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SOME COMMON COUPLED FIXED POINT RESULTS FOR THE MAPPINGS WITH A NEW CONTRACTIVE CONDITION IN A MENGER PBM-METRIC SPACE

XIAORUI KANG[1], NANNAN FANG[2,*]

[1]Institute of Fundamental and Frontier Sciences,
University of Electronic Science and Technology of China, Chengdu, China

[2]Department of General Education, Anhui Xinhua University, Hefei, China

Abstract. In the setting of Menger PbM-metric spaces, a new contraction condition is presented and some new coupled coincidence point and common coupled fixed-point theorems of ω-weakly compatible self-maps pairs are proved. In addition, some examples are presented to demonstrate the validity of the fixed-point theorems. Finally, an existence theorem for a common solution of integral equations is considered.

Keywords. Couple coincidence point; Common couple fixed point; Integral equations; Menger PbM-spaces; ω-compatible mapping pairs.

1. Introduction and Preliminaries

Recently, the research of fixed points of various nonlinear operators is under the spotlight due to their wide real applications, such as image reconstruction and image processing; see, for example, [6, 7, 8, 14, 15, 16, 19] and the references therein. In particular, numerous convex optimization problems in various frameworks have been investigated via fixed-point methods in metric spaces. In 1942, the notion of Menger probabilistic metric spaces (briefly, Menger PM-space) was introduced by Menger in [17]. Since then, the theory of Menger PM-spaces has developed in many directions. It is known that a Menger PM-space is a significant generalization of ordinary metric spaces. Numerous fixed-point results were obtained by authors with respect to different contractive conditions for self-mappings in Menger PM-spaces; see, e.g., [4, 11, 20, 22]. The b-metric space was first investigated by Czerwik in [9, 10]. Recently, numerous results on fixed points on b-metric spaces have been obtained; see, e.g., [2, 3, 18, 21]

*Corresponding author.
E-mail addresses: ruikang@uestc.edu.cn (X. Kang), 1186375240@qq.com (N. Fang).
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and references cited therein. In 2015, Hasanvand and Khanehgir [12] presented a Menger probabilistic b-metric space (briefly, Menger PbM-space), which is an extension of generalized β-type contractive mappings, introduced by Gopal et al. [11] in Menger PM-spaces, and proved some fixed-point results.

Inspired and motivated by the recent studies mentioned above, we consider a new contraction condition on Menger PbM-spaces, and prove some new coupled coincidence point and common coupled fixed-point theorems for ω-weakly compatible self-map pairs. We also present some examples to demonstrate the validity of the fixed point results.

From now on, we assume that X is a nonempty set. If the function $d : X \times X \to \mathbb{R}^+$ satisfies the following conditions: (b1) $d(x,y) = 0$, if and only if $x = y$; (b2) $d(x,y) = d(y,x), \forall x,y \in X$; and (b3) there exists a real number $s \geq 1$ such that $d(x,z) \leq s[d(x,y)+d(y,z)]$, $\forall x,y,z \in X$, then d is called a b-metric on X and a pair (X,d) is called a b-metric space with the coefficient $s \geq 1$. The mapping $f : \mathbb{R} \to \mathbb{R}^+$ is said to be a distribution function if it is non-decreasing and left-continuous with $\sup_{t\in\mathbb{R}} f(t) = 1$ and $\inf_{t\in\mathbb{R}} f(t) = 0$. Recall that the mapping $\triangle : [0,1]\times[0,1] \to [0,1]$ is said to be a triangular norm (briefly, a t-norm) if, for all $a,b,c,d \in [0,1]$,

($\triangle$-1) $\triangle(a,1) = a$;

($\triangle$-2) $\triangle(a,b) = \triangle(b,a)$;

($\triangle$-3) $\triangle(c,d) \geq \triangle(a,b)$ whenever $c \geq a$ and $d \geq b$;

($\triangle$-4) $\triangle(\triangle(a,b),c) = \triangle(a,\triangle(b,c))$.

Typical examples of the t-norm are $\triangle_1(a,b) = \max\{a+b-1,0\}$, $\triangle_2(a,b) = a\cdot b$, and $\triangle_3(a,b) = \min\{a,b\}$. Clearly, the above three t-norm examples satisfy that $\triangle_1 \leq \triangle_2 \leq \triangle_3$.

Remark 1.1. Assume that there exists a t-norm $\triangle$ such that $\triangle(t,t) \geq t$ for all $t \in [0,1]$. Then $\triangle \geq \triangle_3 \geq \triangle_2 \geq \triangle_1$.

Proof. By virtue of $\triangle$-3, for any $a,b \in [0,1]$, one sees that $\triangle(a,b) \geq \triangle(\min\{a,b\},\min\{a,b\}) \geq \min\{a,b\} = \triangle_3(a,b)$. It follows that $\triangle \geq \triangle_3 \geq \triangle_2 \geq \triangle_1$. □

Throughout the rest of the paper, $\mathscr{D}$ is borrowed to stand for the set of all distribution functions, while $\mathscr{H}(\cdot)$ stands for the specific distribution function defined by $\mathscr{H}(t) = 1$, $t > 0$, and $\mathscr{H}(t) = 0$, $t \leq 0$.

Recall that a Menger probabilistic metric space (briefly, Menger PM-space) is a triple $(X,F,\triangle)$, where $\triangle$ is a continuous t-norm and F is a mapping from $X\times X$ into $\mathscr{D}$. Let $F_{x,y}$ denote the value of F at the pair (x,y). Suppose that the following conditions are satisfied

(PM1) $F_{x,y}(t) = \mathscr{H}(t)$, $\forall t \in \mathbb{R}$, if and only if $x = y$;

(PM2) $F_{x,y}(t) = F_{y,x}(t)$, $\forall x,y \in X$ and $t \in \mathbb{R}$;

(PM3) $F_{x,y}(t+s) \geq \triangle(F_{x,z}(t),F_{z,y}(s))$, $\forall x,y,z \in X$ and $s,t \in \mathbb{R}^+$.

If $(X,F,\triangle)$ is a Menger probabilistic b-metric space, then the following conclusion hold:

(PbM4) there exists a real number $k \in (0,1]$ such that $F_{x,y}(t+s) \geq \triangle(F_{x,z}(kt),F_{z,y}(ks))$, $\forall x,y,z \in X$ and $s,t \in \mathbb{R}^+$.

The class of Menger PbM-spaces is larger than that of Menger PM-spaces. In fact, a Menger PbM-space is a Menger PM-space when $k = 1$. There are examples of Menger PbM-spaces which are not Menger PM-spaces. Let $(X,F,\triangle)$ be a Menger PbM-space and let $\triangle$ be a continuous t-norm. Recall that sequence $\{x_n\}$ in X is said to be convergent to $x^* \in X$ (the symbol $\to$ denotes the convergence), if, for any $\lambda \in (0,1)$ and $\varepsilon > 0$, there exists a positive integer $N = N(\varepsilon,\lambda)$ with $F_{x_n,x^*}(\varepsilon) > 1-\lambda$ whenever $n \geq N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x^*}(t) = 1$

for all $t > 0$. Recall that sequence $\{x_n\}$ in X is said to be a Cauchy sequence if there exists a positive integer $N = N(\varepsilon, \lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n, m \geq N$. If every Cauchy sequence in X is convergent, then Menger PbM-space $(X, F, \triangle)$ is said to be complete. Moreover, in a Menger PbM-space $(X, F, \triangle)$, a convergent sequence has a unique limit point. One knows that a Menger PbM-space is not always a topological space. Let $X = \mathbb{R}^+$. By [12], define $F : X \times X \to \mathscr{D}$ by, $\forall x, y \in X$,

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|^2}, & t > 0; \\ 0, & t \leq 0. \end{cases}$$

Observe that $(X, F, \triangle_3)$ is a complete Menger-PbM space with the coefficient $k = \frac{1}{2}$. However, it is not a Menger PM-space. Assume that $(X, F, \triangle)$ is a Menger-PbM space with $k \in (0, 1]$. From [12], one has that F is a lower semi-continuous function of points, i.e., for every fixed $t > 0$ and every two convergent sequences $\{x_n\}$ and $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, $\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t)$. In particular, if $y_n = y$, then $\liminf_{n\to\infty} F_{x_n,y}(t) = F_{x,y}(t)$.

The following example is also due to [12].

Example 1.2. Let (X, d) be a b-metric space with the coefficient $s \geq 1$. Define the distribution function $F : X \times X \to \mathscr{D}$ by $F(x, y)(t) = F_{x,y}(t) = \mathscr{H}(t - d(x, y))$, $\forall x, y \in X, t \in \mathbb{R}$. Then a triple $(X, F, \triangle_3)$ is a Menger PbM-space with the coefficient $k = \frac{1}{s}$. Furthermore, if (X, d) is complete, then $(X, F, \triangle_3)$ is also complete.

Let X be a nonempty set. By [5], the element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping $T : X \times X \to X$ if $T(x, y) = x$ and $T(y, x) = y$.

Recall the following definitions.

(i) From [13], the element $(x, y) \in X \times X$ is said to be a coupled coincidence point of mappings $T : X \times X \to X$ and $g : X \to X$ if $g(x) = T(x, y)$ and $g(y) = T(y, x)$, while (gx, gy) is called a coupled point of coincidence of mappings g and T.

(ii) From [13], the element $(x, y) \in X \times X$ is said to be a common coupled fixed point of mappings $T : X \times X \to X$ and $g : X \to X$ if $T(x, y) = gx = x, T(y, x) = gy = y$.

(iii) In [1], the mappings $T : X \times X \to X$ and $g : X \to X$ are said to be ω-compatible if $g(T(x, y)) = T(gx, gy)$ whenever $F(x, y) = gx$ and $F(y, x) = gy$.

Let Φ be the set function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions [11]: $\phi(t) = 0$ if and only if $t = 0$; $\phi(t)$ is nondecreasing and $\phi(t) \to \infty$ as $t \to \infty$; $\phi(t)$ is continuous in $(0, \infty)$, and $\phi(t)$ is continuous in 0. Let $(X, F, \triangle)$ be a Menger PbM-space. Let $g : X \to X$ and $T : X \times X \to X$ be two given mappings, and let $\alpha : X \times X \times (0, \infty) \to (0, \infty)$ be a function. Recall from [12] that (g, T) is α-admissible if $\alpha(gx, gy, t) \leq 1 \Rightarrow \alpha(T(x, y), T(y, x), t) \leq 1$, $\forall x, y \in X$, $\forall t > 0$.

Lemma 1.3. [12] *Let $(X, F, \triangle)$ be a Menger PbM-space with the coefficient $k \in (0, 1]$, and let $\phi \in \Phi$. Let $c \in (0, 1)$ and $r \in \mathbb{N}$. Then, for any $t > 0$, $F_{x,y}(k^r\phi(t)) \geq F_{x,y}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)$, $\forall x, y \in X$. Then $x = y$.*

2. Main Results

Theorem 2.1. *Let $(X, F, \triangle)$ be a Menger PbM-space with $k \in (0, 1]$. Let $\triangle$ be a continuous t-norm such that $\triangle(v, v) \geq v, \forall v \in [0, 1]$. Assume that $g : X \to X$ and $T : X \times X \to X$ are two given functions with (g, T) being α-admissible. Additionally, assume that the following conditions hold* *(i) $g(X)$ is complete;*

(ii) $T(X \times X) \subseteq g(X)$;

(iii) there exists $(x_0, y_0) \in X \times X$ such that $\alpha(gx_0, gy_0, t) \leq 1$, $\forall t > 0$;

(iv) $\{x_n\}$ and $\{y_n\}$ are two sequences in X such that $\alpha(gx_n, gy_n, t) \leq 1$, $\forall t > 0$, and there exists $(\mu, \nu) \in X \times X$ such that $gx_n \to g\mu$ and $gy_n \to g\nu$ as $n \to \infty$ and $\alpha(g\mu, g\nu, t) \leq 1$ for all $t > 0$;

(v) for any coupled coincidence point (μ, ν) associated with T and g, $\alpha(g\mu, g\nu, t) \leq 1$, $\forall t > 0$.

Moreover, let, for any $x, y, u, v \in X$, $t > 0$, $c \in (0,1)$, and $r \in \mathbb{N}$,

$$\alpha(gx, gy, k^r t)\alpha(gu, gv, k^r t)F_{T(x,y),T(u,v)}(k^r \phi(t)) \geq M(x, y, u, v), \tag{2.1}$$

where $M(x,y,u,v) = \min\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_5, \mathcal{F}_6\}$ with

$$\mathcal{F}_1 = F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \ \mathcal{F}_2 = F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),$$

$$\mathcal{F}_3 = \frac{F_{gx,T(x,y)}(k^{r-1}\phi(\frac{t}{c})) + F_{gu,T(x,y)}(2k^{r-2}\phi(\frac{t}{c}))}{1 + F_{gx,gu}(k^{r-1}\phi(\frac{t}{c}))},$$

$$\mathcal{F}_4 = \frac{F_{gy,T(y,x)}(k^{r-1}\phi(\frac{t}{c})) + F_{gv,T(y,x)}(2k^{r-2}\phi(\frac{t}{c}))}{1 + F_{gy,gv}(k^{r-1}\phi(\frac{t}{c}))},$$

$$\mathcal{F}_5 = \frac{F_{gu,T(u,v)}(k^{r-1}\phi(\frac{t}{c})) + F_{gx,T(u,v)}(2k^{r-2}\phi(\frac{t}{c}))}{1 + F_{gx,gu}(k^{r-1}\phi(\frac{t}{c}))},$$

and

$$\mathcal{F}_6 = \frac{F_{gv,T(v,u)}(k^{r-1}\phi(\frac{t}{c})) + F_{gy,T(v,u)}(2k^{r-2}\phi(\frac{t}{c}))}{1 + F_{gy,gv}(k^{r-1}\phi(\frac{t}{c}))}.$$

Then, T and g have a unique coupled point of coincidence. Moreover, if T and g are ω-compatible, then T and g have a unique common coupled fixed point of the form (u,u).

Proof. Invoking (iii), one sees that there exists $(x_0, y_0) \in X \times X$ such that $\alpha(gx_0, gy_0, t) \leq 1$ for all $t > 0$. Combining this with (ii), one has that there exists $(x_1, y_1) \in X \times X$ such that $gx_1 = T(x_0, y_0)$ and $gy_1 = T(y_0, x_0)$. Since (g, T) is α-admissible, one obtains that $\alpha(gx_1, gy_1, t) \leq 1$ for all $t > 0$. There exists $(x_2, y_2) \in X \times X$ such that $gx_2 = T(x_1, y_1)$ and $gy_2 = T(y_1, x_1)$. In a similarly way, one has that $\alpha(gx_2, gy_2, t) \leq 1$ for all $t > 0$. By induction, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ respectively as $gx_{n+1} = T(x_n, y_n)$, $gy_{n+1} = T(y_n, x_n)$ with

$$\alpha(gx_n, gy_n, t) \leq 1, \ \forall t > 0, \ n = 0, 1, 2, \cdots. \tag{2.2}$$

Suppose that $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$. Otherwise, there exists $n \in \mathbb{N}$ such that $gx_n = gx_{n+1}$ and $gy_n = gy_{n+1}$, that is, $gx_n = T(x_n, y_n)$ and $gy_n = T(y_n, x_n)$. In this case, (x_n, y_n) is a coupled coincidence point of T and g. In (2.1), by taking $(x, y) = (x_n, y_n)$ and $(u, v) = (x_{n+1}, y_{n+1})$, (2.2) yields

$$F_{gx_{n+1}, gx_{n+2}}(k^r \phi(t)) \geq M(x_n, y_n, x_{n+1}, y_{n+1}), \tag{2.3}$$

where

$$\begin{aligned}M(x_n,y_n,x_{n+1},y_{n+1}) = \min\Bigg\{ & F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),\\ & \frac{F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))+F_{gx_{n+1},gx_{n+1}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))},\\ & \frac{F_{gy_n,gy_{n+1}}(k^{r-1}\phi(\frac{t}{c}))+F_{gy_{n+1},gy_{n+1}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gy_n,gy_{n+1}}(k^{r-1}\phi(\frac{t}{c}))},\\ & \frac{F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))+F_{gx_n,gx_{n+2}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))},\\ & \frac{F_{gy_{n+1},gy_{n+2}}(k^{r-1}\phi(\frac{t}{c}))+F_{gy_n,gy_{n+2}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gy_n,gy_{n+1}}(k^{r-1}\phi(\frac{t}{c}))}\Bigg\}.\end{aligned} \tag{2.4}$$

Using conditions (PbM3) and (PbM1), we calculate that

$$\begin{aligned}&\frac{F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)+F_{gx_{n+1},gx_{n+1}}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))}\\ =\ & \frac{F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))+\mathscr{H}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))} = \frac{F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))+1}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))} = 1.\end{aligned} \tag{2.5}$$

By similar arguments as above, we can obtain that

$$\frac{F_{gy_n,gy_{n+1}}(k^{r-1}\phi(\frac{t}{c}))+F_{gy_{n+1},gy_{n+1}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gy_n,gy_{n+1}}(k^{r-1}\phi(\frac{t}{c}))} = 1. \tag{2.6}$$

Invoking condition (PbM3) and $\triangle(v,v) \geq v$ for all $v \in [0,1]$ yields from Remark 1.1 that

$$\begin{aligned}&\frac{F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))+F_{gx_n,gx_{n+2}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))}\\ \geq & \frac{F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))\cdot F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))}\\ = & F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right).\end{aligned}$$

Thus,

$$\frac{F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))+F_{gx_n,gx_{n+2}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gx_n,gx_{n+1}}(k^{r-1}\phi(\frac{t}{c}))} \geq F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right). \tag{2.7}$$

In a similar way, we have that

$$\frac{F_{gy_{n+1},gy_{n+2}}(k^{r-1}\phi(\frac{t}{c}))+F_{gy_n,gy_{n+2}}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)} \geq F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right). \tag{2.8}$$

By substituting (2.5)-(2.8) into (2.4), we deduce that

$$\begin{aligned} M(x_n,y_n,x_{n+1},y_{n+1}) \geq \min\Big\{ & F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\Big\}. \end{aligned} \tag{2.9}$$

By substituting (2.9) into (2.3), we see that

$$\begin{aligned} F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) \geq \min\Big\{ & F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\Big\}. \end{aligned} \tag{2.10}$$

Similarly, we can prove that

$$\begin{aligned} F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) \geq \min\Big\{ & F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\Big\}. \end{aligned} \tag{2.11}$$

Combining (2.10) with (2.11), we immediately have that

$$\begin{aligned} & \min\left\{F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)), F_{gy_{n+1},gy_{n+2}}(k^r\phi(t))\right\} \geq \min\Big\{F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\Big\}. \end{aligned} \tag{2.12}$$

Now the rest of proof is divided into two cases

Case 1 Assume that

$$\begin{aligned} & \min\left\{F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\} \\ \geq\ & \min\left\{F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned}$$

As a result, (2.12) leads to

$$\begin{aligned} & \min\left\{F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)), F_{gy_{n+1},gy_{n+2}}(k^r\phi(t))\right\} \\ \geq\ & \min\left\{F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned} \tag{2.13}$$

Next, one discusses the following four subcases

(a) Suppose that $F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) \geq F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))$. From this, Lemma 1.3 implies that

$$gx_{n+1} = gx_{n+2}. \tag{2.14}$$

Thus, $F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) = F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c})) = 1$, which together with (2.13) obtains that

$$\begin{aligned} & F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) = \min\left\{1, F_{gy_{n+1},gy_{n+2}}(k^r\phi(t))\right\} \\ \geq\ & \min\left\{1, F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\} = F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right). \end{aligned}$$

Clearly, we have that

$$F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) \geq F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right).$$

Again by using Lemma 1.3, one finds that

$$gy_{n+1} = gy_{n+2}. \tag{2.15}$$

By putting together (2.14) and (2.15), it yields that $gx_{n+1} = gx_{n+2} = T(x_{n+1}, y_{n+1})$ and $gy_{n+1} = gy_{n+2} = T(y_{n+1}, x_{n+1})$, which further implies that (gx_{n+1}, gy_{n+1}) is a coupled point of coincidence with respect to g and T.

(b) Suppose that $F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) \geq F_{gy_{n+1},gy_{n+2}}(k^{r-1}\phi(\frac{t}{c}))$. By similar arguments as (a), we find that (gx_{n+1}, gy_{n+1}) is a coupled point of coincidence associated with g and T .

(c) Suppose that $F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) \geq F_{gy_{n+1},gy_{n+2}}(k^{r-1}\phi(\frac{t}{c}))$. Then

$$F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) \geq F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) \geq F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right).$$

With the help of Lemma 1.3, we obtain that $gy_{n+1} = gy_{n+2}$, and hence $F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) = F_{gy_{n+1},gy_{n+2}}(k^{r-1}\phi(\frac{t}{c})) = 1$. This together with (2.13) indicates that

$$\begin{aligned}
&F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) = \min\left\{F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)), 1\right\} \\
&\geq \min\left\{F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), 1\right\} = F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right).
\end{aligned}$$

That is, $F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)) \geq F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))$. Again by applying Lemma 1.3, one sees that $gx_{n+1} = gx_{n+2}$. By similar arguments as (b), we can show that, (gx_{n+1}, gy_{n+1}) is a coupled point of coincidence with respect to g and T.

(d) Suppose that $F_{gy_{n+1},gy_{n+2}}(k^r\phi(t)) \geq F_{gx_{n+1},gx_{n+2}}(k^{r-1}\phi(\frac{t}{c}))$. By similar arguments as (c), we immediately find that (gx_{n+1}, gy_{n+1}) is a coupled point of coincidence of g and T.

Considering conditions (a), (b), (c), and (d), we conclude that g and T have a coupled point of coincidence of g and T in **Case 1**.

Case 2 Assume that

$$\begin{aligned}
&\min\left\{F_{gx_{n+1},gx_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_{n+1},gy_{n+2}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\} \\
&\geq \min\left\{F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.
\end{aligned}$$

With the help of (2.12), one immediately finds that

$$\begin{aligned}
&\min\left\{F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)), F_{gy_{n+1},gy_{n+2}}(k^r\phi(t))\right\} \\
&\geq \min\left\{F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.
\end{aligned} \tag{2.16}$$

Repeating the above process, we assert that

$$\begin{aligned}
&\min\left\{F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)), F_{gy_{n+1},gy_{n+2}}(k^r\phi(t))\right\} \\
&\geq \min\left\{F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),\right\} \\
&\geq \cdots \geq \min\left\{F_{gx_0,gx_1}\left(k^{r-n-1}\phi\left(\frac{t}{c^{n+1}}\right)\right), F_{gy_0,gy_1}\left(k^{r-n-1}\phi\left(\frac{t}{c^{n+1}}\right)\right)\right\}.
\end{aligned}$$

From this, we infer that

$$\begin{aligned} &\min\left\{F_{gx_{n+1},gx_{n+2}}(k^r\phi(t)), F_{gy_{n+1},gy_{n+2}}(k^r\phi(t))\right\} \\ &\geq \quad \min\left\{F_{gx_0,gx_1}\left(k^{r-n-1}\phi\left(\frac{t}{c^{n+1}}\right)\right), F_{gy_0,gy_1}\left(k^{r-n-1}\phi\left(\frac{t}{c^{n+1}}\right)\right)\right\}. \end{aligned} \tag{2.17}$$

Now it remains to prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. For any $t > 0$, by the continuity of ϕ at zero, there exists a constant $s > 0$ such that $\phi(s) < t$. Give $m, n \in \mathbb{N}$ with $m > n$. By combining (PbM3), Remark 1.1 with (2.17) and invoking the nondecreasing property of ϕ, one sees that

$$\begin{aligned} &\min\left\{F_{gx_n,gx_m}((m-n)t), F_{gy_n,gy_m}((m-n)t)\right\} \\ \geq &\min\left\{\min\left\{F_{gx_n,gx_{n+1}}(kt), F_{gx_{n+1},gx_m}((m-n-1)kt)\right\},\right. \\ &\left.\min\left\{F_{gy_n,gy_{n+1}}(kt), F_{gy_{n+1},gy_m}((m-n-1)kt)\right\}\right\} \\ = &\min\left\{\begin{array}{c} \min\left\{F_{gx_n,gx_{n+1}}(kt), F_{gy_n,gy_{n+1}}(kt)\right\}, \\ \min\left\{F_{gx_{n+1},gx_m}((m-n-1)kt), F_{gx_{n+1},gx_m}((m-n-1)kt)\right\} \end{array}\right\} \\ \geq \cdots \geq &\min\left\{\begin{array}{c} \min\left\{F_{gx_n,gx_{n+1}}(kt), F_{gy_n,gy_{n+1}}(kt)\right\}, \\ \min\left\{F_{gx_{n+1},gx_{n+2}}(k^2t), F_{gx_{n+1},gx_{n+2}}(k^2t)\right\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \min\left\{F_{gx_{m-2},gx_{m-1}}(k^{m-n-1}t), F_{gx_{m-2},gx_{m-1}}(k^{m-n-1}t)\right\}, \\ \min\left\{F_{gx_{m-1},gx_m}(k^{m-n}t), F_{gx_{m-1},gx_m}(k^{m-n}t)\right\} \end{array}\right\} \\ \geq &\min\left\{F_{gx_0,gx_1}\left(k^{1-n}\phi\left(\frac{s}{c^n}\right)\right), F_{gy_0,gy_1}\left(k^{1-n}\phi\left(\frac{s}{c^n}\right)\right)\right\}. \end{aligned} \tag{2.18}$$

From the fact that $k^{1-n}\phi(\frac{s}{c^n}) \to \infty$ as $n \to \infty$, we find that

$$\lim_{t\to\infty} \min\left\{F_{gx_0,gx_1}\left(k^{1-n}\phi\left(\frac{s}{c^n}\right)\right), F_{gy_0,gy_1}\left(k^{1-n}\phi\left(\frac{s}{c^n}\right)\right)\right\} = 1.$$

Therefore, for any $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$\min\left\{F_{gx_0,gx_1}\left(k^{1-n}\phi\left(\frac{s}{c^n}\right)\right), F_{gy_0,gy_1}\left(k^{1-n}\phi\left(\frac{s}{c^n}\right)\right)\right\} > 1-\lambda, \tag{2.19}$$

for all $n \geq n_0$. By substituting (2.19) into (2.18), we obtain that

$$\min\{F_{gx_n,gx_m}((m-n)t), F_{gy_n,gy_m}((m-n)t)\} > 1-\lambda,$$

for all $m > n \geq n_0$. Since $t > 0$ and $\lambda \in (0,1)$ are arbitrary, one finds that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. With the help of the completeness of $g(X)$, there exists $(\mu,\nu) \in X \times X$ such that $(gx_n, gy_n) \to (g\mu, g\nu)$ as $n \to \infty$.

Next, we claim that $(g\mu, g\nu)$ is the coupled point of coincidence of T and g. In fact, for any $\varepsilon > 0$, by continuity of ϕ at zero, there exists a constant $t > 0$ such that $\phi(t) < \varepsilon$. From the fact $k \in (0,1]$, we have that $k\varepsilon - k^r\phi(t) > 0$. As a classical result, one infers that $F_{g\mu,T(\mu,\nu)}(\varepsilon) \geq \min\{F_{g\mu,gx_{n+1}}(k\varepsilon - k^r\phi(t)), F_{gx_{n+1},T(\mu,\nu)}(k^r\phi(t))\}$. Recalling that $gx_n \to g\mu$ as $n \to \infty$, for any $\varepsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $F_{g\mu,gx_{n+1}}(kr - k^r\phi(t)) > 1-\varepsilon$, $\forall n \geq n_0$. Hence, $F_{g\mu,T(\mu,\nu)}(\varepsilon) \geq \min\{1-\varepsilon, F_{gx_{n+1},T(\mu,\nu)}(k^r\phi(t))\}$, $\forall n \geq n_0$. Since $\varepsilon \in (0,1)$ is arbitrary, we get that $F_{g\mu,T(\mu,\nu)}(\varepsilon) \geq F_{gx_{n+1},T(\mu,\nu)}(k^r\phi(t))$. By putting $(x,y) = (x_n, y_n)$ and $(u,v) = (\mu,\nu)$

in (2.1), with the help of condition (iv) and (2.2), we successively find that

$$\begin{aligned} F_{g\mu,T(\mu,\nu)}(\varepsilon) \geq & F_{gx_{n+1},T(\mu,\nu)}(k^r\phi(t)) = F_{T(x_n,y_n),T(\mu,\nu)}(k^r\phi(t)) \\ \geq & M(x_n,y_n,\mu,\nu), \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} M(x_n,y_n,\mu,\nu) = \min\Bigg\{ & F_{gx_n,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \\ & \frac{F_{gx_n,T(x_n,y_n)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right) + F_{g\mu,T(x_n,y_n)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gx_n,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)}, \\ & \frac{F_{gy_n,T(y_n,x_n)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right) + F_{g\nu,T(y_n,x_n)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gy_n,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)}, \\ & \frac{F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right) + F_{gx_n,T(\mu,\nu)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gx_n,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)}, \\ & \frac{F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right) + F_{gy_n,T(\nu,\mu)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gy_n,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)} \Bigg\}. \end{aligned} \tag{2.21}$$

By combining (PbM3) with Remark 1.1, we check that

$$F_{g\mu,T(x_n,y_n)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right) \geq F_{g\mu,gx_n}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\cdot F_{gx_n,T(x_n,y_n)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \tag{2.22}$$

$$F_{g\nu,T(y_n,x_n)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right) \geq F_{g\nu,gy_n}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\cdot F_{gy_n,T(y_n,x_n)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \tag{2.23}$$

$$F_{gx_n,T(\mu,\nu)}(2k^{r-2}\phi(\frac{t}{c})) \geq F_{gx_n,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\cdot F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \tag{2.24}$$

and

$$F_{gy_n,T(\nu,\mu)}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right) \geq F_{gy_n,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\cdot F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right). \tag{2.25}$$

By substituting (2.22)-(2.25) into (2.21), one asserts that

$$\begin{aligned} & M(x_n,y_n,\mu,\nu) \\ \geq & \min\Bigg\{F_{gx_n,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy_n,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gx_n,gx_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{gy_n,gy_{n+1}}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\Bigg\}. \end{aligned} \tag{2.26}$$

By putting together (2.20) and (2.26), it follows that

$$\begin{aligned} & F_{g\mu,T(\mu,\nu)}(\varepsilon) \geq \liminf_{n\to\infty} F_{gx_{n+1},T(u,v)}(k^r\phi(t)) \\ = & F_{g\mu,T(\mu,\nu)}(k^r\phi(t)) \geq \liminf_{n\to\infty} M(x_n,y_n,\mu,\nu) \\ = & \min\left\{F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}, \end{aligned}$$

which implies that

$$\begin{aligned} &F_{g\mu,T(\mu,\nu)}(\varepsilon) \geq F_{g\mu,T(\mu,\nu)}(k^r\phi(t)) \\ \geq &\min\left\{F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned} \tag{2.27}$$

Similarly, it can be proved that

$$\begin{aligned} &F_{g\nu,T(\nu,\mu)}(\varepsilon) \geq F_{g\nu,T(\nu,\mu)}(k^r\phi(t)) \\ \geq &\min\left\{F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned} \tag{2.28}$$

By combining (2.27) with (2.28), we obtain that

$$\begin{aligned} &\min\{F_{g\mu,T(\mu,\nu)}(\varepsilon), F_{g\nu,T(\nu,\mu)}(\varepsilon)\} \geq \min\{F_{g\mu,T(\mu,\nu)}(k^r\phi(t)), F_{g\nu,T(\nu,\mu)}(k^r\phi(t))\} \\ \geq &\min\left\{F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned}$$

By repeating this process n times, we calculate that

$$\begin{aligned} &\min\{F_{g\mu,T(\mu,\nu)}(\varepsilon), F_{g\nu,T(\nu,\mu)}(\varepsilon)\} \geq \min\{F_{g\mu,T(\mu,\nu)}(k^r\phi(t)), F_{g\nu,T(\nu,\mu)}(k^r\phi(t))\} \\ \geq &\min\left\{F_{g\mu,T(\mu,\nu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\} \\ \geq &\cdots \geq \min\left\{F_{g\mu,T(\mu,\nu)}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right)\right\}. \end{aligned} \tag{2.29}$$

Invoking $\lim_{t\to\infty}\min\left\{F_{g\mu,T(\mu,\nu)}(t), F_{g\nu,T(\nu,\mu)}(t)\right\} = 1$, with the help of $k^{r-n}\phi\left(\frac{t}{c^n}\right) \to \infty$ as $n \to \infty$, for any $\lambda \in (0,1)$, one sees that there exists $n_0 \in \mathbb{N}$ such that

$$\min\left\{F_{g\mu,T(\mu,\nu)}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right), F_{g\nu,T(\nu,\mu)}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right)\right\} > 1-\lambda,\ \forall n \geq n_0.$$

By virtue of (2.29) and the above inequality, for any $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\min\{F_{g\mu,T(\mu,\nu)}(\varepsilon)), F_{g\nu,T(\nu,\mu)}(\varepsilon)\} > 1-\lambda$, $\forall n \geq n_0$. From this, we obtain that $g\mu = T(\mu,\nu)$ and $g\nu = T(\nu,\mu)$. As a result, $(g\mu, g\nu)$ is a coupled point of coincidence, associated with mappings T and g. Thus it follows from (b) that g and T have a coupled fixed point of coincidence.

Next, we prove the uniqueness of the coupled point of coincidence. Assume that there exists another pair $(\mu^*,\nu^*) \in X \times X$ with $T(\mu^*,\nu^*) = g\mu^*$ and $T(\nu^*,\mu^*) = g\nu^*$. For any $\xi > 0$, by continuity of ϕ at zero, there exists a constant $t > 0$ such that $\phi(t) < \xi$. As $k \in (0,1]$, we have that $\xi > k^r\phi(t)$. With the help of (2.1), we obtain that

$$\begin{aligned} &F_{g\mu,g\mu^*}(\xi) = F_{T(\mu,\nu),T(\mu^*,\nu^*)}(\xi) \geq F_{T(\mu,\nu),T(\mu^*,\nu^*)}(k^r\phi(t)) \\ \geq \quad &\alpha(g\mu, g\nu, k^r t)\alpha(g\mu^*, g\nu^*, k^r t)F_{T(\mu,\nu),T(\mu^*,\nu^*)}(k^r\phi(t)) \geq M(\mu,\nu,\mu^*,\nu^*), \end{aligned} \tag{2.30}$$

where

$$\begin{aligned}
&M(\mu,\nu,\mu^*,\nu^*)\\
=&\min\left\{F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),\right.\\
&\frac{F_{g\mu,g\mu}(k^{r-1}\phi(\frac{t}{c}))+F_{g\mu^*,g\mu}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{g\mu,g\mu^*}(k^{r-1}\phi(\frac{t}{c}))},\frac{F_{g\nu,g\nu}(k^{r-1}\phi(\frac{t}{c}))+F_{g\nu^*,g\nu}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{g\nu,g\nu^*}(k^{r-1}\phi(\frac{t}{c}))},\\
&\left.\frac{F_{g\mu^*,g\mu^*}(k^{r-1}\phi(\frac{t}{c}))+F_{g\mu,g\mu^*}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{g\mu,g\mu^*}(k^{r-1}\phi(\frac{t}{c}))},\frac{F_{g\nu^*,g\nu^*}(k^{r-1}\phi(\frac{t}{c}))+F_{g\nu,g\nu^*}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{g\nu,g\nu^*}(k^{r-1}\phi(\frac{t}{c}))}\right\}.
\end{aligned}\tag{2.31}$$

On the other hand, by using Remark 1.1, we have that

$$\begin{aligned}
F_{g\mu,g\mu^*}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)\geq&\Delta\left(F_{g\mu,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right)\\
\geq&F_{g\mu,g\mu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\cdot F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),
\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}
F_{g\nu,g\nu^*}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)\geq&\Delta\left(F_{g\nu,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right)\\
\geq&F_{g\nu,g\nu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\cdot F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right).
\end{aligned}\tag{2.33}$$

By substituting (2.32) and (2.33) into (2.31), we obtain that

$$M(\mu,\nu,\mu^*,\nu^*)\geq\min\left\{F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.\tag{2.34}$$

From this, we find that

$$M(\mu,\nu,\mu^*,\nu^*)\geq\min\left\{F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.\tag{2.35}$$

By substituting (2.35) into (2.30), we obtain that

$$F_{g\mu,g\mu^*}(\xi)\geq F_{g\mu,g\mu^*}\left(k^r\phi(t)\right)\geq\min\left\{F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.$$

Similarly, we can show that

$$F_{g\nu,g\nu^*}(\xi)\geq F_{g\nu,g\nu^*}(k^r\phi(t))\geq\min\left\{F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.$$

Combining the above two inequalities, we have that

$$\begin{aligned}
\min\left\{F_{g\mu,g\mu^*}(\xi),F_{g\nu,g\nu^*}(\xi)\right\}\geq&\min\left\{F_{g\mu,g\mu^*}\left(k^r\phi(t)\right),F_{g\nu,g\nu^*}\left(k^r\phi(t)\right)\right\}\\
\geq&\min\left\{F_{g\mu,g\mu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.
\end{aligned}\tag{2.36}$$

Via n-iterations, we deduce that

$$\begin{aligned}
\min\left\{F_{g\mu,g\mu^*}(\xi),F_{g\nu,g\nu^*}(\xi)\right\}\geq&\min\left\{F_{g\mu,g\mu^*}(k^r\phi(t)),F_{g\nu,g\nu^*}(k^r\phi(t))\right\}\\
\geq\quad\cdots\geq&\min\left\{F_{g\mu,g\mu^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right),F_{g\nu,g\nu^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right)\right\}.
\end{aligned}$$

This implies that

$$\min\left\{F_{g\mu,g\mu^*}(\xi),F_{gv,gv^*}(\xi)\right\}\geq\min\left\{F_{g\mu,g\mu^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right),F_{gv,gv^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right)\right\}. \tag{2.37}$$

By the fact $k^{r-n}\phi(\frac{t}{c^n})\to\infty$ as $n\to\infty$, one sees that

$$\lim_{n\to\infty}\min\left\{F_{g\mu,g\mu^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right),F_{gv,gv^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right)\right\}=1.$$

Therefore, for any $\lambda\in(0,1)$, there exists $n_0\in\mathbb{N}$ such that

$$\min\left\{F_{g\mu,g\mu^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right),F_{gv,gv^*}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right)\right\}\geq 1-\lambda,\ \forall\, n\geq n_0. \tag{2.38}$$

Combining (2.37) with (2.38), we have that $\min\left\{F_{g\mu,gv}(\varepsilon),F_{g\mu^*,gv^*}(\varepsilon)\right\}\geq 1-\lambda,\ \forall\, n\geq n_0$, which implies that $g\mu=g\mu^*$ and $gv=gv^*$.

Next, we prove that g and T have a unique common coupled fixed point. For that purpose, we set $u=g\mu=T(\mu,v), v=gv=T(v,\mu)$. By virtue of g and T are ω-compatible, we have that

$$gu=gT(\mu,v)=T(g\mu,gv)=T(u,v),gv=gT(v,\mu)=T(gv,g\mu)=T(v,u).$$

Therefore, (gu,gv) is a coupled point of coincidence of g and T, which together with its uniqueness, obtains that $gu=g\mu=u$ and $gv=gv=v$. Therefore, we conclude that $gu=T(u,v)=u$ and $gv=T(v,u)=v$. As a classical result, (u,v) is the unique common coupled fixed point of g and T.

Next, we demonstrate that $u=v$. In fact, for any $\zeta>0$, by continuity of ϕ at zero, we can find a $t>0$, which satisfies $\zeta>\phi(t)$. Since $k\in(0,1]$, we have that $\zeta>k^r\phi(t)$. It follows from (2.1) that

$$F_{gu,gv}(\zeta)\geq\alpha(gu,gv,k^rt)\alpha(gv,gu,k^rt)F_{T(u,v),T(v,u)}(k^r\phi(t))\geq M(u,v,v,u), \tag{2.39}$$

where

$$M(u,v,v,u)=\quad\min\left\{F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),\frac{1+F_{gv,gu}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gv,gu}(k^{r-1}\phi(\frac{t}{c}))}\right\}. \tag{2.40}$$

Invoking Remark 1.1, we have that

$$\begin{aligned}F_{gu,gv}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)&\geq\triangle\left(F_{gu,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right)\\ &\geq\min\left\{\mathscr{H}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}=F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),\end{aligned}$$

which implies that

$$F_{gu,gv}\left(2k^{r-2}\phi\left(\frac{t}{c}\right)\right)\geq F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right). \tag{2.41}$$

It follows from (2.41) that

$$\frac{1+F_{gv,gu}(2k^{r-2}\phi(\frac{t}{c}))}{1+F_{gv,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)}\geq\frac{1+F_{gv,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)}{1+F_{gv,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)}\geq 1. \tag{2.42}$$

By combining (2.40) with (2.42), we obtain that

$$M(u,v,v,u)\geq F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right). \tag{2.43}$$

By substituting (2.43) into (2.39), one sees that

$$F_{gu,gv}(\zeta) \geq F_{gu,gv}(k^r\phi(t)) \geq F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right).$$

By repeating the above process, it is easily seen that

$$F_{gu,gv}(\zeta) \geq F_{gu,gv}(k^r\phi(t)) \geq F_{gu,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right) \geq \cdots \geq F_{gu,gv}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right).$$

That is

$$F_{gu,gv}(\zeta) \geq F_{gu,gv}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right). \tag{2.44}$$

Recalling that $k^{r-n}\phi(\frac{t}{c^n}) \to \infty$ as $n \to \infty$, we have that $\lim_{n\to\infty} F_{gu,gv}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right) = 1$. For any $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that

$$F_{gu,gv}(\zeta) \geq F_{gu,gv}\left(k^{r-n}\phi\left(\frac{t}{c^n}\right)\right) \geq 1-\lambda,\ \forall n \geq n_0.$$

This yields that $gu = gv$, and so $u = gu = gv = v$, that is, $u = gu = T(u,u)$. Consequently, T and g have a unique common coupled fixed point of the form (u,u). This completes the proof of Theorem 2.1. □

By similar arguments to those in Theorem 2.1, we derive the following result.

Theorem 2.2. *Let $(X,F,\triangle)$ be a Menger PbM-space with $k \in (0,1]$. Let $\triangle$ be a continuous t-norm satisfying $\triangle(v,v) \geq v, \forall v \in [0,1]$. Suppose that $g : X \to X$ and $T : X \times X \to X$ are two given functions with (g,T) being α-admissible. Suppose that the following conditions hold*

(i) $g(X)$ is complete;

(ii) $T(X \times X) \subseteq g(X)$;

(iii) there exists $(x_0,y_0) \in X \times X$ such that $\alpha(gx_0,gy_0,t) \leq 1, \forall t > 0$;

(iv) $\{x_n\}$ and $\{y_n\}$ are two sequences in X satisfying $\alpha(gx_n,gy_n,t) \leq 1$, $\forall n \in \mathbb{N}$, $t > 0$ and there exists a pair $(\mu,\nu) \in X \times X$ such that $gx_n \to g\mu$, $gy_n \to g\nu$ as $n \to \infty$ and $\alpha(g\mu,g\nu,t) \leq 1$, $\forall t > 0$;

(v) for any coupled coincidence point (μ,ν) of T and g, we have that $\alpha(g\mu,g\nu,t) \leq 1$, $\forall t > 0$.

Furthermore, for all $x,y,u,v \in X$, $t > 0$, $c \in (0,1)$, and $r \in \mathbb{N}$, it holds

$$\alpha(gx,gy,k^r t)\alpha(gu,gv,k^r t)F_{T(x,y),T(u,v)}(k^r\phi(t)) \geq M(x,y,u,v),$$

where

$$M(x,y,u,v) = \min\Big\{F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gx,T(x,y)}(k^{r-1}\phi(\frac{t}{c})),$$

$$F_{gu,T(x,y)}(2k^{r-2}\phi(\frac{t}{c})), F_{gy,T(y,x)}(k^{r-1}\phi(\frac{t}{c})),\ F_{gv,T(y,x)}(2k^{r-2}\phi(\frac{t}{c})), F_{gu,T(u,v)}(k^{r-1}\phi(\frac{t}{c})),$$

$$F_{gx,T(u,v)}(2k^{r-2}\phi(\frac{t}{c})), F_{gv,T(v,u)}(k^{r-1}\phi(\frac{t}{c})),\ F_{gy,T(v,u)}(2k^{r-2}\phi(\frac{t}{c}))\Big\}.$$

Then T and g have a unique coupled point of coincidence. Moreover, if T and g are ω-compatible, T and g have a unique common coupled fixed point of the form (u,u).

By setting $g = Id$ (Id denotes the identity mapping) and $k = 1$ (k is the coefficient of the Menger PM-space) in Theorem 2.1, we have the following corollary.

Corollary 2.3. *Let $(X,F,\triangle)$ be a complete Menger PM-space with $\triangle$ a continuous t-norm satisfying $\triangle(v,v)\geq v,\forall v\in[0,1]$. Suppose that $T:X\times X\to X$ is an α-admissible function. Suppose that the following conditions hold*

(i) there exists $(x_0,y_0)\in X\times X$ such that $\alpha(x_0,y_0,t)\leq 1,\forall t>0$;

(ii) $\{x_n\}$ and $\{y_n\}$ are two sequences in X satisfying $\alpha(x_n,y_n,t)\leq 1$, $\forall n\in\mathbb{N}$, $t>0$. There exists $(\mu,\nu)\in X\times X$ such that $x_n\to\mu$, $y_n\to\nu$ as $n\to\infty$ and $\alpha(\mu,\nu,t)\leq 1$, $\forall t>0$;

(iii) for any coupled coincidence point (μ,ν) of T, we find that $\alpha(\mu,\nu,t)\leq 1,\forall t>0$.

Furthermore, for all $x,y,u,v\in X$, $t>0$, and $c\in(0,1)$, it holds that

$$\alpha(x,y,t)\alpha(u,v,t)F_{T(x,y),T(u,v)}(\phi(t))\geq M(x,y,u,v),$$

where

$$\begin{aligned}M(x,y,u,v)=\min\Bigg\{&F_{x,u}\left(\phi\left(\frac{t}{c}\right)\right),F_{y,v}\left(\phi\left(\frac{t}{c}\right)\right),\\
&\frac{F_{x,T(x,y)}(\phi(\frac{t}{c}))+F_{u,T(x,y)}(2\phi(\frac{t}{c}))}{1+F_{x,u}(\phi(\frac{t}{c}))},\frac{F_{y,T(y,x)}(\phi(\frac{t}{c}))+F_{v,T(y,x)}(2\phi(\frac{t}{c}))}{1+F_{y,v}(\phi(\frac{t}{c}))},\\
&\frac{F_{u,T(u,v)}(\phi(\frac{t}{c}))+F_{x,T(u,v)}(2\phi(\frac{t}{c}))}{1+F_{x,u}(\phi(\frac{t}{c}))},\frac{F_{v,T(v,u)}(\phi(\frac{t}{c}))+F_{y,T(v,u)}(2\phi(\frac{t}{c}))}{1+F_{y,v}(\phi(\frac{t}{c}))}\Bigg\}.\end{aligned}$$

Then T have a unique coupled fixed point of the form (u,u).

We can state another generalization of Theorem 2.1 as the following corollary.

Corollary 2.4. *Let $(X,F,\triangle)$ be a Menger PbM-space with the coefficient $k\in(0,1]$. Let $\triangle$ be a continuous t-norm satisfying $\triangle(v,v)\geq v,\forall v\in[0,1]$. Suppose that $g:X\to X$ and $T:X\times X\to X$ be two given functions such that (g,T) is α-admissible. Suppose that the following conditions hold*

(i) $g(X)$ is complete;

(ii) $T(X\times X)\subseteq g(X)$;

(iii) there exists $(x_0,y_0)\in X\times X$ such that $\alpha(gx_0,gy_0,t)\leq 1,\forall t>0$;

(iv) $\{x_n\}$ and $\{y_n\}$ are two sequences in X satisfying $\alpha(gx_n,gy_n,t)\leq 1$, $\forall n\in\mathbb{N}$, $t>0$. There exists $(\mu,\nu)\in X\times X$ such that $gx_n\to g\mu$, $gy_n\to g\nu$ as $n\to\infty$ and $\alpha(g\mu,g\nu,t)\leq 1$, $\forall t>0$;

(v) for any coupled coincidence point (μ,ν) of T and g, we infer that $\alpha(g\mu,g\nu,t)\leq 1$, $\forall t>0$.

Moreover, for all $x,y,u,v\in X$, $t>0$, $c\in(0,1)$, and $r\in\mathbb{N}$, it holds that

$$\begin{aligned}&\alpha(gx,gy,k^rt)\alpha(gu,gv,k^rt)F_{T(x,y),T(u,v)}(k^r\phi(t))\\
&\geq\min\left\{F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.\end{aligned}\tag{2.45}$$

Then T and g have a unique coupled point of coincidence. Moreover, if T and g are ω-compatible, then T and g have a unique common coupled fixed point of the form (u,u).

Example 2.5. Let $X=\mathbb{R}^+$. Let F be given as in Example 1.2. Thus $(X,F,\triangle_3)$ is a complete Menger PbM-space with $k=\frac{1}{2}$. Define the functions $T:X\times X\to X$, $g:X\to X$, $\alpha:X\times X\times(0,\infty)\to(0,\infty)$, and $\phi:\mathbb{R}^+\to\mathbb{R}^+$ as $T(x,y)=|x-y|$, $g(x)=x$, $\alpha(x,y,t)=\frac{1}{4}$, $\phi(t)=t$. By setting $c=\frac{1}{2}$. It is obvious that $\frac{1}{4}(|x-u|-|y-v|)^2 \leq \max\{(x-u)^2,(y-v)^2\}$

$$\Leftrightarrow \frac{1}{16}\frac{\frac{1}{2^r}t}{\frac{1}{2^r}t+[|x-u|-|y-v|]^2} \quad \geq \quad \min\left\{\frac{\frac{t}{2^{r-2}}}{\frac{t}{2^{r-2}}+(x-u)^2},\frac{\frac{t}{2^{r-2}}}{\frac{t}{2^{r-2}}+(y-v)^2}\right\}.$$

$$\alpha(gx,gu,k^rt)\alpha(gy,gv,k^rt)\frac{\frac{1}{2^r}t}{\frac{1}{2^r}t+(|x-u|-|y-v|)^2}$$

$$\geq \min\left\{\frac{\frac{1}{2^{r-1}}\frac{t}{c}}{\frac{1}{2^{r-1}}\frac{t}{c}+(x-u)^2},\frac{\frac{1}{2^{r-1}}\frac{t}{c}}{\frac{1}{2^{r-1}}\frac{t}{c}+(y-v)^2}\right\}.$$

Therefore

$$\alpha(gx,gy,k^rt)\alpha(gu,gv,k^rt)F_{T(x,y),T(u,v)}(k^r\phi(t))$$

$$\geq \min\left\{F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.$$

Then mappings T and g satisfy (2.45). Clearly, other conditions are also satisfied in Corollary 2.3, which by Corollary 2.4 asserts that g and T have a unique common coupled fixed in X. Indeed, $(0,0)$ is the unique common coupled fixed of g and T.

Example 2.6. Let $X=\mathbb{R}$ and let (X,d) be a b-metric space with $d(x,y)=|x-y|^p$ $(p>1)$ for all $x,y\in X$ with the coefficient $s=2^{p-1}>1$. Define the distribution function by $\mathscr{D}(t)=0, t\leq 0$ and $\mathscr{D}(t)=1-e^{-t}, t>0$. For any $t>0$, we define a function $F: X\times X\to\mathbb{R}^+$ by

$$F_{x,y}(t)=\begin{cases}\mathscr{H}(t), & x=y,\\ \mathscr{D}\left(\frac{t}{d(x,y)}\right), & x\neq y.\end{cases}$$

By taking $\triangle=\triangle_3$, one has that $(X,F,\triangle_3)$ is a Menger-PbM space with the coefficient $k=\frac{1}{s}$. Obviously, F satisfies conditions (PbM1) and (PbM2). Now it remains to prove that F satisfies (PbM3).

(a) If $x\neq y\neq z$, it follows from $d(x,y)\leq sd(x,z)+sd(y,z)$ that

$$\frac{t_1+t_2}{d(x,y)}\geq\frac{t_1+t_2}{sd(x,z)+sd(y,z)}\geq\min\left\{\frac{t_1}{sd(x,z)},\frac{t_2}{sd(y,z)}\right\},\ \forall\, t_1,t_2>0.$$

By the increasing monotonicity of $\mathscr{D}(t)$, we derive that

$$\mathscr{D}\left(\frac{t_1+t_2}{d(x,y)}\right)\geq\min\left\{\mathscr{D}\left(\frac{t_1}{sd(x,z)}\right),\mathscr{D}\left(\frac{t_2}{sd(y,z)}\right)\right\},$$

which implies that $F_{x,y}(t_1+t_2)\geq\triangle_3\left(F_{x,z}\left(\frac{t_1}{s}\right),F_{y,z}\left(\frac{t_2}{s}\right)\right)$.

(b) Otherwise, we have that

$$F_{x,y}(t_1+t_2)\geq\triangle_3\left(F_{x,z}\left(\frac{t_1}{s}\right),F_{y,z}\left(\frac{t_2}{s}\right)\right)=\triangle_3\left(F_{x,z}(kt_1),F_{y,z}(kt_2)\right).$$

By the above analysis, we prove that F satisfies the condition (PbM3). Therefore, $(X,F,\triangle_3)$ is a Menger PbM-space with the coefficient $k=\frac{1}{s}$.

Example 2.7. Let $(X,F,\triangle_3)$ be a Menger PbM-space defined in Example 1.2 with $k=\frac{1}{s}$ $(s=2^{p-1},\ p\geq 1)$. Give the mappings $T: X\times X\to X$ and $g: X\to X$ respectively by $T(x,y)=\sin|x-y|$ and $gx=8sx$. Define the functions $\phi:\mathbb{R}^+\to\mathbb{R}^+$ and $\alpha: X\times X\times(0,\infty)\to(0,\infty)$ by $\phi(x)=2x$

and $\alpha(x,y,t)=1$, $\forall x,y\in X$, $t>0$. Let $c=\frac{1}{2}$. It is easily to claim that $T(X\times X)\subseteq g(X)$. Next, we prove that g and T are $\omega-$compatible. Indeed, by a directly calculation, we have that

$$\begin{cases} T(x,y)=gx \\ T(y,x)=gy \end{cases} \Leftrightarrow \begin{cases} \sin|x-y|=8px \\ \sin|y-x|=8py \end{cases} \Leftrightarrow 8px=8py \Leftrightarrow x=y=0.$$

This implies that $(g(0),g(0))$ is the unique coupled point of coincidence for mappings T and g. In addition, we have that $T(g(0),g(0))=g(T(0,0))=0$. Therefore, the mapping pair (T,g) is ω-compatible. Next, we prove that the mappings T and g satisfy (2.45). For this, we consider the following two cases

(i) If $x=y$, then $F_{x,y}(t)=\mathscr{H}(t)=1,\forall t>0$. Clearly, (2.45) in Corollary 2.4 is satisfied.

(ii) If $x\neq y$, we calculate that

$$\begin{aligned} &|\sin|x-y|-\sin|u-v|\,|\leq|\,|x-y|-|u-v|\,| \\ \leq\ &|\,(x-y)-(u-v)\,|\leq|x-y|+|u-v|. \end{aligned}$$

Therefore, $|\sin|x-y|-\sin|u-v||^p\leq(|x-y|-|u-v|)^p\leq 2^p\left(|x-u|^p+|y-v|^p\right)$. Thus, for all $t>0$,

$$\frac{2t}{|\sin|x-y|-\sin|u-v||^p}\geq\frac{t+t}{2^p|x-u|^p+2^p|y-v|^p}\geq\min\left\{\frac{t}{2^p|x-u|^p},\frac{t}{2^p|y-v|^p}\right\}.$$

Then

$$\begin{aligned} \frac{2t\cdot k^r}{|\sin|x-y|-\sin|u-v|^p} &\geq \min\left\{\frac{t\cdot k^r}{2^p|x-u|^p},\frac{t\cdot k^r}{2^p|y-v|^p}\right\} \\ = \quad \min\left\{\frac{4t\cdot k^{r-1}}{(8p)^p|x-u|^p},\frac{4t\cdot k^{r-1}}{(8p)^p|y-v|^p}\right\} &= \min\left\{\frac{4t\cdot k^{r-1}}{|8px-8pu|^p},\frac{4t\cdot k^{r-1}}{|8py-8pv|^p}\right\}, \end{aligned}$$

that is,

$$\frac{2t\cdot k^r}{|\sin|x-y|-\sin|u-v||^p}\geq\min\left\{\frac{4t\cdot k^{r-1}}{|8px-8pu|^p},\frac{4t\cdot k^{r-1}}{|8py-8pv|^p}\right\},$$

which implies that

$$\frac{k^r\cdot\phi(t)}{d(T(x,y),T(u,v))}\geq\min\left\{\frac{k^{r-1}\cdot\phi(\frac{t}{c})}{d(gx,gu)},\frac{k^{r-1}\cdot\phi(\frac{t}{c})}{d(gy,gv)}\right\}.$$

Consequently,

$$\mathscr{D}\left(\frac{k^r\cdot\phi(t)}{d(T(x,y),T(u,v))}\right)\geq\min\left\{\mathscr{D}\left(\frac{k^{r-1}\cdot\phi(\frac{t}{c})}{d(gx,gu)}\right),\mathscr{D}\left(\frac{k^{r-1}\cdot\phi(\frac{t}{c})}{d(gy,gv)}\right)\right\},$$

that is, $F_{T(x,y),T(u,v)}(k^r\phi(t))\geq\min\left\{F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}$. By taking $\alpha(x,y,t)=1,\forall x,y\in X,t>0$, we have that, $\forall x,y,u,v\in X$,

$$\begin{aligned} &\alpha(gx,gu,k^rt)\alpha(gy,gv,k^rt)F_{T(x,y),T(u,v)}(k^r\phi(t)) \\ &\geq\min\left\{F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right),F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned}$$

Considering the above cases, one has that T and g satisfy (2.45) of Corollary 2.4. Clearly, all the conditions of Corollary 2.4 are satisfied. Indeed, $(0,0)$ is the unique common coupled fixed point of g and T.

3. Applications to Integral Equations

Throughout this section, we assume that $X = C[0,1]$ is the set of all continuous functions. Consider the integral equation in the following system, $\forall x, y \in X$

$$x(p) = h(p) + \omega \int_0^1 G_1(p,q)S_1(q,x(q))dq \int_0^1 G_2(p,q)S_2(q,y(q))dq, \tag{3.1}$$

where $q \in I = [0,1]$ and $\omega \in [0,+\infty)$. Now we assert that the system satisfies the following assumptions: $S_i : I \times x \to \mathbb{R}$ and $M_i(q) : I \to \mathbb{R}$ $(i = 1,2)$ are continuous with $S_i(q,x) \leq M_i(q)(i = 1,2)$; $h : I \to \mathbb{R}$ is a continuous function and $G_i : I \times I \to \mathbb{R}^+$ $(i = 1,2)$ is continuous at $(p,q) \in I \times I$. Furthermore, for any $p \in I$, it is measurable with respect of $q \in I$, which means that $\int_0^1 G_i(p,q)M_i(q)dq \leq K(i = 1,2)$. There exists $L_i \in [0,1)(i = 1,2)$ such that, $\forall x,y \in X$, $| S_i(q,x) - S_i(q,y) | \leq L_i | x - y |$ $(i = 1,2)$. For the real number $k \in (0,1]$, there exists $c \in (0,1)$ such that $4\omega^2 K^2 \max\{L_1^2, L_2^2\} \parallel m \parallel_\infty^2 \leq kc$, where $\parallel m \parallel_\infty = \sup G_i(p,q), \forall p,q \in I(i = 1,2)$. Define the function $d : X \times X \to \mathbb{R}^+$ by $d(x,y) = \sup_{q\in I} | x(q) - y(q) |^2$, $x,y \in X$. It is obvious that (X,d) is a complete $b-$metric space with $a = 2$. The mapping $F : X \times X \to \mathscr{D}$ is given by $F_{x,y}(t) = \mathscr{H}(t - d(x,y)), t > 0, \forall x,y \in X$. The t-norm $\triangle : X \times X \to X$ is given by $\triangle(x,y) = \min\{x,y\}, \forall x,y \in X$. Hence, $(X,F,\triangle)$ is a Menger-PbM metric space with the coefficient $s = \frac{1}{2}$.

Theorem 3.1. *Under hypothesises, (3.1) has a unique solution.*

Proof. Define the functions $T : X \times X \to X$, $g : X \to X$, $\phi : \mathbb{R}^+ \to \mathbb{R}^+$, and $\alpha : X \times X \times (0,+\infty) \to (0,+\infty)$ respectively by

$$T(x,y)(p) = h(p) + \omega \int_0^1 G_1(p,q)S_1(q,x(q))dq \int_0^1 G_2(p,q)S_2(q,y(q))dq, \forall q \in I;$$

$$g(x) = x, \forall x \in X; \phi(t) = t, \forall t \in \mathbb{R}^+; \alpha(x,y,t) = 1, \forall x,y \in X, \forall t > 0.$$

Then it easy to prove that (g,T) is an α-style admissible contractive mapping pair. By using the conditions, we have that

$$\begin{aligned}
&|T(x,y)(p) - T(u,v)(p)| \\
\leq& \omega K \int_0^1 G_2(p,q)L_2 \mid y(q) - v(q) \mid dq + \omega K \int_0^1 G_1(p,q)L_1 \mid x(q) - u(q) \mid dq \\
\leq& \omega K \max\{L_1,L_2\} \left(\left(\int_0^1 G_2^2(p,q)dq \right)^{\frac{1}{2}} \left(\int_0^1 \mid y(q) - v(q) \mid^2 dq \right)^{\frac{1}{2}} \right. \\
&\left. + \left(\int_0^1 G_1^2(p,q)dq \right)^{\frac{1}{2}} \left(\int_0^1 \mid x(q) - u(q) \mid^2 dq \right)^{\frac{1}{2}} \right) \\
\leq& \omega K \max\{L_1,L_2\} \parallel m \parallel_\infty (\sup_{q\in I} \mid y(q) - v(q) \mid + \sup_{q\in I} \mid x(q) - u(q) \mid).
\end{aligned}$$

which indicates that

$$\begin{aligned}
&d(T(x,y),T(u,v)) = \sup_{q\in I} \mid T(x,y)(q) - T(u,v)(q) \mid^2 \\
\leq&\{\omega K \max\{L_1, L_2\} \parallel m \parallel_\infty (\sup_{q\in I} \mid y(q) - v(q) \mid + \sup_{q\in I} \mid x(q) - u(q) \mid)\}^2 \\
\leq&4\omega^2 K^2 \max\{L_1^2, L_2^2\} \parallel m \parallel_\infty^2 \max\{d(gy,gv), d(gx,gu)\} \\
\leq&kc \max\{d(gy,gv), d(gx,gu)\}.
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
\alpha(gx,gu,t)\alpha(gy,gv,t)F_{T(x,y),T(u,v)}(k^r t) &= \mathscr{H}(k^r t - d(T(x,y),T(u,v))) \\
&\geq \mathscr{H}(k^r t - kc\max\{d(gy,gv),d(gx,gu)\}) \\
&= \min\left\{F_{gx,gu}\left(\frac{k^{r-1}t}{c}\right), F_{gy,gv}\left(\frac{k^{r-1}t}{c}\right)\right\} \\
&= \min\left\{F_{gx,gu}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right), F_{gy,gv}\left(k^{r-1}\phi\left(\frac{t}{c}\right)\right)\right\}.
\end{aligned}$$

In addition, other conditions of Corollary 2.4 hold. Invoking Corollary 2.4, we have that g and T have a unique common coupled fixed point, that is, there exists $x^* \in X$ such that $x^* = gx^* = T(x^*, x^*)$. Consequently,

$$x^*(p) = h(p) + \omega \int_0^1 G_1(p,q)S_1(q,x^*(q))dq \int_0^1 G_2(p,q)S_2(q,x^*(q))dq, q \in I = [0,1], \omega \geq 0.$$

Thus, we conclude that x^* is the unique common coupled fixed point of g and T, which is the unique solution to (3.1). □

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