach fixed point theorem, can obtain that $P_\Omega f$ is a contraction, so there exists a unique point $p \in \Omega$ such that $p = P_\Omega f(p)$. By the conditions (D_1), (D_3), and (D_4), there exists $r \in (0,1)$ and a large enough number $n_0 > 0$, for any $n > n_0$, $\lambda_n^{(0)} > r + \tau$, $\lambda_n^{(i)} > r$, $0 < \beta_n < 1 - \alpha$. Taking $\omega_n^{(i)} = \frac{\lambda_n^{(i)}}{1-\lambda_n^{(0)}}\,(1 \leq i \leq s)$, we have $\sum_{i=1}^{s}\omega_n^{(i)} = 1$. Observe that

$$\begin{aligned} \lambda_n^{(0)}u_n + \sum_{i=1}^{s}\lambda_n^{(i)}U_i(u_n) &= \lambda_n^{(0)}u_n + \left(1-\lambda_n^{(0)}\right)\sum_{i=1}^{s}\omega_n^{(i)}U_i(u_n) \\ &= \sum_{i=1}^{s}\omega_n^{(i)}\left(\lambda_n^{(0)}u_n + \left(1-\lambda_n^{(0)}\right)U_i(u_n)\right). \end{aligned} \tag{3.2}$$

From (3.1), (3.2), the convexity of $\|.\|^2$, the condition (D_3), and Lemma 2.6, we obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\sum_{i=1}^{s} \omega_n^{(i)} (\lambda_n^{(0)} u_n + \left(1-\lambda_n^{(0)}\right) U_i(u_n)) - p\right\|^2 \\
&\le \sum_{i=1}^{s} \omega_n^{(i)} \left\|\lambda_n^{(0)} u_n + \left(1-\lambda_n^{(0)}\right) U_i(u_n) - p\right\|^2 \\
&\le \sum_{i=1}^{s} \omega_n^{(i)} \left(\|u_n - p\|^2 - \left(1-\lambda_n^{(0)}\right)\left(\lambda_n^{(0)} - \tau\right)\|U_i(u_n) - u_n\|^2\right) \\
&= \|u_n - p\|^2 - \left(1-\lambda_n^{(0)}\right)\left(\lambda_n^{(0)} - \tau\right)\sum_{i=1}^{s} \omega_n^{(i)} \|U_i(u_n) - u_n\|^2 \\
&= \|u_n - p\|^2 - \left(\lambda_n^{(0)} - \tau\right)\sum_{i=1}^{s} \lambda_n^{(i)} \|U_i(u_n) - u_n\|^2 \qquad (3.3)\\
&\le \|u_n - p\|^2 \qquad (3.4)
\end{aligned}$$

for all $n \ge n_0$, and

$$\|u_n - p\| \le \|x_n - p\| + \rho_n \|x_n - x_{n-1}\|. \qquad (3.5)$$

From (3.4) and (3.5), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\| \\
\le& \beta_n \|f(x_n) - p\| + (1-\beta_n)\|y_n - p\| \\
\le& \beta_n \|f(x_n) - p\| + (1-\beta_n)\|u_n - p\| \\
\le& \beta_n \|f(x_n) - f(p)\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|u_n - p\| \\
\le& \beta_n \alpha \|x_n - p\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|u_n - p\| \\
\le& \beta_n \alpha \|x_n - p\| + \beta_n \|f(p) - p\| + (1-\beta_n)\|x_n - p\| + \rho_n \|x_n - x_{n-1}\| \\
=& [1-(1-\alpha)\beta_n]\|x_n - p\| + (1-\alpha)\beta_n \left[\frac{\|f(p) - p\|}{1-\alpha} + \frac{\rho_n \|x_n - x_{n-1}\|}{(1-\alpha)\beta_n}\right] \qquad (3.6)
\end{aligned}$$

for all $n \ge n_0$. By condition (D_2), we let

$$M := 2\max\left\{\frac{\|f(p) - p\|}{1-\alpha}, \sup_{n\ge 0} \frac{\rho_n \|x_n - x_{n-1}\|}{(1-\alpha)\beta_n}\right\}.$$

It follows from (3.6) that

$$\begin{aligned}
\|x_{n+1} - p\| &\le [1-(1-\alpha)\beta_n]\|x_n - p\| + (1-\alpha)\beta_n M \\
&\le \max\{\|x_n - p\|, M\} \\
&\vdots \\
&\le \max\{\|x_{n_0} - p\|, M\},
\end{aligned}$$

for all $n \ge n_0$. Hence $\{x_n\}$ is bounded. From conditions (D_1) and (D_2), we can know that $\lim_{n\to\infty} \rho_n \|x_n - x_{n-1}\| = 0$, which implies that $\{u_n\}$ is bounded. Furthermore, it follows from

(3.4) that $\{y_n\}$ is also bounded. Therefore, it follows from (3.3), (a_1), and condition (D_3) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\beta_n)^2\|y_n-p\|^2+2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle \\
&\leq (1-\beta_n)^2\|u_n-p\|^2+2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle \\
&\quad -\left(\lambda_n^{(0)}-\tau\right)(1-\beta_n)^2\textstyle\sum_{i=1}^s \lambda_n^{(i)}\|U_i(u_n)-u_n\|^2 \\
&= (1-\beta_n)^2\left\|x_n+\rho_n\left(x_n-x_{n-1}\right)-p\right\|^2+2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle \\
&\quad -\left(\lambda_n^{(0)}-\tau\right)(1-\beta_n)^2\textstyle\sum_{i=1}^s \lambda_n^{(i)}\|U_i(u_n)-u_n\|^2 \\
&= (1-\beta_n)^2\|x_n-p\|^2+\rho_n^2(1-\beta_n)^2\left\|x_n-x_{n-1}\right\|^2 \\
&\quad +2\rho_n(1-\beta_n)^2\langle x_n-x_{n-1}, x_n-p\rangle+2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle \\
&\quad -\left(\lambda_n^{(0)}-\tau\right)(1-\beta_n)^2\textstyle\sum_{i=1}^s \lambda_n^{(i)}\|U_i(u_n)-u_n\|^2 \qquad (3.7)\\
&\leq (1-\beta_n)^2\|x_n-p\|^2+\rho_n^2(1-\beta_n)^2\left\|x_n-x_{n-1}\right\|^2 \\
&\quad +2\rho_n(1-\beta_n)^2\langle x_n-x_{n-1}, x_n-p\rangle+2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle \qquad (3.8)
\end{aligned}$$

for every $n \geq n_0$. We also have

$$\begin{aligned}
&2\beta_n\langle f(x_n)-p, x_{n+1}-p\rangle \\
=&2\beta_n\langle f(x_n)-f(p), x_{n+1}-p\rangle+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
\leq&2\beta_n\|f(x_n)-f(p)\|\,\|x_{n+1}-p\|+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
\leq&\beta_n\left[\|f(x_n)-f(p)\|^2+\|x_{n+1}-p\|^2\right]+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle \\
\leq&\beta_n\alpha\|x_n-p\|^2+\beta_n\|x_{n+1}-p\|^2+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle. \qquad (3.9)
\end{aligned}$$

From (3.8) and (3.9), we have, for all $n \geq n_0$,

$$\begin{aligned}
(1-\beta_n)\|x_{n+1}-p\|^2 \leq& \left[(1-\beta_n)^2+\alpha\beta_n\right]\|x_n-p\|^2+\rho_n^2(1-\beta_n)^2\|x_n-x_{n-1}\|^2 \\
&+2\rho_n(1-\beta_n)^2\langle x_n-x_{n-1}, x_n-p\rangle+2\beta_n\langle f(p)-p, x_{n+1}-p\rangle.
\end{aligned} \tag{3.10}$$

From (D_1), (3.10) can be re-written as

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq& \left(1-\frac{\beta_n(1-\alpha-\beta_n)}{1-\beta_n}\right)\|x_n-p\|^2+\rho_n^2(1-\beta_n)\|x_n-x_{n-1}\|^2 \\
&+2\rho_n(1-\beta_n)\langle x_n-x_{n-1}, x_n-p\rangle+\frac{2\beta_n}{1-\beta_n}\langle f(p)-p, x_{n+1}-p\rangle.
\end{aligned} \tag{3.11}$$

From (3.7) and (3.9), for large enough number n, we see that

$$\begin{aligned}
\|x_{n+1}-p\|^2 \leq& \|x_n-p\|^2+\rho_n^2(1-\beta_n)\|x_n-x_{n-1}\|^2 \\
&+2\rho_n(1-\beta_n)\langle x_n-x_{n-1}, x_n-p\rangle+\frac{2\beta_n}{1-\beta_n}\langle f(p)-p, x_{n+1}-p\rangle \\
&-\left(\lambda_n^{(0)}-\tau\right)(1-\beta_n)\textstyle\sum_{i=1}^s \lambda_n^{(i)}\|U_i(u_n)-u_n\|^2.
\end{aligned} \tag{3.12}$$

Let $S_n = \|x_n - p\|^2$, $\pi_n = z_n \sigma_n$, $z_n = \frac{\beta_n(1-\alpha-\beta_n)}{1-\beta_n}$,

$$\eta_n = \left(\lambda_n^{(0)} - \tau\right)(1-\beta_n)\sum_{i=1}^{s} \lambda_n^{(i)} \|U_i(u_n) - u_n\|^2,$$

and

$$\sigma_n = \frac{\rho_n^2 (1-\beta_n)^2}{\beta_n(1-\alpha-\beta_n)}\|x_n - x_{n-1}\|^2 + \frac{2\rho_n(1-\beta_n)^2}{\beta_n(1-\alpha-\beta_n)}\langle x_n - x_{n-1}, x_n - p\rangle$$
$$+ \frac{2}{1-\alpha-\beta_n}\langle f(p) - p, x_{n+1} - p\rangle.$$

From (3.11) and (3.12), we have $S_{n+1} \le (1-z_n)S_n + z_n\sigma_n$ and $S_{n+1} \le S_n - \eta_n + \pi_n$. From conditions (D_1) and (D_2), we have $\sum_{n=0}^{\infty} z_n = \infty$ and $\lim_{n\to\infty}\pi_n = 0$. In order to use Lemma 2.9, it remains to prove that $\lim_{k\to\infty}\eta_{n_k} = 0$. Thus we can deduce $\limsup_{k\to 0}\sigma_{n_k} \le 0$ for any subsequence $\{\eta_{n_k}\}$ of $\{\eta_n\}$. Indeed, let $\{\eta_{n_k}\}$ be a subsequence of $\{\eta_n\}$ such that $\lim\limits_{k\to\infty}\eta_{n_k} = 0$, which implies that

$$\lim_{k\to\infty}\|U_i(u_{n_k}) - u_{n_k}\| = 0\,(1 \le i \le s), \tag{3.13}$$

which is due to conditions (D_1), (D_3), and (D_4). From conditions (D_1) and (D_2), we have

$$\|u_{n_k} - x_{n_k}\| = \rho_{n_k}\|x_{n_k} - x_{n_k-1}\| \to 0, \ as\ k \to \infty. \tag{3.14}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\left\{x_{n_{k_j}}\right\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup \bar{x}$ and $\limsup\limits_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim\limits_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle$. From (3.14) , we have $u_{n_{k_j}} \rightharpoonup \bar{x}$. Since $I - U_i\,(1 \le i \le s)$ is demiclosed at 0, we obtain from (3.13) that $\bar{x} \in \Omega = \bigcap_{i=1}^{s} F(U_i)$. Combining the property of projections and $p = P_\Omega f(p)$, we obtain

$$\limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle = \lim_{j\to\infty}\langle f(p) - p, x_{n_{k_j}} - p\rangle = \langle f(p) - p, \bar{x} - p\rangle \le 0. \tag{3.15}$$

On the other hand, we find from (3.13) that

$$\|y_{n_k} - u_{n_k}\| \le \left(1 - \lambda_{n_k}^{(0)}\right)\sum_{i=1}^{s}\omega_{n_k}^{(i)}\|U_i(u_{n_k}) - u_{n_k}\| \to 0. \tag{3.16}$$

It follows that $\|y_{n_k} - x_{n_k}\| \le \|y_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0$ as $k \to \infty$. Further, using condition (D_1), we see that

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &\le \beta_{n_k}\|f(x_{n_k}) - f(p) + f(p) - x_{n_k}\| + (1-\beta_{n_k})\|y_{n_k} - x_{n_k}\| \\ &\le \beta_{n_k}\|f(x_{n_k}) - f(p)\| + \beta_{n_k}\|f(p) - x_{n_k}\| + (1-\beta_{n_k})\|y_{n_k} - x_{n_k}\| \\ &\le \beta_{n_k}\alpha\|x_{n_k} - p\| + \beta_{n_k}\|f(p) - x_{n_k}\| + (1-\beta_{n_k})\|y_{n_k} - x_{n_k}\| \to 0\end{aligned}$$

as $k \to \infty$. From (3.15), we have

$$\begin{aligned}&\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - p\rangle \\ =&\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - x_{n_k} + x_{n_k} - p\rangle \\ \le&\limsup_{k\to\infty}\langle f(p) - p, x_{n_k+1} - x_{n_k}\rangle + \limsup_{k\to\infty}\langle f(p) - p, x_{n_k} - p\rangle \\ \le&\limsup_{k\to\infty}\|f(p) - p\|\,\|x_{n_k+1} - x_{n_k}\| + \langle f(p) - p, \bar{x} - p\rangle \le 0.\end{aligned}$$

This together with the conditions (D_1), (D_2), and

$$\begin{cases} \limsup\limits_{k\to\infty} \frac{\rho_{n_k}^2\left(1-\beta_{n_k}\right)^2}{\beta_{n_k}\left(1-\alpha-\beta_{n_k}\right)} \left\|x_{n_k}-x_{n_k-1}\right\| = 0, \\ \limsup\limits_{k\to\infty} \frac{2\rho_{n_k}\left(1-\beta_{n_k}\right)^2}{\beta_{n_k}\left(1-\alpha-\beta_{n_k}\right)} \langle x_{n_k}-x_{n_k-1}, x_{n_k}-p\rangle = 0, \\ \limsup\limits_{k\to\infty} \frac{2\beta_{n_k}}{\left(1-\alpha-\beta_{n_k}\right)} \langle f(p)-p, x_{n_k+1}-p\rangle \le 0 \end{cases}$$

implies that $\limsup_{k\to\infty}\sigma_{n_k} \le 0$. From Lemma 2.9, we observe that $\lim_{n\to\infty} S_n = 0$. Hence $x_n \to p$ as $n\to\infty$. This completes the proof. □

Remark 3.2. [14] In particular, we can set

$$0 \le \rho_n \le \bar{\rho}_n,\ \bar{\rho}_n = \begin{cases} \min\left\{\frac{\xi_n}{\|x_n-x_{n-1}\|}, \frac{n-1}{n+\eta-1}\right\}, & if\ x_n \ne x_{n-1}; \\ \frac{n-1}{n+\eta-1}, & otherwise, \end{cases}$$

where $\eta \ge 3$ and $\{\xi_n\}$ is a positive sequence such that $\lim_{n\to\infty}\frac{\xi_n}{\beta_n} = 0$.

Remark 3.3. Theorem 3.1 improves [14, Algorithm 1.2] to viscosity approximation and extends a single nonexpansive to a finite family of demicontractive mappings.

If $U_1 = U_2 = \cdots = U_s = U$, we have from Theorem 3.1 the following corollary.

Corollary 3.4. *Let H a real Hilbert space. Let $U: H\to H$ be a τ-demicontractive mapping and let $f: H\to H$ be a contraction with constant $\alpha \in \left(0, \frac{1}{\sqrt{2}}\right)$. Suppose that $I-U$ is demiclosed at 0 and $F(U)\ne\emptyset$. Let $\left\{\lambda_n^{(i)}\right\}$ $(0\le i\le 1)$ be a sequence in $[0,1]$ such that $\sum_{i=0}^{1}\lambda_n^{(i)} = 1$. Let $\{\beta_n\}$ be a sequence in $(0,1)$ and the following conditions hold:*
(D_1) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;
(D_2) $\lim_{n\to\infty}\frac{\rho_n}{\beta_n}\|x_n-x_{n-1}\| = 0$;
(D_3) $\liminf_{n\to\infty}\lambda_n^{(0)} > \tau$;
(D_4) $\liminf_{n\to\infty}\lambda_n^{(1)} > 0$.
Let $x_0, x_1 \in H$ be arbitrary initials. Define a sequence via the following algorithm:

$$\begin{cases} u_n = x_n + \rho_n(x_n - x_{n-1}), \\ y_n = \lambda_n^{(0)} u_n + \lambda_n^{(1)} U(u_n), \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n. \end{cases} \tag{3.17}$$

Then the sequence $\{x_n\}$ defined by (3.17) converges strongly to $p = P_{F(U)}f(p)$.

Since every nonexpansive mapping is a 0-demicontractive mapping, we can obtain by Lemma 2.7 and Corollary 3.4 the following result immediately.

Corollary 3.5. *Let H be a real Hilbert space. Let $U: H\to H$ be a nonexpansive mapping, and let $f: H\to H$ be a contraction with constant $\alpha\in(0,\frac{1}{\sqrt{2}})$. Suppose that $F(U)\ne\emptyset$. Let $\left\{\lambda_n^{(i)}\right\}$ $(0\le i\le 1)$ be a sequence in $[0,1]$ such that $\sum_{i=0}^{1}\lambda_n^{(i)} = 1$. Let $\{\beta_n\}$ be a sequence in $(0,1)$ and the following conditions hold:*
(D_1) $\lim_{n\to\infty}\beta_n = 0$ *and* $\sum_{n=0}^{\infty}\beta_n = \infty$;

(D_2) $\lim_{n\to\infty}\frac{\rho_n}{\beta_n}\|x_n - x_{n-1}\| = 0$;
(D_3) $\liminf_{n\to\infty}\lambda_n^{(0)} > \tau$;
(D_4) $\liminf_{n\to\infty}\lambda_n^{(1)} > 0$.
Let $x_0, x_1 \in H$ be arbitrary initials. Define a sequence by

$$\begin{cases} u_n = x_n + \rho_n(x_n - x_{n-1}), \\ y_n = \lambda_n^{(0)} u_n + \lambda_n^{(1)} U(u_n), \\ x_{n+1} = \beta_n f(x_n) + (1-\beta_n) y_n. \end{cases} \tag{3.18}$$

Then the sequence $\{x_n\}$ defined by (3.18) converges strongly to $p = P_{F(U)} f(p)$.

4. Numerical Experiment

In this section, in order to demonstrate the realization and convergence of algorithm (3.1), we consider the following example in $(R, |\cdot|)$.

Example 4.1. Let $H = R$. Let $f : R \to R$ be defined by $f(x) = \frac{1}{2}x$. Let $U_i(x) = -2ix$ for $i = 1,2,3$. Choose $\xi_n = \frac{1}{(n+1)^2}$, $\beta_n = \frac{1}{n+1}$, $\lambda_n^{(0)} = \frac{6n}{7(n+1)}$, and $\lambda_n^{(i)} = \frac{n+7}{21(n+1)}$ for $i = 1,2,3$. Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $p = P_\Omega f(p)$, where $\Omega = \bigcap_{i=1}^3 F(U_i)$.

It is easy to see that $\Omega = \cap_{i=1}^3 F(U_i) = \{0\} \neq \emptyset$, and f is a contraction with constant $\frac{1}{2}$. From Example 2.3, we obtain that U_i $(1 \le i \le 3)$ is a $\frac{2i-1}{2i+1}$-demicontractive mapping and $I - U_i$ is demiclosed at 0. It can be observed that all the assumptions of Theorem 3.1 and conditions $(D_1) \sim (D_4)$ are satisfied. Algorithm (3.1) is reduced to the following: $x_{n+1} = \frac{1}{2(n+1)}x_n + \frac{n}{n+1}y_n$, where $y_n = \frac{2n-28}{7(n+1)}(x_n + \rho_n(x_n - x_{n-1}))$. Hence, from Theorem 3.1, the sequence defined above converges strongly to $0 \in \Omega = \{0\}$.

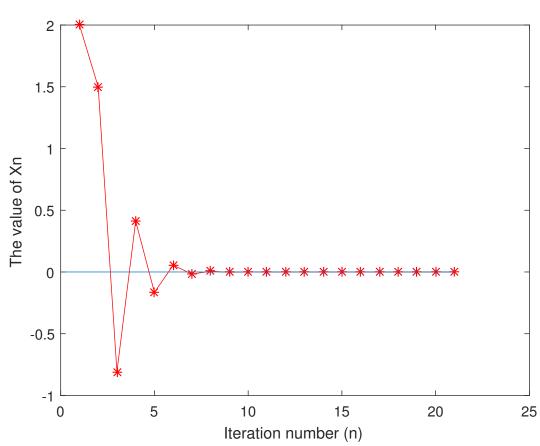

FIGURE 1. Numerical result for Example 4.1 with initial values $x_0 = 2$, $x_1 = 1.5$

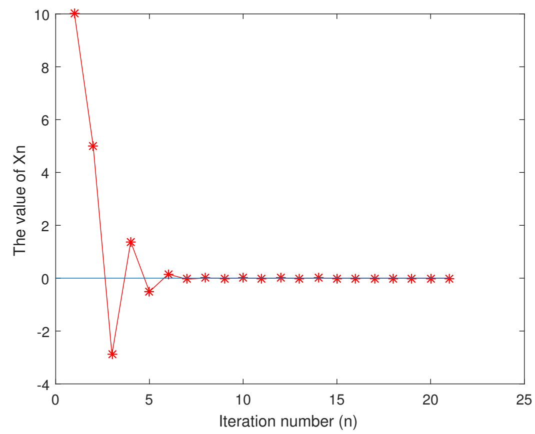

FIGURE 2. Numerical result for Example 4.1 with initial values $x_0 = 10$, $x_1 = 5$

TABLE 1. Computational results for Example 4.1

	$x_0 = 2$, $x_1 = 1.5$	$x_0 = 10$, $x_1 = 5$
n	x_n	x_n
$n = 0$	2	10
$n = 1$	1.5	5
$n = 3$	-0.8082	-2.89153
$n = 4$	0.41206	1.379328
$n = 5$	-0.16545	-0.5109
$n = 6$	0.055223	0.419811
$n = 7$	-0.01722	-0.03694
$n = 8$	0.006109	0.00919
$n = 9$	-0.00279	-0.00314
$n = 10$	0.001051	0.001399
$n = 11$	-0.00032	-0.0039
$n = 12$	7.69E-05	9.94E-05
$n = 13$	-1.29E-05	-1.58E-05
$n = 14$	1.14E-06	1.48E-06
$n = 15$	3.82E-08	4.94E-08
$n = 16$	-1.34E-08	-1.74E-08
$n = 17$	-2.18E-09	-2.82E-09
$n = 18$	2.67E-10	3.46E-10
$n = 19$	1.41E-10	1.82E-10
$n = 20$	5.71E-12	7.39E-12
$n = 21$	-8.49E-12	-1.10E-11
$n = 22$	0.0000	0.0000
$n = 23$	0.0000	0.0000
$n = 24$	0.0000	0.0000
$n = 25$	0.0000	0.0000

Funding

Yaqin Wang was supported by the National Natural Science Foundation under Grant No.11975156.

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