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EXISTENCE OF ONE WEAK SOLUTION FOR A STEKLOV PROBLEM INVOLVING THE WEIGHTED $p(\cdot)$-LAPLACIAN

SHAPOUR HEIDARKHANI[1,*], SHAHIN MORADI[1], GHASEM A. AFROUZI[2]

[1]Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran

[2]Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

Abstract. In this study, we investigate the existence of at least one weak solution for a nonlinear Steklov boundary-value problem involving weighted $p(\cdot)$-Laplacian. Our technical approach is based on variational methods. In addition, an example to illustrate our results is given.

Keywords. Steklov problem; Variational methods; Weak solution; Weighted variable exponent.

1. INTRODUCTION

The aim of this paper is to establish the existence of at least one weak solution for the following Steklov problem of the type

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) = b(x)|u|^{p(x)-2}u, & x \in \Omega, \\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = f(x,u(x)), & x \in \partial\Omega, \end{cases} \tag{P^f}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded smooth domain, $\frac{\partial u}{\partial v}$ is the outer unit normal derivative on $\partial\Omega$, p is a continuous function on $\overline{\Omega}$, i.e., $p \in C(\overline{\Omega})$ with $\inf_{y\in\overline{\Omega}} p(y) > N$, the function $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $a(x)$ and $b(x)$ are weight functions.

The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, which is mainly due to the fact that it is not homogeneous. The study of differential and partial differential equations with variable exponent has been received considerable attention in recent years due to their various physical applications. In fact, there are applications concerning elastic mechanics [35], electrorheological fluids [33], image restoration [12], and continuum mechanics [7]. Materials, which require such advanced theories, have been under experimental studies since 1950. The first important discovery on electrorheological fluids was contributed

*Corresponding author.

E-mail address: sh.heidarkhani@razi.ac.ir (S. Heidarkhani) sh.moradi@razi.ac.ir (S. Moradi), afrouzi@umz.ac.ir (G.A. Afrouzi).

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by Willis Winslow in 1949. The viscosity of these fluids depends upon the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. For a general account of the underlying physics; see [18] and for some technical applications [31]. Electrorheological fluids also have functions in robotics and space technology. Moreover, the $p(x)$-Laplacian operator was studied extensively; see [10, 13, 15, 16, 19, 23, 25] and the references therein.

Nonlinear Steklov boundary-value problem involving weighted $p(x)$-Laplacian has captured a special attention in the recent last years. We refer the reader to [3, 5, 6, 8, 9, 26, 34] and the references therein. For example, in [6], Allaoui et al., by using the variational method, under appropriate assumptions on f, obtained some results on the existence and multiplicity of solutions for the following nonlinear Steklov boundary-value problem

$$\begin{cases} \Delta_{p(x)}u = |u|^{p(x)-2}u, & x \in \Omega, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu} = \lambda f(x,u(x)), & x \in \partial\Omega. \end{cases}$$

Hsini et al. [26], based on the variational method and the Ekeland's principle, established a nontrivial weak solution under appropriate conditions for a weighted Steklov problem involving the $p(x)$-Laplacian operator in Sobolev spaces with variable exponents. Aydin and Unal [9], by using the Ricceri's variational principle, obtained the existence of at least three weak solutions for a parametric version of problem (P^f).

Inspired by the above results, we are interested to discuss the existence of at least one weak solution for problem (P^f) in the present paper. Precisely, in Theorem 3.1, we establish the existence of at least one weak solution for problem (P^f). We present an example in which the hypotheses of Theorems 3.1 are fulfilled. Also, in Theorem 3.3, a parametric version of this result is successively discussed in which, for small values of the parameter and requiring an additional asymptotical behaviour of the potential at zero, the existence of at least one weak solution is established. We also list some consequences the main results. As special cases of Theorem 3.3, we obtain Theorem 3.11 by considering the case $p(x) = p > N$.

2. Preliminaries

We prove the existence of at least one weak solution for problem (P^f). The key argument in our results is the following version of Ricceri's variational principle [32, Theorem 2.1], which was given by Bonanno and Molica Bisci in [11].

Theorem 2.1. *Let X be a reflexive real Banach space. Let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let I_λ be the functional defined as $I_\lambda := \Phi - \lambda\Psi$, where $\lambda \in \mathbb{R}$. For every $r > \inf_X \Phi$, let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in \left(0, \frac{1}{\varphi(r)}\right)$, the restriction of the functional I_λ to $\Phi^{-1}(-\infty, r)$ admits a global minimum, which is a critical point (precisely a local minimum) of I_λ in X.

We refer the interested reader to the papers [2, 4, 14, 17, 20, 21, 22, 24] in which Theorem 2.1 was successfully employed to ensure the existence of at least one solution for boundary value problems.

Suppose that Ω is a bounded open domain of $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $p \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \inf_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

For any $p \in C_+(\overline{\Omega})$, we denote

$$1 < p^- = \inf_{x\in\Omega} p(x) \le p^+ = \sup_{x\in\Omega} p(x) < \infty.$$

Let $p \in C_+(\overline{\Omega})$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ consists of all measurable functions u such that $\rho_{p(\cdot)}(u) < \infty$, equipped with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf\left\{ \lambda > 0 : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1 \right\},$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

The space $L_{p(\cdot)}(\Omega)$ is a Banach space with respect to $\|\cdot\|_{p(\cdot)}$. If $p(\cdot) = p$ is a constant function, then the norm $\|\cdot\|_{p(\cdot)}$ coincides with the usual Lebesgue norm $\|\cdot\|_p$; see [28]. A measurable and locally integrable function $a : \Omega \to (0, \infty)$ is called a weight function. Define the weighted variable exponent Lebesgue space by

$$L_a^{p(\cdot)}(\Omega) = \left\{ u \,\middle|\, u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} a(x) dx < \infty \right\}$$

with the Luxemburg norm

$$\|u\|_{p(\cdot),a} = \inf\left\{ \tau > 0 : \rho_{p(\cdot),a}\left(\frac{u}{\tau}\right) \le 1 \right\},$$

where

$$\rho_{p(\cdot),a}(u) = \int_\Omega |u(x)|^{p(x)} a(x) dx.$$

The space $L_a^{p(\cdot)}(\Omega)$ is a Banach space with respect to $\|\cdot\|_{p(\cdot),a}$. Moreover, $u \in L_a^{p(\cdot)}(\Omega)$ if and only if $\|u\|_{p(\cdot),a} = \|u a^{\frac{1}{p(\cdot)}}\|_{p(\cdot)} < \infty$. It is known that the relationships between $\rho_{p(\cdot),a}$ and $\|\cdot\|_{p(\cdot),a}$ as

$$\min\left\{ \rho_{p(\cdot),a}(u)^{\frac{1}{p^-}}, \rho_{p(\cdot),a}(u)^{\frac{1}{p^+}} \right\} \le \|u\|_{p(\cdot),a} \le \max\left\{ \rho_{p(\cdot),a}(u)^{\frac{1}{p^-}}, \rho_{p(\cdot),a}(u)^{\frac{1}{p^+}} \right\}$$

and

$$\min\left\{ \|u\|_{p(\cdot),a}^{p^-}, \|u\|_{p(\cdot),a}^{p^+} \right\} \le \rho_{p(\cdot),a}(u) \le \max\left\{ \|u\|_{p(\cdot),a}^{p^-}, \|u\|_{p(\cdot),a}^{p^+} \right\}$$

are satisfied. Also, if $0 < C_1 \leq a(x)$ for all $x \in \Omega$, then $L_a^{p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ since one easily sees that

$$C_1 \int_\Omega |u(x)|^{p(x)} dx \leq \int_\Omega |u(x)|^{p(x)} a(x) dx$$

and $C_1 \|u\|_{p(\cdot)} \leq \|u\|_{p(\cdot),a}$. Moreover, the dual space of $L_a^{p(\cdot)}(\Omega)$ is $L_{a^*}^{q(\cdot)}(\Omega)$, where $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ and $a^* = a^{1-q(\cdot)} = a^{-\frac{1}{p(\cdot)-1}}$. Let $a : \Omega \to (0,\infty)$. In addition, the space $L_a^{p(\cdot)}(\partial\Omega)$ can be defined by

$$L_a^{p(\cdot)}(\partial\Omega) = \left\{ u \,\middle|\, u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} a(x) d\sigma < \infty \right\}$$

equipped with the Luxemburg norm, where $d\sigma$ is the measure on the boundary. Then space $L_a^{p(\cdot)}(\partial\Omega)$ is a Banach space with respect to $\|\cdot\|_{p(\cdot),a}$. If $a \in L^\infty(\Omega)$, then $L_a^{p(\cdot)} = L^{p(\cdot)}$ (see [1]).

Theorem 2.2. [8] *If $a^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\Omega)$, then $L_a^{p(\cdot)}(\Omega) \hookrightarrow L^1_{loc}(\Omega) \hookrightarrow D'(\Omega)$, that is, every function in $L_a^{p(\cdot)}(\Omega)$ has distributional (weak) derivative, where $D'(\Omega)$ is distribution space.*

Remark 2.3. If $a^{-\frac{1}{p(\cdot)-1}} \notin L^1_{loc}(\Omega)$, then the embedding $L_a^{p(\cdot)}(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ need not hold.

Remark 2.4. Let $a^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\Omega)$. We set the weighted variable exponent Sobolev space $W_a^{k,p(\cdot)}(\Omega)$ by

$$W_a^{k,p(\cdot)}(\Omega) = \left\{ u \in L_a^{p(\cdot)}(\Omega) : D^\alpha u \in L_a^{p(\cdot)}(\Omega), 0 \leq |\alpha| \leq k \right\}$$

equipped with the norm

$$\|u\|_{k,p(\cdot),a} = \sum_{0 \leq |\alpha| \leq k} \|D^\alpha u\|_{p(\cdot),a},$$

where $\alpha \in \mathbb{N}_0^N$ is a multi-index, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_N$, and $D^\alpha = \dfrac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_N}^{\alpha_N}}$.

It is known that $W_a^{k,p(\cdot)}(\Omega)$ is a reflexive Banach space. In particular, the space $W_a^{k,p(\cdot)}(\Omega)$ is defined by

$$W_a^{k,p(\cdot)}(\Omega) = \left\{ u \in L_a^{p(\cdot)}(\Omega) : |\nabla u| \in L_a^{p(\cdot)}(\Omega) \right\}.$$

The function $\rho_{1,p(\cdot),a} : W_a^{1,p(\cdot)}(\Omega) \to [0,\infty)$ is shown as

$$\rho_{1,p(\cdot),a}(u) = \rho_{p(\cdot),a}(u) + \rho_{p(\cdot),a}(\nabla u).$$

Also, the norm $\|u\|_{1,p(\cdot),a} = \|u\|_{p(\cdot),a} + \|\nabla u\|_{p(\cdot),a}$ makes the space $W_a^{1,p(\cdot)}(\Omega)$ a Banach space. Let $a^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\Omega)$ and $b^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\Omega)$. The double weighted variable exponent Sobolev space $W_{a,b}^{1,p(\cdot)}(\Omega)$ is defined by

$$W_{a,b}^{1,p(\cdot)}(\Omega) = \left\{ u \in L_b^{p(\cdot)}(\Omega) : |\nabla u| \in L_a^{p(\cdot)}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,p(\cdot),a,b} = \|u\|_{p(\cdot),b} + \|\nabla u\|_{p(\cdot),a}.$$

Since $a^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\Omega)$ and $b^{-\frac{1}{p(\cdot)-1}} \in L^1_{loc}(\Omega)$, then it can be seen that $L^{p(\cdot)}_a(\Omega) \hookrightarrow L^1_{loc}(\Omega)$ and $L^{p(\cdot)}_b(\Omega) \hookrightarrow L^1_{loc}(\Omega)$. Therefore, the double weighted variable exponent Sobolev space $W^{1,p(\cdot)}_{a,b}(\Omega)$ is well-defined. The dual space of $W^{1,p(\cdot)}_{a,b}(\Omega)$ is $W^{-1,q(\cdot)}_{a^*,b^*}(\Omega)$, where $\dfrac{1}{p(\cdot)} + \dfrac{1}{q(\cdot)} = 1$ and $a^* = a^{-\frac{1}{p(\cdot)-1}}$ and $b^* = b^{-\frac{1}{p(\cdot)-1}}$. Moreover, the space $W^{1,p(\cdot)}_{a,b}(\Omega)$ is a separable and reflexive Banach space (see [9]).

Proposition 2.5. [29] *Let $J(u) = \int_\Omega \left(a(x)|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)} \right) dx$. For all $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$,*

(i) if $\|u\|_{1,p(\cdot),a,b} \geq 1$, then $\|u\|^{p^-}_{1,p(\cdot),a,b} \leq J(u) \leq \|u\|^{p^+}_{1,p(\cdot),a,b}$ is satisfied;
(ii) if $\|u\|_{1,p(\cdot),a,b} \leq 1$, then $\|u\|^{p^+}_{1,p(\cdot),a,b} \leq J(u) \leq \|u\|^{p^-}_{1,p(\cdot),a,b}$ is satisfied.

The following a compact embedding theorem of $W^{1,p(\cdot)}_{a,b}(\Omega)$ into $C(\overline{\Omega})$ plays an important role in this paper. For the proof, we use the method in [27, Theorem 2.11].

Theorem 2.6. [9, Theorem 5] *Let $a^{-\alpha(\cdot)} \in L^1(\Omega)$ with $\alpha(x) \in \left(\dfrac{N}{p(x)}, \infty \right) \cap \left[\dfrac{1}{p(x)-1}, \infty \right)$. If we define the variable exponent $p_*(x) = \dfrac{\alpha(x)p(x)}{\alpha(x)+1}$ with $N < p_*^-$, then we have the compact embedding $W^{1,p(\cdot)}_{a,b}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$.*

Corollary 2.7. *Since $W^{1,p(\cdot)}_{a,b}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega})$, then there exists a $C_2 > 0$ such that*

$$\|u\|_\infty \leq C_2 \|u\|_{1,p(\cdot),a,b} \tag{2.1}$$

for any $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$, where $\|u\|_\infty = \sup_{x\in\overline{\Omega}} u(x)$ for every $u \in C(\overline{\Omega})$.

Proposition 2.8. [29] *Consider the functional*

$$S(u) = \int_\Omega \frac{1}{p(x)} \left(a(x)|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)} \right) dx$$

for all $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$. Then,

(i) $S : W^{1,p(\cdot)}_{a,b}(\Omega) \to \mathbb{R}$ is sequentially weakly lower semicontinuous and $S \in C^1(W^{1,p(\cdot)}_{a,b}(\Omega), \mathbb{R})$. Moreover, the derivative operator S' of S is defined as

$$S'(u)(v) = \int_\Omega \left(a(x)|\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) + b(x)|u(x)|^{p(x)-2} u(x)v(x) \right) dx;$$

for all $u, v \in W^{1,p(\cdot)}_{a,b}(\Omega)$.

(ii) $S' : W^{1,p(\cdot)}_{a,b}(\Omega) \to W^{-1,q(\cdot)}_{a^*,b^*}(\Omega)$ is a continuous, bounded, and strictly monotone operator;

(iii) S' is a mapping of type (S_+), i.e., if $u_n \rightharpoonup u$ in $W^{1,p(\cdot)}_{a,b}(\Omega)$ and $\limsup_{n\to\infty} \langle S'(u_n) - S'(u), u_n - u \rangle \leq 0$, then $u_n \to u$ in $W^{1,p(\cdot)}_{a,b}(\Omega)$;

(iv) $S' : W^{1,p(\cdot)}_{a,b}(\Omega) \to W^{-1,q(\cdot)}_{a^*,b^*}(\Omega)$ is a homeomorphism.

For the reader's convenience, we recall some background facts concerning the Lebesgue-Sobolev spaces variable exponent and introduce some notation.

We say that a function $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$ is a weak solution to problem (P^f) if

$$\int_\Omega a(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x)dx + \int_\Omega b(x)|u(x)|^{p(x)-2}u(x)v(x)dx \\ - \int_{\partial\Omega} f(x,u(x))v(x)d\sigma = 0$$

holds for all $v \in W^{1,p(\cdot)}_{a,b}(\Omega)$. In the sequel meas$(\partial\Omega)$ denotes the Lebesgue measure of the set $\partial\Omega$.

3. Main Results

We state our main result as follows.

Theorem 3.1. *Assume that*

$$\sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_{\partial\Omega} \sup_{0\le t\le\gamma} F(x,t)d\sigma} > p^+ C_2^{p^-} \qquad (D_F)$$

where C_2 is the constant defined in (2.1). Then, problem (P^f) admits at least one weak solution in $W^{1,p(\cdot)}_{a,b}(\Omega)$.

Proof. Our aim is to apply Theorem 2.1 to problem (P^f). Consider the functionals Φ, Ψ for every $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$, defined by

$$\Phi(u) = \int_\Omega \frac{1}{p(x)} \left(a(x)|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)} \right) dx \qquad (3.1)$$

and

$$\Psi(u) = \int_{\partial\Omega} F(x,u(x))d\sigma, \qquad (3.2)$$

where

$$F(x,t) = \int_0^t f(x,y)dy$$

and put $I(u) = \Phi(u) - \Psi(u)$ for every $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$. Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. Recalling (3.1) and Proposition 2.5, we have

$$\frac{1}{p^+}\|u\|^{p^-}_{1,p(\cdot),a,b} \le \frac{1}{p^+}J(u) = \frac{1}{p^+}\int_\Omega \left(a(x)|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)} \right) dx \le \Phi(u) \\ \le \frac{1}{p^-}\|u\|^{p^+}_{1,p(\cdot),a,b}, \qquad (3.3)$$

for all $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$ such that $\|u\|_{1,p(\cdot),a,b} > 1$, so Φ is coercive. By [30, Proposition 3.1] and Proposition 2.5, we obtain $\Phi, \Psi \in C^1\left(W^{1,p(\cdot)}_{a,b}(\Omega), \mathbb{R}\right)$ with the derivatives given by

$$\Psi'(u)(v) = \int_{\partial\Omega} f(x,u(x))v(x)d\sigma$$

and

$$\Phi'(u)(v) = \int_\Omega \Big(a(x)|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + b(x)|u(x)|^{p(x)-2}u(x)v(x) \Big)\, dx$$

for every $v \in W^{1,p(\cdot)}_{a,b}(\Omega)$. Furthermore, Φ is sequentially weakly lower semicontinuous. Moreover, Ψ is sequentially weakly upper semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1, are verified. We note that operator I is a $C^1\left(W^{1,p(\cdot)}_{a,b}(\Omega),\mathbb{R}\right)$ functional and the critical points of I are weak solutions to problem (P^f). We now look on the existence of a critical point of the functional I in $W^{1,p(\cdot)}_{a,b}(\Omega)$. By using the condition (D_F), there exists $\bar{\gamma} > 0$ such that

$$\frac{\bar{\gamma}^{p^-}}{\displaystyle\int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma} > p^+ C_2^{p^-}. \tag{3.4}$$

Let $\bar{\gamma}$ be a real number such that $0 < \bar{\gamma} < \min\{1, C_2\}$. Choosing

$$r = \frac{1}{p^+}\left(\frac{\bar{\gamma}}{C_2}\right)^{p^-},$$

one has $r \in (0,1)$. For all $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$ with $\Phi(u) \le r$, owing to Proposition 2.8, one has

$$\min\left\{ \|u\|^{p^-}_{1,p(\cdot),a,b}, \|u\|^{p^+}_{1,p(\cdot),a,b} \right\} < rp^+.$$

Then

$$\|u\|_{1,p(\cdot),a,b} < \max\left\{ (p^+ r)^{\frac{1}{p^-}}, (p^+ r)^{\frac{1}{p^+}} \right\} = \frac{\bar{\gamma}}{C_2}.$$

It follows from (2.1) that $|u(x)| \le C_2\|u\|_{1,p(\cdot),a,b} \le C_2(p^+r)^{\frac{1}{p^-}} = \bar{\gamma}$ for all $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$ and $x \in \Omega$ with $\Phi(u) \le r$. This follows that

$$\Phi^{-1}(-\infty, r) = \{u \in X,\ \Phi(u) < r\} \subseteq \{u \in X,\ |u| \le \bar{\gamma}\}.$$

Hence, we have

$$\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) \le \int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma.$$

By simple calculations and from the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty,r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{(\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)) - \Psi(u)}{r - \Phi(u)} \le \frac{\sup_{v\in\Phi^{-1}(-\infty,r)}\Psi(v)}{r} \\
&\le \frac{\displaystyle\int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma}{\frac{1}{p^+}(\frac{\bar{\gamma}}{C_2})^{p^-}}.
\end{aligned}$$

At this point, we see that

$$\varphi(r) \leq \frac{\displaystyle\int_{\partial\Omega} \sup_{0\leq t\leq \bar{\gamma}} F(x,t)d\sigma}{\displaystyle\frac{1}{p^+}\left(\frac{\bar{\gamma}}{C_2}\right)^{p^-}}. \tag{3.5}$$

Consequently, from (3.4) and (3.5), one has $\varphi(r) < 1$. Hence, since $1 \in \left(0, \dfrac{1}{\varphi(r)}\right)$, applying Theorem 2.1, we see that the functional I admits at least one critical point (local minima) $\tilde{u} \in \Phi^{-1}(-\infty, r)$. The proof is complete. $\square$

Here we present an example in which the hypotheses of Theorem 3.1 are satisfied.

Example 3.2. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : \ x^2+y^2 \leq 9\}$. Consider the autonomous problem

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p(x,y)-2}\nabla u\right) = b(x)|u|^{p(x,y)-2}u, & x,y \in \Omega, \\ a(x)|\nabla u|^{p(x,y)-2}\dfrac{\partial u}{\partial v} = f(u(x)), & x,y \in \partial\Omega, \end{cases} \tag{3.6}$$

where $p(x,y) = x^2+y^2+4$ for all $x,y \in \overline{\Omega}$. We have

$$F(t) = \frac{1}{10^4 C_2^4} t^4 \left(t^6 + e^{2t} - 1\right)$$

for every $t \in \mathbb{R}$. By simple calculations, we obtain $meas(\partial\Omega) = 6\pi$, $p^- = 4$ and $p^+ = 13$. Since

$$\sup_{\gamma>0} \frac{\gamma^4}{meas(\partial\Omega)\sup_{0\leq|t|\leq\gamma} F(t)} = \sup_{\gamma>0} \frac{\gamma^4}{\dfrac{6\pi}{10^4C_2^4}\gamma^4(\gamma^6+e^{2\gamma}-1)} > \frac{\gamma^4}{\dfrac{6\pi}{10^4C_2^4}\gamma^4} = \frac{10^4C_2^4}{6\pi}$$

and

$$\frac{10^4C_2^4}{6\pi} > 13C_2^4 = p^+C_2^{p^-},$$

then the condition (D_F) is automatically satisfied. We observe that all assumptions of Theorem 3.1 are fulfilled. Hence, Theorem 3.1 implies that problem (3.6) admits at least one weak solution in $W_{a,b}^{1,p(x,y)}(\Omega)$.

We note that Theorem 3.1 can be exploited to demonstrate the existence of at least one solution for the following parametric version of the problem (P^f)

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) = b(x)|u|^{p(x)-2}u, & x \in \Omega, \\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = \lambda f(x,u(x)), & x \in \partial\Omega \end{cases} \tag{P_λ^f}$$

where λ is a positive parameter. More precisely, we have the following result.

Theorem 3.3. *For every λ small enough, i.e.,*

$$\lambda \in \left(0, \frac{1}{p^+C_2^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\displaystyle\int_{\partial\Omega}\sup_{0\leq t\leq\gamma} F(x,t)d\sigma}\right),$$

where C_2 is the constant defined in (2.1), problem (P_λ^f) admits at least one weak solution $u_\lambda \in W_{a,b}^{1,p(\cdot)}(\Omega)$.

Proof. Fix λ as in the conclusion. Take Φ and Ψ as given in the proof of Theorem 3.1, and put $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for every $u \in W_{a,b}^{1,p(\cdot)}(\Omega)$. Let us pick

$$0 < \lambda < \frac{1}{p^+ C_2^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_{\partial\Omega} \sup_{0\le t\le\gamma} F(x,t)d\sigma}.$$

Hence, there exists $\bar{\gamma} > 0$ such that

$$\lambda p^+ C_2^{p^-} < \frac{\bar{\gamma}^{p^-}}{\int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma}.$$

Choose $r = \frac{1}{p^+}(\frac{\bar{\gamma}}{C_2})^{p^-}$. By the same notations as in the proof of Theorem 3.1, one has

$$\varphi(r) \le \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)}{r} \le \frac{\int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma}{\frac{1}{p^+}(\frac{\bar{\gamma}}{C_2})^{p^-}} < \frac{1}{\lambda}.$$

So, since $\lambda \in (0, \frac{1}{\varphi(r)})$, Theorem 2.1 ensures that the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$ and since the critical points of the functional I_λ are the solutions to problem (P_λ^f), we have the conclusion. □

Now, we give some remarks for our results.

Remark 3.4. In Theorem 3.3, we looked for the critical points of the functional I_λ naturally associated with the problem (P_λ^f). We note that, in general, I_λ can be unbounded from the following in $W_{a,b}^{1,p(\cdot)}(\Omega)$. Indeed, for example, in the case that $f(\xi) = 1 + |\xi|^{\gamma-p^+}\xi^{p^+-1}$ for each $\xi \in \mathbb{R}$, for any fixed $u \in W_{a,b}^{1,p(\cdot)}(\Omega)\backslash\{0\}$ and $\iota \in \mathbb{R}$. By the expression of f, we have

$$F(t) = \int_0^t f(\xi)d\xi = \int_0^t (1 + |\xi|^{\gamma-p^+}\xi^{p^+-1})d\xi = t + \frac{t^\gamma}{\gamma}.$$

By using (3.3) and $\gamma > p^+$, we obtain

$$\begin{aligned}
I_\lambda(\iota u) =& \Phi(\iota u) - \lambda \int_{\partial\Omega} F(\iota u(x))d\sigma \\
\le& \frac{1}{p^-}\|\iota u\|_{1,p(\cdot),a,b}^{p^+} - \lambda \int_{\partial\Omega} |\iota u(x)|d\sigma - \lambda \int_{\partial\Omega} \left|\frac{(\iota u(x))^\gamma}{\gamma}\right| d\sigma \\
=& \iota^{p^+}\frac{1}{p^-}\|u\|_{1,p(\cdot),a,b}^{p^+} - \lambda\iota \int_{\partial\Omega} |u(x)|d\sigma - \lambda\frac{\iota^\gamma}{\gamma}\int_{\partial\Omega} |u(x)|^\gamma d\sigma \\
=& \iota^{p^+}\frac{1}{p^-}\|u\|_{1,p(\cdot),a,b}^{p^+} - \lambda\iota\|u\|_{L^1(\partial\Omega)} - \lambda\frac{\iota^\gamma}{\gamma}\|u\|_{L^\gamma(\partial\Omega)}^\gamma \to -\infty
\end{aligned}$$

as $\iota \to +\infty$. Hence, we can not use direct minimization to find critical points of the functional I_λ.

Remark 3.5. For fixed $\bar{\gamma} > 0$, let

$$\frac{\bar{\gamma}^{p^-}}{\int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma} > p^+ C_2^{p^-}.$$

Then the result of Theorem 3.3 holds with $\|u_\lambda\|_\infty \le \bar{\gamma}$ is the ensured weak solution in $W_{a,b}^{1,p(\cdot)}(\Omega)$.

Remark 3.6. If, in Theorem 3.1, the function $f(x,\xi) \ge 0$ for every $x \in \partial\Omega$ and $\xi \in \mathbb{R}$, the condition (D_F) takes the following more simple and significative form

$$\sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_{\partial\Omega} F(x,\gamma)d\sigma} > p^+ C_2^{p^-}. \tag{D_F'}$$

Moreover, if the following assumption holds

$$\limsup_{\gamma\to+\infty} \frac{\gamma^{p^-}}{\int_{\partial\Omega} F(x,\gamma)d\sigma} > p^+ C_2^{p^-},$$

then the condition (D_F') is automatically satisfied.

Remark 3.7. If, in Theorem 3.3, $f(x,0) = 0$ for all $x \in \partial\Omega$, then the ensured weak solution may be trivial. So, if $f(x,0) \ne 0$ for all $x \in \partial\Omega$, then the ensured weak solution is obviously non-trivial. On the other hand, the non-triviality of the weak solution can be achieved also in the case $f(x,0) = 0$ for a.e. $x \in \partial\Omega$ requiring the extra condition at zero, that is, there are a non-empty open set $D \subseteq \partial\Omega$ and $B \subset D$ of positive Lebesgue measure such that

$$\limsup_{\xi\to0^+} \frac{ess\inf_{x\in B} F(x,\xi)}{|\xi|^{p^-}} = +\infty \tag{3.7}$$

and

$$\liminf_{\xi\to0^+} \frac{ess\inf_{x\in D} F(x,\xi)}{|\xi|^{p^-}} > -\infty. \tag{3.8}$$

Indeed, let $0 < \bar{\lambda} < \lambda^*$, where

$$\lambda^* = \frac{1}{p^+ C_2^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_{\partial\Omega} \sup_{0\le t\le\gamma} F(x,t)d\sigma}.$$

Then, there exists $\bar{\gamma} > 0$ such that

$$\bar{\lambda} p^+ C_2^{p^-} < \frac{\bar{\gamma}^{p^-}}{\int_{\partial\Omega} \sup_{0\le t\le\bar{\gamma}} F(x,t)d\sigma}.$$

Let Φ and Ψ be as given in (3.1) and (3.2), respectively. Thanks to Theorem 2.1, for every $\lambda \in (0,\bar{\lambda})$, there exists a critical point of $I_\lambda = \Phi - \lambda\Psi$ such that $u_\lambda \in \Phi^{-1}(-\infty, r_\lambda)$ where $r_\lambda = \frac{1}{p^+}\left(\frac{\bar{\gamma}}{C_2}\right)^{p^-}$. In particular, u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r_\lambda)$.

We next demonstrate that the function u_λ cannot be trivial. Let us prove that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty. \tag{3.9}$$

Owing to the assumptions (3.7) and (3.8), we can consider a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that

$$\lim_{n\to+\infty} \frac{ess\inf_{x\in B} F(x,\xi_n)}{|\xi_n|^{p^-}} = +\infty$$

and

$$ess\inf_{x\in D} F(x,\xi) \geq \kappa|\xi|^{p^-}$$

for every $\xi \in [0,\sigma]$. We consider a set $\mathscr{G} \subset B$ of positive measure and a function $v \in W^{1,p(\cdot)}_{a,b}(\Omega)$ such that

(k_1) $v(x) \in [0,1]$ for every $x \in \partial\Omega$,
(k_2) $v(x) = 1$ for every $x \in \mathscr{G}$,
(k_3) $v(x) = 0$ for every $x \in \partial\Omega \setminus D$.

Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta \; meas(\mathscr{G}) + \kappa \displaystyle\int_{D\setminus\mathscr{G}} |v(x)|^{p^-} d\sigma}{\dfrac{1}{p^+}\|v\|^{p^-}_{1,p(\cdot),a,b}}.$$

Then, there is $n_0 \in \mathbb{N}$ such that $\xi_n < \sigma$ and $ess\inf_{x\in B} F(x,\xi_n) \geq \eta|\xi_n|^{p^-}$, for every $n > n_0$. Now, for every $n > n_0$, by considering the properties of the function v (that is, $0 \leq \xi_n v(x) < \sigma$ for n large enough), by (3.3), we have

$$\begin{aligned}\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} &= \frac{\displaystyle\int_{\mathscr{G}} F(x,\xi_n)d\sigma + \int_{D\setminus\mathscr{G}} F(x,\xi_n v(x))d\sigma}{\Phi(\xi_n v)} \\ &> \frac{\eta \; meas(\mathscr{G}) + \kappa \displaystyle\int_{D\setminus\mathscr{G}} |v(x)|^{p^-} d\sigma}{\dfrac{1}{p^+}\|v\|^{p^-}_{1,p(\cdot),a,b}} > M.\end{aligned}$$

Since M could be taken arbitrarily large, it is concluded that

$$\lim_{n\to\infty} \frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} = +\infty,$$

from which (3.9) clearly follows. Hence, there exists a sequence $\{w_n\} \subset W^{1,p(\cdot)}_{a,b}(\Omega)$ strongly converging to zero such that, for n large enough, $w_n \in \Phi^{-1}(-\infty, r)$ and

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) < 0.$$

Since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we obtain

$$I_\lambda(u_\lambda) < 0. \tag{3.10}$$

Hence, u_λ is not trivial.

Remark 3.8. From (3.10), we easily observe that the map

$$(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.11}$$

is negative. Also, one has

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{1,p(\cdot),a,b} = 0.$$

Indeed, bearing in mind that Φ is coercive and for every $\lambda \in (0,\lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty,r)$, one has that there exists a positive constant L such that $\|u_\lambda\|_{1,p(\cdot),a,b} \le L$ for every $\lambda \in (0,\lambda^*)$. After that, it is easy to see that there exists a positive constant N such that

$$\left| \int_{\partial\Omega} f(x,u_\lambda(x))u_\lambda(x)d\sigma \right| \le N\|u_\lambda\|_{1,p(\cdot),a,b} \le NL \tag{3.12}$$

for every $\lambda \in (0,\lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for every $v \in W^{1,p(\cdot)}_{a,b}(\Omega)$ and every $\lambda \in (0,\lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_{\partial\Omega} f(x,u_\lambda(x))u_\lambda(x)d\sigma \tag{3.13}$$

for every $\lambda \in (0,\lambda^*)$. For $\|u_\lambda\|_{1,p(\cdot),a,b} \ge 1$, from Proposition(2.5), we have

$$0 \le \|u_\lambda\|^{p^-}_{1,p(\cdot),a,b} \le \Phi'(u_\lambda)(u_\lambda).$$

From (3.13), we have

$$0 \le \|u_\lambda\|^{p^-}_{1,p(\cdot),a,b} \le \lambda \int_{\partial\Omega} f(x,u_\lambda(x))u_\lambda(x)d\sigma \tag{3.14}$$

for any $\lambda \in (0,\lambda^*)$. Letting $\lambda \to 0^+$, by (3.14) together with (3.12), we have

$$\lim_{\lambda\to 0^+} \|u_\lambda\|_{1,p(\cdot),a,b} = 0.$$

The proof of the case $\|u_\lambda\|_{1,p(\cdot),a,b} \le 1$ is similar to case $\|u_\lambda\|_{1,p(\cdot),a,b} \ge 1$. Then, we have the desired conclusion immediately.

Finally, we have to demonstrate that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0,\lambda^*)$. For our goal, we see that, for any $u \in W^{1,p(\cdot)}_{a,b}(\Omega)$,

$$I_\lambda(u) = \lambda\left(\frac{\Phi(u)}{\lambda} - \Psi(u)\right). \tag{3.15}$$

Now, let us consider $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty,r)$ for $i = 1,2$. Also, set

$$m_{\lambda_i} = \left(\frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i})\right) = \inf_{v\in\Phi^{-1}(-\infty,r)} \left(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\right),$$

for every $i = 1,2$. Clearly, (3.11) together with (3.15) and the positivity of λ implies that

$$m_{\lambda_i} < 0 \quad \text{for} \quad i = 1,2. \tag{3.16}$$

Moreover

$$m_{\lambda_2} \le m_{\lambda_1}, \tag{3.17}$$

which is due to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.15)-(3.17) and the fact that $0 < \lambda_1 < \lambda_2$, we obtain that $I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \le \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1})$, so the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0,\lambda^*)$. The arbitrariness of $\lambda < \lambda^*$ demonstraes that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0,\lambda^*)$.

Remark 3.9. We observe that Theorem 3.3 is a bifurcation result in the sense that the pair $(0,0)$ belongs to the closure of the set

$$\left\{(u_\lambda,\lambda) \in W^{1,p(\cdot)}_{a,b}(\Omega)\times(0,+\infty) : u_\lambda \text{ is a non-trivial weak solution of } (P^f_\lambda)\right\}$$

in $W^{1,p(\cdot)}_{a,b}(\Omega)\times\mathbb{R}$. Indeed, by Theorem 3.3, we have that

$$\|u_\lambda\|_{1,p(\cdot),a,b} \to 0 \quad \text{as} \quad \lambda \to 0.$$

Hence, there exist two sequences $\{u_j\}$ in $W^{1,p(\cdot)}_{a,b}(\Omega)$ and $\{\lambda_j\}$ in $\mathbb{R}^+$ (here $u_j = u_{\lambda_j}$) such that

$$\lambda_j \to 0^+ \quad \text{and} \quad \|u_j\|_{1,p(\cdot),a,b} \to 0,$$

as $j \to +\infty$. Moreover, we emphasis that due to the fact that the map

$$(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing, for every $\lambda_1,\lambda_2 \in (0,\lambda^*)$, with $\lambda_1 \neq \lambda_2$, the weak solutions u_{λ_1} and u_{λ_2} ensured by Theorem 3.3 are different.

When f does not depend on x, we can obtain the following autonomous version of Theorem 3.3.

Theorem 3.10. *Let $f:\mathbb{R}\to\mathbb{R}$ be a non-negative continuous function. Put $F(\xi)=\int_0^\xi f(t)dt$ for all $\xi\in\mathbb{R}$. Assume that*

$$\lim_{\xi\to 0^+}\frac{F(\xi)}{\xi^{p^-}} = +\infty.$$

Then, for each

$$\lambda \in \left(0, \frac{1}{meas(\partial\Omega)p^+C_2^{p^-}}\sup_{\gamma>0}\frac{\gamma^{p^-}}{F(\gamma)}\right),$$

where C_2 is the constant defined in (2.1), the problem

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p(x)-2}\nabla u\right) = b(x)|u|^{p(x)-2}u, & x\in\Omega,\\ a(x)|\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial v} = \lambda f(u(x)), & x\in\partial\Omega \end{cases}$$

admits at least one non-trivial weak solution $u_\lambda \in W^{1,p(\cdot)}_{a,b}(\Omega)$ such that $\lim_{\lambda\to 0^+}\|u_\lambda\|_{1,p(\cdot),a,b}=0$ and the real function

$$\lambda \to \int_\Omega \frac{1}{p(x)}\left(a(x)|\nabla u(x)|^{p(x)} + b(x)|u(x)|^{p(x)}\right)dx - \int_{\partial\Omega} F(u(x))d\sigma$$

is negative and strictly decreasing in $\left(0, \dfrac{1}{meas(\partial\Omega)p^+C_2^{p^-}}\sup_{\gamma>0}\dfrac{\gamma^{p^-}}{F(\gamma)}\right)$.

A special case of Theorem 3.3 is the following theorem.

Theorem 3.11. *Let $p(x) = p > N$ for every $x \in \Omega$. For every λ small enough, i.e.,*

$$\lambda \in \left(0, \frac{1}{pC_2^p} \sup_{\gamma>0} \frac{\gamma^p}{\int_{\partial\Omega} \sup_{0\leq t\leq\gamma} F(x,t)d\sigma}\right),$$

where C_2 is the constant defined in (2.1), the problem

$$\begin{cases} \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) = b(x)|u|^{p-2}u, & x \in \Omega, \\ a(x)|\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda f(x,u(x)), & x \in \partial\Omega \end{cases}$$

admits at least one weak solution $u_\lambda \in W^{1,p}_{a,b}(\Omega)$.

REFERENCES

[1] R.A. Adams, J.J.F. Fournier, Sobolev Spaces (2nd Ed.), Academic Press, 2003.

[2] G.A. Afrouzi, M. Bohner, G. Caristi, S. Heidarkhani, S. Moradi, An existence result for impulsive multi-point boundary value systems using a local minimization principle, J. Optim. Theory Appl. 177 (2018) 1-20.

[3] G.A. Afrouzi, A. Hadijan, S. Heidarkhani, Steklov problem involving the $p(x)$-Laplacian, Electron. J. Differ. Equ. 2014 (2014) 134.

[4] G.A. Afrouzi, A. Hadjian, G. Molica Bisci, Some remarks for one-dimensional mean curvature problems through a local minimization principle, Adv. Nonlinear Anal. 2 (2013) 427-441.

[5] M. Allaoui, Coontinuos spectrum of steklov nonhomogeneous elliptic problem, Opuscula Math. 35 (2015) 853-866.

[6] M. Allaoui, A.R.El Amrouss, A. Ourraoui, Existence and multiplicity of solutions for a Steklov problem involving the $p(x)$-Laplace operator, Electron. J. Differ. Equ. 2012 (2012) 1-12.

[7] S. Antontsev, S. Shmarev, Chapter 1 Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions, Handbook of Differential Equations, Stationary Partial Differential Equations, 3 (2006) 1-100.

[8] I. Aydin, Weighted variable Sobolev spaces and capacity, J. Funct. Spaces Appl. 2012 (2012) 132690.

[9] I. Aydin, C. Unal, Three solutions to a Steklov problem involving the weighted $p(\cdot)$-Laplacian, Rocky Mountain J. Math. (2021) 67-76.

[10] A. Ayoujil, A.R. El Amrouss, On the spectrum of a fourth order elliptic equation with variable exponent, Nonlinear Anal. 71 (2009) 4916-4926.

[11] G. Bonanno, G. Molica Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, Bound. Value Probl. 2009 (2009) 670675.

[12] Y. Chen, S. Levine, R. Rao, Variable exponent, linear growth functionals in image restoration, SIAM J. Appl. Math. 66 (2006) 1383-1406.

[13] R. Demarque, O. Miyagaki, Radial solutions of inhomogeneous fourth order elliptic equations and weighted Sobolev embeddings, Adv. Nonlinear Anal. 4 (2015) 135-151.

[14] M. Galewski, G. Molica Bisci, Existence results for one-dimensional fractional equations, Math. Meth. Appl. Sci. 39 (2016) 1480-1492.

[15] B. Ge, Q.-M. Zhou, Y.-H. Wu, Eigenvalues of the $p(x)$-biharmonic operator with indefinite weight, Zeitschrift für angewandte Mathematik und Physik, 66 (2015) 1007-1021.

[16] J.R. Graef, S. Heidarkhani, L. Kong, Multiple solutions for a class of $(p_1,\cdots,p_n)$-biharmonic systems, Commun. Pur. Appl. Anal. 12 (2013) 1393-1406.

[17] J.R. Graef, S. Heidarkhani, L. Kong, S. Moradi, Existence results for impulsive fractional differential equations with p-Laplacian via variational methods, Math. Bohemica, to appear.

[18] T.C. Halsey, Electrorheological fluids, Science 258 (1992) 761-766.

[19] S. Heidarkhani, Existence of non-trivial solutions for systems of n fourth order partial differential equations, Math. Slovaca 64 (2014) 1249-1266.

[20] S. Heidarkhani, G.A. Afrouzi, M. Ferrara, G. Caristi, S. Moradi, Existence results for impulsive damped vibration problems, Bull. Malays. Math. Sci. Soc. 41 (2018) 1409-1428.

[21] S. Heidarkhani, G.A. Afrouzi, S. Moradi, Variational approaches to systems of Sturm-Liouville boundary value problems, Asian-European J. Math. 14 (2021) 2150032.

[22] S. Heidarkhani, G.A. Afrouzi, S. Moradi, G. Caristi, B. Ge, Existence of one weak solution for $p(x)$-biharmonic equations with Navier boundary conditions, Zeitschrift fuer Angewandte Mathematik und Physik, 67 (2016) 1-13.

[23] S. Heidarkhani, M. Ferrara, A. Salari, G. Caristi, Multiplicity results for $p(x)$-biharmonic equations with Navier boundary, Compl. Var. Ellip. Equ. 61 (2016) 1494-1516.

[24] S. Heidarkhani, S. Moradi, D. Barilla, Existence results for second-order boundary-value problems with variable exponents, Nonlinear Anal. 44 (2018) 40-53.

[25] S. Heidarkhani, Y. Tian, C.-L. Tang, Existence of three solutions for a class of $(p_1,\cdots,p_n)$-biharmonic systems with Navier boundary conditions, Ann. Polon. Math. 104 (2012) 261-277.

[26] M. Hsini, N. Irzi, Kh. Kefi, Nonhomogeneous $p(x)$-Laplacian Steklov problem with weights, Complex Var. Ellip. Equ. 65 (2020) 440-45.

[27] Y.H. Kim, L. Wang, C. Zhang, Global bifurcation for a class of degenerate elliptic equations with variable exponents, J. Math. Anal. Appl. 371 (2010) 624-637.

[28] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (1991) 59-618.

[29] Q. Liu, D. Liu, Existence and multiplicity of solutions to a $p(x)$-Laplacian equation with nonlinear boundary condition on unbounded domain, Differ. Equ. Appl. 5 (2013) 595-611.

[30] M. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. Lond. Ser. A 10.1098/rspa.2005.1633.

[31] C. Pfeiffer, C. Mavroidis, Y. Bar-Cohen, B. Dolgin, Electrorheological fluid based force feedback device, in Proceedings of the 1999 SPIE Telemanipulator and Telepresence Technologies VI Conference (Boston, MA), 3840 (1999) 88-99.

[32] B. Ricceri, A general variational principle and some of its applications, J. Comput. Appl. Math. 113 (2000) 401-410.

[33] M.Ružiška, Electro-rheological Fluids: Modeling and Mathematical Theory, Lecture Notes in Math., 1784, Springer, Berlin, 2000.

[34] C. Unal, I. Aydın, Compact embeddings of weighted variable exponent Sobolev spaces and existence of solutions for weighted $p(\cdot)$-Laplacian, Comples Var. Ellip. Equ. 66 (2021) 1755-1773.

[35] V.V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv. 29 (1987) 33-66.