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CRANK-NICOLSON SPLITTING POSITIVE DEFINITE MIXED ELEMENT DISCRETIZATION OF PARABOLIC CONTROL PROBLEMS

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Abstract. In this paper, we propose the Crank-Nicolson splitting positive definite mixed finite element approximation of parabolic control problems with control constraints. For the state and co-state variables, the Crank-Nicolson scheme is used for time discretization and the first-order Raviart-Thomas mixed element is applied for space discretization. The numerical solution of the control variable is obtain by variational discretization. Based on some regularity assumptions, we derive optimal priori error estimates of the control, state and co-state. Some numerical examples confirm the theoretical investigations.

Keywords. A priori error estimates; Crank-Nicolson splitting positive definite mixed finite element; Parabolic optimal control problems.

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1. Introduction

Optimal control problems (OCPs) are widely used in scientific and engineering problems, such as water pollution control [1], air pollution control [2], and nuclear pollution control. In the past decades, the research on numerical methods for OCPs governed by different partial differential equations (PDEs) was under the spotlight; see [3, 4, 5, 6] for a systematic introduction. Many numerical methods of OCPs were proposed recently, such as finite difference methods (FDMs) [7], finite element methods (FEMs) [8, 9], space-time FEMs [10, 11], characteristic FEMs [12, 13], mixed FEMs [14], finite volume methods [15, 16], spectral method [17, 18], least-squares method [19, 20], multigrid method [21, 22], and so on.

In temperature OCPs [23] and flow OCPs [24], their objective functionals contain the flux of the primal state variables. Thus people pay more attention on the accuracy of the gradient. Mixed FEMs is more appropriate in these cases [14] because numerical solutions for both the state and its flux can be obtained with the same accuracy. Unfortunately, mixed finite element

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(MFE) spaces have to satisfy the Ladyženskaja-Babuška-Brezzi (LBB) condition [25], which brings little available discretization spaces and expensive computing costs.

In order to free of LBB condition, Pani presented an H^1-Galerkin mixed finite element method (MFEM) in [26] and Yang proposed a splitting positive definite MFEM in [27]. Recently, fully discrete splitting positive definite MFEM and H^1-Galerkin MFEM for parabolic optimal control problems (POCPs) were investigated in [28] and [29], respectively. However, their optimal priori error estimates is $\mathscr{O}(h+k)$. We will develop the Crank-Nicolson splitting positive definite mixed finite element (CNSPDMFE) combined with variational discretization (VD) [30, 31] approximation of POCPs and establish the optimal priori error estimates $\mathscr{O}(h^2+k^2)$.

We consider the following POCPs:

$$\min_{u\in K}\frac{1}{2}\int_0^T\left(\|\boldsymbol{Y}-\boldsymbol{Y}_d\|^2+\|y-y_d\|^2+\nu\|u\|^2\right)dt,$$
$$\begin{cases} y_t+\mathrm{div}\boldsymbol{Y}=u+f, \quad t\in J, x\in\Omega,\\ \boldsymbol{Y}=-\nabla y, \quad t\in J, x\in\Omega,\\ y=0, \quad t\in J, x\in\partial\Omega,\\ y(0,x)=y_0(x), \quad x\in\Omega, \end{cases} \tag{1.1}$$

where $J=(0,T]$, $\Omega\subset\mathbf{R}^2$ is a rectangle, and $\nu>0$. Set $U=L^2(J;L^2(\Omega))$, $\boldsymbol{Y}_d\in U^2$, $y_d,f\in U$, $y_0\in H^1(\Omega)$, and $K\subset U$ is defined by

$$K=\left\{v\in U: a\le v(t,x)\le b, \text{ a.e. in } J\times\Omega,\, a,b\in\mathbf{R}\right\}.$$

In this paper, $W^{m,p}(\Omega)$ denotes standard Sobolev spaces on Ω with a semi-norm

$$|v|_{m,p}=\Big(\sum_{|\alpha|=m}\|D^\alpha v\|^p_{L^p(\Omega)}\Big)^{1/p}$$

and a norm

$$\|v\|_{m,p}=\Big(\sum_{|\alpha|\le m}\|D^\alpha v\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

$L^s(J;W^{m,p}(\Omega))$ denotes all L^s integrable functions from J into $W^{m,p}(\Omega)$ with a norm

$$\|v\|_{L^s(J;W^{m,p}(\Omega))}=\Big(\int_0^T||v||^s_{W^{m,p}(\Omega)}dt\Big)^{\frac{1}{s}}.$$

For simplicity, we denote $H^m(\Omega)=W^{m,2}(\Omega)$, $H^m_0(\Omega)=\{v\in W^{m,p}(\Omega): v|_{\partial\Omega}=0\}$, $\|\cdot\|=\|\cdot\|_{0,2}$, $\|\cdot\|_m=\|\cdot\|_{m,2}$, $\|v\|_{L^s(W^{m,p})}=\|v\|_{L^s(J;W^{m,p}(\Omega))}$ with $L^s(W^{m,p})=L^s(J;W^{m,p}(\Omega))$, and $C>0$ is a generic constant independent of k or h.

The structure of this paper is as follows. We present the CNSPDMFE approximation and the corresponding discrete optimality conditions of POCPs (1.1) in Section 2. In Section 3, we introduce some auxiliary variables and important error estimates. Optimal priori error estimates are established in Section 4. In Section 5, the last section, we provided a projection gradient algorithm and some numerical experiments.

2. CNSPDMFE Approximation of POCPs

We give the CNSPDMFE approximation of POCPs (1.1) in this section. Let $Q = H^1(J;W)$, $\boldsymbol{L} = H^1(J;\boldsymbol{V})$ with

$$W = L^2(\Omega), \boldsymbol{V} = H(\text{div};\Omega) = \left\{\boldsymbol{\upsilon} \in (L^2(\Omega))^2, \text{div}\boldsymbol{\upsilon} \in L^2(\Omega)\right\}$$

and the inner products

$$(f_1, f_2) = \int_\Omega f_1 f_2, \quad \forall f_1, f_2 \in W,$$
$$(\boldsymbol{\psi}, \boldsymbol{\upsilon}) = \sum_{i=1}^{2} (\boldsymbol{\psi}_i, \boldsymbol{\upsilon}_i), \quad \forall \boldsymbol{\psi}, \boldsymbol{\upsilon} \in W^2.$$

In addition, $K' = \{\omega \in W : a \leq \omega \leq b, \text{ a.e. in } \Omega\}$.

As in [28], the POCPs (1.1) can be restated as the following splitting positive definite mixed weak form:

$$\min_{u\in K} \frac{1}{2}\int_0^T \left(\|\boldsymbol{Y} - \boldsymbol{Y}_d\|^2 + \|y - y_d\|^2 + \nu\|u\|^2\right) dt$$
$$\begin{cases} (\boldsymbol{Y}_t, \boldsymbol{\upsilon}) + (\text{div}\boldsymbol{Y}, \text{div}\boldsymbol{\upsilon}) = (u + f, \text{div}\boldsymbol{\upsilon}), \quad \forall t \in J, \boldsymbol{\upsilon} \in \boldsymbol{V}, \\ \boldsymbol{Y}(x,0) = \boldsymbol{Y}_0(x), \quad x \in \Omega, \\ (y_t, \omega) + (\text{div}\boldsymbol{Y}, \omega) = (u + f, \omega), \quad \forall t \in J, \omega \in W, \\ y(x,0) = y_0(x), \quad x \in \Omega, \end{cases} \tag{2.1}$$

where $\boldsymbol{Y}_0(x) = -\nabla y_0(x)$.

It follows from [3] that POCPs (2.1) has a unique solution $(\boldsymbol{Y}, y, u) \in \boldsymbol{L} \times Q \times K$, and that $(\boldsymbol{Y}, y, u)$ is the solution to (2.1) if and only if there is a adjoint state $(\boldsymbol{Z}, z) \in \boldsymbol{L} \times Q$ such that $(\boldsymbol{Y}, y, \boldsymbol{Z}, z, u)$ satisfies:

$$(\boldsymbol{Y}_t, \boldsymbol{\upsilon}) + (\text{div}\boldsymbol{Y}, \text{div}\boldsymbol{\upsilon}) = (u + f, \text{div}\boldsymbol{\upsilon}), \quad \forall t \in J, \boldsymbol{\upsilon} \in \boldsymbol{V}, \tag{2.2}$$
$$\boldsymbol{Y}(x,0) = \boldsymbol{Y}_0(x), \quad x \in \Omega, \tag{2.3}$$
$$(y_t, \omega) + (\text{div}\boldsymbol{Y}, \omega) = (u + f, \omega), \quad \forall t \in J, \omega \in W, \tag{2.4}$$
$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{2.5}$$
$$(z_t, \omega) = (y - y_d, \omega), \quad \forall t \in J, \omega \in W, \tag{2.6}$$
$$z(x,T) = 0, \quad x \in \Omega, \tag{2.7}$$
$$-(\boldsymbol{Z}_t, \boldsymbol{\upsilon}) + (\text{div}\boldsymbol{Z}, \text{div}\boldsymbol{\upsilon}) + (z, \text{div}\boldsymbol{\upsilon}) = -(\boldsymbol{Z} - \boldsymbol{Y}_d, \boldsymbol{\upsilon}), \quad \forall t \in J, \boldsymbol{\upsilon} \in \boldsymbol{V}, \tag{2.8}$$
$$\boldsymbol{Z}(x,T) = 0, \quad x \in \Omega, \tag{2.9}$$
$$(\nu u - z - \text{div}\boldsymbol{Z}, \tilde{u} - u) \geq 0, \quad \forall t \in J, \tilde{u} \in K'. \tag{2.10}$$

We introduce a pointwise projection $P_K : W \to K'$, which satisfies:

$$P_K\varphi(x) = \min\left\{b, \max\left\{a, \frac{\varphi(x)}{\nu}\right\}\right\}, \quad \forall \varphi \in W, x \in \Omega. \tag{2.11}$$

Then (2.10) can be equivalently rewritten as: $u = P_K(z + \text{div}\boldsymbol{Z})$.

Let $\mathscr{T}_h$ be a regular triangulation of Ω, $h = \max_{e\in\mathscr{T}_h}\{h_e\}$ with the diameter h_e of e and $P_m(e)$ be the space of polynomials of total degree at most m on e. Let $W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$ denote the order

$m=1$ Raviart-Thomas MFE spaces [32, 25], namely,

$$\begin{aligned}W_h &:= \{\omega_h \in W : \omega_h|_e \in P_m(e), \forall e \in \mathscr{T}_h\},\\ \boldsymbol{V}_h &:= \{\boldsymbol{\upsilon}_h \in \boldsymbol{V} : \boldsymbol{\upsilon}_h|_e \in (P_m(e))^2 + x\cdot P_m(e), \forall e \in \mathscr{T}_h\}.\end{aligned}$$

Let $L^2(\Omega)$-projection [32] $R_h:\ W \to W_h$, which satisfies:

$$(R_h\omega - \omega, \omega_h) = 0, \quad \forall \omega_h \in W_h, \omega \in W, \tag{2.12}$$

$$\|\omega - R_h\omega\|_{0,\rho} \le Ch^r\|\omega\|_{r,\rho}, \quad 0 \le \rho \le \infty, \quad \forall \omega \in W^{r,\rho}(\Omega), \quad 1 \le r \le 1+m. \tag{2.13}$$

We recall the Fortin projection [32] $\Pi_h:\ \boldsymbol{V} \to \boldsymbol{V}_h$, which satisfies:

$$(\mathrm{div}(\Pi_h\boldsymbol{\upsilon} - \boldsymbol{\upsilon}), \omega_h) = 0, \quad \forall \omega_h \in W_h, \boldsymbol{\upsilon} \in \boldsymbol{V}, \tag{2.14}$$

$$\|\boldsymbol{\upsilon} - \Pi_h\boldsymbol{\upsilon}\| \le Ch^r\|\boldsymbol{\upsilon}\|_r, \quad \forall \boldsymbol{\upsilon} \in (H^r(\Omega))^2, \quad 1 \le r \le 1+m, \tag{2.15}$$

$$\|\mathrm{div}(\boldsymbol{\upsilon} - \Pi_h\boldsymbol{\upsilon})\| \le Ch^r\|\mathrm{div}\boldsymbol{\upsilon}\|_r, \quad \forall \mathrm{div}\boldsymbol{\upsilon} \in H^r(\Omega), \quad 1 \le r \le 1+m. \tag{2.16}$$

Let $N \in \mathbf{Z}^+$, $k = T/N$, and $I_n = [t_n, t_{n+1}]$ with $t_n = nk, n = 0, 1, \cdots, N$. We define $\psi^n = \psi(t_n, x)$,

$$\begin{aligned}\psi^{n+\frac{1}{2}} &= \left(\psi^{n+1} + \psi^n\right)/2,\\ d_t\psi^n &= \left(\psi^{n+1} - \psi^n\right)/k,\end{aligned}$$

and discrete time-dependent norms

$$\|\psi\|_{l^s(W^{m,q})} = \left(\sum_{n=0}^{N-1} k\left\|\psi^{n+\frac{1}{2}}\right\|_{W^{m,q}}^s\right)^{1/s}, \quad 1 \le s < \infty$$

and standard definition for $s = \infty$. For convenience, we denote $|||\psi|||_{l^s(W^{m,p})}$ by $|||\psi|||_{s,m}$.

Then the CNSPDMFE approximation of (2.1) is:

$$\min_{u_h^n \in K'} \frac{1}{2}\sum_{n=0}^{N-1} k\left(\left\|\boldsymbol{Y}_h^{n+\frac{1}{2}} - \boldsymbol{Y}_d^{n+\frac{1}{2}}\right\|^2 + \left\|y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}\right\|^2 + \nu\left\|u_h^{n+\frac{1}{2}}\right\|^2\right)$$

$$\begin{cases}(d_t\boldsymbol{Y}_h^n, \boldsymbol{\upsilon}_h) + \left(\mathrm{div}\boldsymbol{Y}_h^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{\upsilon}_h\right) = \left(u_h^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{\upsilon}_h\right), \quad \forall\ \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h,\\ \boldsymbol{Y}_h^0(x) = \Pi_h\boldsymbol{Y}_0(x), \quad x \in \Omega,\\ (d_t y_h^n, \omega_h) + \left(\mathrm{div}\boldsymbol{Y}_h^{n+\frac{1}{2}}, \omega_h\right) = \left(u_h^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}, \omega_h\right), \forall\ \omega_h \in W_h,\\ y_h^0(x) = R_h y_0(x), \quad x \in \Omega,\end{cases} \tag{2.17}$$

Similar to [33, 34], (2.17) has a unique solution $(\boldsymbol{Y}_h^n, y_h^n, u_h^n) \in \boldsymbol{\upsilon}_h \times W_h \times K', n = 1, 2, \cdots, N$, and that $(\boldsymbol{Y}_h^n, y_h^n, u_h^n)$ is the solution to (2.17) if and only if there is a adjoint state $(\boldsymbol{Z}_h^n, z_h^n) \in$

$\boldsymbol{\upsilon}_h \times W_h, n = N-1, \cdots, 1, 0$ such that $(\boldsymbol{Y}_h^n, y_h^n, \boldsymbol{Z}_h^n, z_h^n, u_h^n)$ satisfies:

$$(d_t \boldsymbol{Y}_h^n, \boldsymbol{\upsilon}_h) + \left(\text{div}\boldsymbol{Y}_h^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right) = \left(u_h^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right), \quad \forall \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \tag{2.18}$$

$$\boldsymbol{Y}_h^0(x) = \Pi_h \boldsymbol{Y}_0(x), \quad x \in \Omega, \tag{2.19}$$

$$(d_t y_h^n, \omega_h) + \left(\text{div}\boldsymbol{Y}_h^{n+\frac{1}{2}}, \omega_h\right) = \left(u_h^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}, \omega_h\right), \quad \forall \omega_h \in W_h, \tag{2.20}$$

$$y_h^0(x) = R_h y_0(x), \quad x \in \Omega, \tag{2.21}$$

$$(d_t z_h^n, \omega_h) = \left(y_h^{n+\frac{1}{2}} - y_d^{n+\frac{1}{2}}, \omega_h\right), \quad \forall \omega_h \in W_h, \tag{2.22}$$

$$z_h^N(x) = 0, \quad x \in \Omega, \tag{2.23}$$

$$\begin{aligned} &-(d_t \boldsymbol{Z}_h^n, \boldsymbol{\upsilon}_h) + \left(\text{div}\boldsymbol{Z}_h^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right) + \left(z_h^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right) \\ &= -\left(\boldsymbol{Y}_h^{n+\frac{1}{2}} - \boldsymbol{Y}_d^{n+\frac{1}{2}}, \boldsymbol{\upsilon}_h\right), \quad \forall \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \end{aligned} \tag{2.24}$$

$$\boldsymbol{Z}_h^N(x) = 0, \quad x \in \Omega, \tag{2.25}$$

$$\left(\nu u_h^{n+\frac{1}{2}} - z_h^{n+\frac{1}{2}} - \text{div}\boldsymbol{Z}_h^{n+\frac{1}{2}}, \tilde{u} - u_h^{n+\frac{1}{2}}\right) \geq 0, \quad \forall \tilde{u} \in K'. \tag{2.26}$$

It should be noted that we use VD for the control in (2.26) and (2.26) can be equivalently expressed as: $u_h^{n+\frac{1}{2}} = P_K\left(z_h^{n+\frac{1}{2}} + \text{div}\boldsymbol{Z}_h^{n+\frac{1}{2}}\right), \quad n = 0, 1, \cdots, N-1.$

3. Auxiliary Variables and Error Estimates

We introduce some useful auxiliary variables and important error estimates in this section. For any $\tilde{u} \in K$, we define variables $(\boldsymbol{Y}_h(\tilde{u}), y_h(\tilde{u}), \boldsymbol{Z}_h(\tilde{u}), z_h(\tilde{u}))$, which satisfies

$$(d_t \boldsymbol{Y}_h^n(\tilde{u}), \boldsymbol{\upsilon}_h) + \left(\text{div}\boldsymbol{Y}_h^{n+\frac{1}{2}}(\tilde{u}), \text{div}\boldsymbol{\upsilon}_h\right) = \left(\tilde{u}^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right), \quad \forall \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \tag{3.1}$$

$$\boldsymbol{Y}_h^0(\tilde{u})(x) = \Pi_h \boldsymbol{Y}_0(x), \quad x \in \Omega, \tag{3.2}$$

$$(d_t y_h^n(\tilde{u}), \omega_h) + \left(\text{div}\boldsymbol{Y}_h^{n+\frac{1}{2}}(\tilde{u}), \omega_h\right) = \left(\tilde{u}^{n+\frac{1}{2}} + f^{n+\frac{1}{2}}, \omega_h\right), \quad \forall \omega_h \in W_h, \tag{3.3}$$

$$y_h^0(\tilde{u})(x) = R_h y_0(x), \quad x \in \Omega, \tag{3.4}$$

$$(d_t z_h^n(\tilde{u}), \omega_h) = \left(y_h^{n+\frac{1}{2}}(\tilde{u}) - y_d^{n+\frac{1}{2}}, \omega_h\right), \quad \forall \omega_h \in W_h, \tag{3.5}$$

$$z_h^N(\tilde{u})(x) = 0, \quad x \in \Omega, \tag{3.6}$$

$$\begin{aligned} &-(d_t \boldsymbol{Z}_h^n(\tilde{u}), \boldsymbol{\upsilon}_h) + \left(\text{div}\boldsymbol{Z}_h^{n+\frac{1}{2}}(\tilde{u}), \text{div}\boldsymbol{\upsilon}_h\right) + \left(z_h^{n+\frac{1}{2}}(\tilde{u}), \text{div}\boldsymbol{\upsilon}_h\right) \\ &= -\left(\boldsymbol{Y}_h^{n+\frac{1}{2}}(\tilde{u}) - \boldsymbol{Y}_d^{n+\frac{1}{2}}, \boldsymbol{\upsilon}_h\right), \quad \forall \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \end{aligned} \tag{3.7}$$

$$\boldsymbol{Z}_h^N(\tilde{u})(x) = 0, \quad x \in \Omega. \tag{3.8}$$

Next, we derive the following error estimates on the above auxiliary variables.

Lemma 3.1. *Let* $(\boldsymbol{Y}_h, y_h, \boldsymbol{Z}_h, z_h, u_h)$ *and* $(\boldsymbol{Y}_h(u), y_h(u), \boldsymbol{Z}_h(u), z_h(u))$ *be the solutions to (2.18)-(2.26) and (3.1)-(3.8) with* $\tilde{u} = u$*. There hold*

$$|||y_h(u) - y_h|||_{\infty,0} + |||\boldsymbol{Y}_h(u) - \boldsymbol{Y}_h|||_{\infty,0} \le C|||u_h - u|||_{2,0}, \tag{3.9}$$

$$|||z_h(u) - z_h|||_{\infty,0} + |||\boldsymbol{Z}_h(u) - \boldsymbol{Z}_h|||_{\infty,0} \le C|||u_h - u|||_{2,0}, \tag{3.10}$$

$$|||div(\boldsymbol{Y}_h(u)) - \boldsymbol{Y}_h|||_{2,0} + |||div(\boldsymbol{Z}_h(u)) - \boldsymbol{Z}_h|||_{2,0} \le C|||u_h - u|||_{2,0}. \tag{3.11}$$

Proof. Let $\boldsymbol{\beta} = \boldsymbol{Y}_h - \boldsymbol{Y}_h(u)$, $\alpha = y_h - y_h(u)$, $\boldsymbol{\theta} = \boldsymbol{Z}_h - \boldsymbol{Z}_h(u)$ and $\eta = z_h - z_h(u)$. From (2.18)-(2.25) and (3.1)-(3.8), for $n = 0, 1, \cdots, N-1$, we obtain error equations

$$(d_t\alpha^n, \omega_h) + \left(\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}, \omega_h\right) = \left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, \omega_h\right), \quad \forall\, \omega_h \in W_h, \tag{3.12}$$

$$(d_t\boldsymbol{\beta}^n, \boldsymbol{v}_h) + \left(\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}, \text{div}\boldsymbol{v}_h\right) = \left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, \text{div}\boldsymbol{v}_h\right), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.13}$$

$$-(d_t\boldsymbol{\theta}^n, \boldsymbol{v}_h) + \left(\text{div}\boldsymbol{\theta}^{n+\frac{1}{2}}, \text{div}\boldsymbol{v}_h\right) + \left(\eta^{n+\frac{1}{2}}, \text{div}\boldsymbol{v}_h\right) = -\left(\boldsymbol{\beta}^{n+\frac{1}{2}}, \boldsymbol{v}_h\right), \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.14}$$

$$(d_t\eta^n, \omega_h) = \left(\alpha^{n+\frac{1}{2}}, \omega_h\right), \quad \forall\, \omega_h \in W_h. \tag{3.15}$$

Taking $\boldsymbol{v}_h = \boldsymbol{\beta}^{n+\frac{1}{2}}$ in (3.13), we have

$$\left(d_t\boldsymbol{\beta}^n, \boldsymbol{\beta}^{n+\frac{1}{2}}\right) + \left(\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}, \text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}\right) = \left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, \text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}\right). \tag{3.16}$$

Note that $\left(d_t\boldsymbol{\beta}^n, \boldsymbol{\beta}^{n+\frac{1}{2}}\right) = \frac{\|\boldsymbol{\beta}^{n+1}\|^2 - \|\boldsymbol{\beta}^n\|^2}{2k}$. From (3.16) and Cauchy inequality with ε, we have

$$\frac{\left\|\boldsymbol{\beta}^{n+1}\right\|^2 - \left\|\boldsymbol{\beta}^n\right\|^2}{2k} + \left\|\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}\right\|^2 \le C(\varepsilon)\left\|u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right\|^2 + \varepsilon\left\|\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}\right\|^2. \tag{3.17}$$

We multiply both sides of (3.17) by $2k$ and sum it over n from 0 to $M (1 \le M \le N-1)$, and then obtain

$$\left\|\boldsymbol{\beta}^{M+1}\right\|^2 + 2\sum_{n=0}^{M}\left\|\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}\right\|^2 \le 2C(\varepsilon)\sum_{n=0}^{M}\left\|u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right\|^2, \tag{3.18}$$

which yields

$$|||\boldsymbol{\beta}|||_{\infty,0} + |||\text{div}\boldsymbol{\beta}|||_{2,0} \le C|||u_h - u|||_{2,0}. \tag{3.19}$$

Selecting $\omega_h = \alpha^{n+\frac{1}{2}}$ in (3.12), we conclude

$$(d_t\alpha^n, \alpha^n) = -\left(\text{div}\boldsymbol{\beta}^{n+\frac{1}{2}}, \alpha^n\right) + \left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, \alpha^n\right). \tag{3.20}$$

According to Cauchy inequality, it is easy to derive

$$|||\alpha|||_{\infty,0} \le C|||\text{div}\boldsymbol{\beta}|||_{2,0} + C|||u_h - u|||_{2,0}. \tag{3.21}$$

Choose $\boldsymbol{v}_h = \boldsymbol{\theta}^{n+\frac{1}{2}}$ in (3.14) and $\omega_h = \eta^{n+\frac{1}{2}}$ in (3.15), respectively. Similarly, one has

$$|||\boldsymbol{\theta}|||_{\infty,0} + |||\text{div}\boldsymbol{\theta}|||_{2,0} \le C|||\boldsymbol{\beta}|||_{2,0} + C|||\eta|||_{2,0}, \tag{3.22}$$

$$|||\eta|||_{\infty,0} \le C|||\alpha|||_{2,0}. \tag{3.23}$$

Collecting (3.19)-(3.23), we obtain (3.9)-(3.11). □

For simplicity, we use the following notations

$$\begin{aligned}
&\psi_y = y - y_h(u), \upsilon_y = R_h y - y_h(u), \omega_y = y - R_h y,\\
&\boldsymbol{\xi_Y} = \boldsymbol{Y} - \boldsymbol{Y}_h(u), \boldsymbol{\vartheta_Y} = \Pi_h \boldsymbol{Y} - \boldsymbol{Y}_h(u), \boldsymbol{\rho_Y} = \boldsymbol{Y} - \Pi_h \boldsymbol{Y}\\
&\psi_z = z - z_h(u), \upsilon_z = R_h z - z_h(u), \omega_z = z - R_h z,\\
&\boldsymbol{\xi_Z} = \boldsymbol{Z} - \boldsymbol{Z}_h(u), \boldsymbol{\vartheta_Z} = \Pi_h \boldsymbol{Z} - \boldsymbol{Z}_h(u), \boldsymbol{\rho_Z} = \boldsymbol{Z} - \Pi_h \boldsymbol{Z}.
\end{aligned}$$

Lemma 3.2. *Let $(\boldsymbol{Y}, y, \boldsymbol{Z}, z, u)$ and $(\boldsymbol{Y}_h(u), y_h(u), \boldsymbol{Z}_h(u), z_h(u))$ be the solutions to (2.2)-(2.10) and (3.1)-(3.8) with $\tilde{u} = u$, respectively. Suppose that $y, z \in L^\infty(H^2)$, $y_{ttt}, z_{ttt} \in L^2(L^2)$, $\boldsymbol{Y}_t, \boldsymbol{Z}_t \in L^2((H^2)^2)$, and $\boldsymbol{Y}_{ttt}, \boldsymbol{Z}_{ttt} \in L^2((L^2)^2)$. There hold*

$$|||y_h(u) - y|||_{\infty,0} + |||\boldsymbol{Y}_h(u) - \boldsymbol{Y}|||_{\infty,0} \le C(h^2 + k^2), \tag{3.24}$$

$$|||z_h(u) - z|||_{\infty,0} + |||\boldsymbol{Z}_h(u) - \boldsymbol{Z}|||_{\infty,0} \le C(h^2 + k^2), \tag{3.25}$$

$$|||div(\boldsymbol{Y}_h(u) - \boldsymbol{Y})|||_{2,0} + |||div(\boldsymbol{Z}_h(u) - \boldsymbol{Z})|||_{2,0} \le C(h^2 + k^2). \tag{3.26}$$

Proof. Let $t = \frac{t_n + t_{n+1}}{2}$ in (2.2), (2.4), (2.6), and (2.8). According to (2.2)-(2.9), (3.1)-(3.8), we find from the definition of R_h and Π_h, for $n = 0, 1, \cdots, N-1$ that

$$\left(d_t \upsilon_y^n, \omega_h\right) + \left(\text{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}, \omega_h\right) = \left(d_t y^n - y_t^{n+\frac{1}{2}}, \omega_h\right), \quad \forall\, \omega_h \in W_h, \tag{3.27}$$

$$\left(d_t \boldsymbol{\vartheta}_{\boldsymbol{Y}}^n, \boldsymbol{\upsilon}_h\right) + \left(\text{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right) = \left(d_t \boldsymbol{Y}^n - \boldsymbol{Y}_t^{n+\frac{1}{2}}, \boldsymbol{\upsilon}_h\right) - \left(d_t \boldsymbol{\rho}_{\boldsymbol{Y}}^n, \boldsymbol{\upsilon}_h\right), \quad \forall\, \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h, \tag{3.28}$$

$$\begin{aligned}
&-\left(d_t \boldsymbol{\vartheta}_{\boldsymbol{Z}}^n, \boldsymbol{\upsilon}_h\right) + \left(\text{div}\boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right) + \left(\upsilon_z^{n+\frac{1}{2}}, \text{div}\boldsymbol{\upsilon}_h\right),\\
&= \left(d_t \boldsymbol{\rho}_{\boldsymbol{Z}}^n, \boldsymbol{\upsilon}_h\right) - \left(d_t \boldsymbol{Z}^n - \boldsymbol{Z}_t^{n+\frac{1}{2}}, \boldsymbol{\upsilon}_h\right) - \left(\boldsymbol{\xi}_{\boldsymbol{Y}}^{n+\frac{1}{2}}, \boldsymbol{\upsilon}_h\right), \quad \forall\, \boldsymbol{\upsilon}_h \in \boldsymbol{V}_h,
\end{aligned} \tag{3.29}$$

$$\left(d_t \upsilon_z^n, \omega_h\right) = \left(d_t z^n - z_t^{n+\frac{1}{2}}, \omega_h\right) + \left(\upsilon_y^{n+\frac{1}{2}}, \omega_h\right), \quad \forall\, \omega_h \in W_h. \tag{3.30}$$

Set $\omega_h = \upsilon_y^{n+\frac{1}{2}}$ in (3.27) and $\boldsymbol{\upsilon}_h = \boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}$ in (3.28), respectively. It follows that

$$\left(d_t \upsilon_y^n, \upsilon_y^{n+\frac{1}{2}}\right) = \left(d_t y^n - y_t^{n+\frac{1}{2}}, \upsilon_y^{n+\frac{1}{2}}\right) - \left(\text{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}, \upsilon_y^{n+\frac{1}{2}}\right), \tag{3.31}$$

$$\left(d_t \boldsymbol{\vartheta}_{\boldsymbol{Y}}^n, \boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}\right) + \left(\text{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}, \text{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}\right) = \left(d_t \boldsymbol{Y}^n - \boldsymbol{Y}_t^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}\right) - \left(d_t \boldsymbol{\rho}_{\boldsymbol{Y}}^n, \boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac{1}{2}}\right). \tag{3.32}$$

From Cauchy-Schwarz inequality and interpolation theory, we have

$$\begin{aligned}\left(d_t\boldsymbol{Y}^n-\boldsymbol{Y}_t^{n+\frac12},\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right)&\le\left\|d_t\boldsymbol{Y}^n-\boldsymbol{Y}_t^{n+\frac12}\right\|\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\\&\le\frac1k\int_{t_n}^{t_{n+1}}\left\|\boldsymbol{Y}_t-\boldsymbol{Y}_t^{n+\frac12}\right\|dt\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\\&\le Ck^{\frac32}\left(\int_{t_n}^{t_{n+1}}\|\boldsymbol{Y}_{ttt}\|^2dt\right)^{\frac12}\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\\&=Ck^{\frac32}\|\boldsymbol{Y}_{ttt}\|_{L^2(I_n;L^2)}\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\end{aligned}\tag{3.33}$$

and

$$\left(d_ty^n-y_t^{n+\frac12},\upsilon_y^{n+\frac12}\right)\le Ck^{\frac32}\|y_{ttt}\|_{L^2(I_n;L^2)}\left\|\upsilon_y^{n+\frac12}\right\|.\tag{3.34}$$

According to (2.15) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left(d_t\boldsymbol{\rho}_{\boldsymbol{Y}}^n,\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right)&\le\|d_t\boldsymbol{\rho}_{\boldsymbol{Y}}^n\|\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\\&\le\frac1k\int_{t_n}^{t_{n+1}}\|\boldsymbol{\rho}_{\boldsymbol{Y},t}\|dt\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\\&\le k^{-\frac12}\left(\int_{t_n}^{t_{n+1}}\|\boldsymbol{\rho}_{\boldsymbol{Y},t}\|^2dt\right)^{\frac12}\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|\\&\le h^2k^{-\frac12}\|\boldsymbol{Y}_t\|_{L^2(I_n;H^2)}\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|.\end{aligned}\tag{3.35}$$

From (3.32), (3.33), (3.35), and Cauchy inequality with ε, we arrive at

$$\begin{aligned}&\frac{\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+1}\right\|^2-\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n}\right\|^2}{2k}+\left\|\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|^2\\&\le C(\varepsilon)k^3\|\boldsymbol{Y}_{ttt}\|^2_{L^2(I_n;L^2)}+\frac{C(\varepsilon)h^4}{k}\|\boldsymbol{Y}_t\|^2_{L^2(I_n;H^2)}+2\varepsilon\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|^2.\end{aligned}\tag{3.36}$$

We multiply both sides of (3.36) by $2k$ and sum it from 0 to $M(1\le M\le N-1)$, and then derive

$$\left\|\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{M+1}\right\|^2+2\sum_{n=0}^{N-1}k\left\|\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}^{n+\frac12}\right\|^2\le 2C(\varepsilon)k^4\|\boldsymbol{Y}_{ttt}\|^2_{L^2(L^2)}+2C(\varepsilon)h^4\|\boldsymbol{Y}_t\|^2_{L^2(H^2)}.\tag{3.37}$$

From (3.37), it is easy to see that

$$|||\boldsymbol{\vartheta}_{\boldsymbol{Y}}|||_{\infty,0}+|||\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}|||_{2,0}\le C\left(h^2\|\boldsymbol{Y}_t\|_{L^2(H^2)}+k^2\|\boldsymbol{Y}_{ttt}\|_{L^2(L^2)}\right).\tag{3.38}$$

According to (3.31) and (3.34), we can similarly obtain

$$|||\upsilon_y|||_{\infty,0}\le C\left(k^2\|y_{ttt}\|_{L^2(L^2)}+|||\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Y}}|||_{2,0}\right).\tag{3.39}$$

Taking $\boldsymbol{\upsilon}_h = \boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}$ and $\omega_h = \upsilon_z^{n+\frac{1}{2}}$ in (3.29) and (3.30), respectively, we have

$$\left(d_t\upsilon_z^n, \upsilon_z^{n+\frac{1}{2}}\right) = \left(d_t z^n - z_t^{n+\frac{1}{2}}, \upsilon_z^{n+\frac{1}{2}}\right) + \left(\upsilon_y^{n+\frac{1}{2}}, \upsilon_z^{n+\frac{1}{2}}\right), \tag{3.40}$$

$$\begin{aligned} &-\left(d_t\boldsymbol{\vartheta}_{\boldsymbol{Z}}^n, \boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}\right) + \left(\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}\right) \\ &= \left(d_t\boldsymbol{\rho}_{\boldsymbol{Z}}^n, \boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}\right) - \left(d_t\boldsymbol{Z}^n - \boldsymbol{Z}_t^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}\right) - \left(\boldsymbol{\xi}_{\boldsymbol{Y}}^{n+\frac{1}{2}}, \boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}\right) - \left(\upsilon_z^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Z}}^{n+\frac{1}{2}}\right). \end{aligned} \tag{3.41}$$

Similarly, we can derive that

$$|||\upsilon_z|||_{\infty,0} \le C\left(k^2\|z_{ttt}\|_{L^2(L^2)} + |||\upsilon_y|||_{2,0}\right), \tag{3.42}$$

$$\begin{aligned} &|||\boldsymbol{\vartheta}_{\boldsymbol{Z}}|||_{\infty,0} + |||\mathrm{div}\boldsymbol{\vartheta}_{\boldsymbol{Z}}|||_{2,0} \\ &\le C\left(h^2\|\boldsymbol{Z}_t\|_{L^2(H^2)} + k^2\|\boldsymbol{Z}_{ttt}\|_{L^2(L^2)} + |||\boldsymbol{\vartheta}_{\boldsymbol{Y}}|||_{2,0} + |||\upsilon_z|||_{\infty,0}\right). \end{aligned} \tag{3.43}$$

It follows from (2.13), (2.15)-(2.16), (3.38)-(3.39), (3.42)-(3.43), and triangle inequality, we can obtain (3.24)-(3.26). □

Lemma 3.3. *Let* $(\boldsymbol{Y}_h, y_h, \boldsymbol{Z}_h, z_h, u_h)$ *and* $(\boldsymbol{Y}, y, \boldsymbol{Z}, z, u)$ *be the solutions to (2.18)-(2.26) and (2.2)-(2.10), respectively. Then*

$$\sum_{n=0}^{N-1} k\left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, div\,\boldsymbol{\theta}^{n+\frac{1}{2}} + \eta^{n+\frac{1}{2}}\right) \le 0. \tag{3.44}$$

Proof. Choose $\omega_h = \eta^{n+\frac{1}{2}}$ in (3.12), $\boldsymbol{\upsilon}_h = \boldsymbol{\theta}^{n+\frac{1}{2}}$ in (3.13), $\boldsymbol{\upsilon}_h = \boldsymbol{\beta}^{n+\frac{1}{2}}$ in (3.14), and $\omega_h = \alpha^{n+\frac{1}{2}}$ in (3.15), respectively. Multiplying both sides of equations (3.12)-(3.15) by $2k$ then summing it from 0 to $N-1$, we arrive at

$$\sum_{n=0}^{N-1} k\left(u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}, \mathrm{div}\boldsymbol{\theta}^{n+\frac{1}{2}} + \eta^{n+\frac{1}{2}}\right) = -|||\alpha|||_{2,0}^2 - |||\boldsymbol{\beta}|||_{2,0}^2. \tag{3.45}$$

Then we have (3.44) from (3.45). □

4. A Priori Error Estimates

We investigate optimal priori error of the CNSPDMFE discretization (2.18)-(2.26) in this section.

Theorem 4.1. *Let* $(\boldsymbol{Y}, y, \boldsymbol{Z}, z, u)$ *and* $(\boldsymbol{Y}_h, y_h, \boldsymbol{Z}_h, z_h, u_h)$ *be the solutions to (2.2)-(2.10) and (2.18)-(2.26), respectively. Suppose that all the conditions in Lemma 3.1-3.3 are satisfied. Then it holds*

$$|||u_h - u|||_{2,0} \le C\left(h^2 + k^2\right). \tag{4.1}$$

Proof. Taking $t = \frac{t_n + t_{n+1}}{2}$, $\tilde{u} = u_h^{n+\frac{1}{2}}$ in (2.10) and selecting $\tilde{u} = u^{n+\frac{1}{2}}$ in (2.26), we have

$$\left(\nu u^{n+\frac{1}{2}} - z^{n+\frac{1}{2}} - \mathrm{div}\boldsymbol{Z}^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}} - u^{n+\frac{1}{2}}\right) \ge 0, \tag{4.2}$$

$$\left(\nu u_h^{n+\frac{1}{2}} - z_h^{n+\frac{1}{2}} - \mathrm{div}\boldsymbol{Z}_h^{n+\frac{1}{2}}, u^{n+\frac{1}{2}} - u_h^{n+\frac{1}{2}}\right) \ge 0. \tag{4.3}$$

It follows from (4.2) and (4.3) that

$$\begin{aligned}\nu|||u_h-u|||_{2,0}^2 &= \sum_{n=1}^{N} k\left(u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}, u_h^{n+\frac{1}{2}}-u^{n+\frac{1}{2}}\right)\\ &\le \sum_{n=1}^{N} k\left(z^{n+\frac{1}{2}}-z_h^{n+\frac{1}{2}}+\operatorname{div}\left(\boldsymbol{Z}^{n+\frac{1}{2}}-\boldsymbol{Z}_h^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right)\\ &= \sum_{n=1}^{N} k\left(z^{n+\frac{1}{2}}-z_h^{n+\frac{1}{2}}(u)+\operatorname{div}\left(\boldsymbol{Z}^{n+\frac{1}{2}}-\boldsymbol{Z}_h^{n+\frac{1}{2}}(u)\right), u^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right)\\ &\quad+\sum_{n=1}^{N} k\left(z_h^{n+\frac{1}{2}}(u)-z_h^{n+\frac{1}{2}}+\operatorname{div}\left(\boldsymbol{Z}_h^{n+\frac{1}{2}}(u)-\boldsymbol{Z}_h^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right)\\ &=: E_1+E_2.\end{aligned} \tag{4.4}$$

For E_1, by using Cauchy inequality with ε, we find that

$$E_1 \le C(\varepsilon)\left(|||z_h(u)-z|||_{2,0}^2+|||\operatorname{div}\left(\boldsymbol{Z}_h(u)-\boldsymbol{Z}\right)|||_{2,0}^2\right)+\varepsilon|||u_h-u|||_{2,0}^2. \tag{4.5}$$

From Lemma 3.3, we know that

$$E_2=\sum_{n=1}^{N} k\left(z_h^{n+\frac{1}{2}}(u)-z_h^{n+\frac{1}{2}}+\operatorname{div}\left(\boldsymbol{Z}_h^{n+\frac{1}{2}}(u)-\boldsymbol{Z}_h^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}-u_h^{n+\frac{1}{2}}\right)\le 0. \tag{4.6}$$

Substituting (3.25)-(3.26) and (4.5)-(4.6) in (4.4), we derive (4.1). □

Theorem 4.2. *Let $(\boldsymbol{Y}, y, \boldsymbol{Z}, z, u)$ and and $(\boldsymbol{Y}_h, y_h, \boldsymbol{Z}_h, z_h, u_h)$ be the solutions to (2.2)-(2.10) and (2.18)-(2.26), respectively. With the same assumption of Theorem 4.1, there hold*

$$|||\boldsymbol{Y}_h-\boldsymbol{Y}|||_{\infty,0}+|||y_h-y|||_{\infty,0}\le C\left(h^2+k^2\right), \tag{4.7}$$

$$|||\boldsymbol{Z}_h-\boldsymbol{Z}|||_{\infty,0}+|||z_h-z|||_{\infty,0}\le C\left(h^2+k^2\right), \tag{4.8}$$

$$|||div(\boldsymbol{Y}_h)-\boldsymbol{Y}|||_{2,0}+|||div(\boldsymbol{Z}_h-\boldsymbol{Z})|||_{2,0}\le C\left(h^2+k^2\right). \tag{4.9}$$

Proof. By using Lemmas 3.1, Lemma 3.2, Theorem 4.1, and the triangle inequality, we obtain (4.7)-(4.9) immediately. □

5. Numerical Experiments

We provide a numerical algorithm for the CNSPDMFE approximation of POCPs and present two examples to validate the previous theoretical analysis results in this section.

Similar to [35], for a convex optimization problem with control pointwise inequality constraint:

$$\min_{u\in K\subset U} J(u).$$

Its iterative scheme reads:

$$\begin{cases} b(u_{n+\frac{1}{2}}, v)=b(u_n, v)-\rho_n\left(J'(u_n), v\right), & \forall v\in U, n=0,1,2,\cdots,\\ u_{n+1}=P_K(u_{n+\frac{1}{2}}), \end{cases} \tag{5.1}$$

where $b(\cdot,\cdot)=\int_0^T(\cdot,\cdot)$, ρ_n is a iteration step size, and P_K is computed as (2.11).

For an acceptable error Tol, by using (5.1) to the CNSPDMFE discretization scheme of POCPs (1.1), we can propose the following numerical algorithm. For convenience, the subscript h has been dropped.

Algorithm 5.1.
1. Set $u_{(0)}^i, i = 1,2,\cdots,N$.
2. I. Solve $\boldsymbol{Y}_{(n)}^i \in \boldsymbol{\upsilon}_h, i = 0,1,\cdots,N-1$ such that

$$\left(d_t \boldsymbol{Y}_{(n)}^i, \boldsymbol{\upsilon}\right) + \left(\mathrm{div}\boldsymbol{Y}_{(n)}^{i+\frac{1}{2}}, \mathrm{div}\boldsymbol{\upsilon}\right) = \left(f^{i+\frac{1}{2}} + u_{(n)}^{i+\frac{1}{2}}, \mathrm{div}\boldsymbol{\upsilon}\right), \quad \boldsymbol{Y}_{(n)}^0 = \Pi_h \boldsymbol{Y}_0.$$

II. Solve $y_{(n)}^i \in \Omega_h, i = 0,1,\cdots,N-1$ such that

$$\left(d_t y_{(n)}^i, \omega\right) = -\left(\mathrm{div}\boldsymbol{Y}_{(n)}^{i+\frac{1}{2}}, \omega\right) + \left(f^{i+\frac{1}{2}} + u_{(n)}^{i+\frac{1}{2}}, \omega\right), \quad y_{(n)}^0 = R_h y_0.$$

III. Solve $z_{(n)}^i \in \Omega_h, i = N-1,\cdots,1,0$ such that

$$\left(d_t z_{(n)}^i, \omega\right) = \left(y_{(n)}^{i+\frac{1}{2}} - y_d^{i+\frac{1}{2}}, \omega\right), \quad z_{(n)}^N = 0.$$

IV. Solve $\boldsymbol{Z}_{(n)}^i \in \boldsymbol{\upsilon}_h, i = N-1,\cdots,1,0$ such that

$$-\left(d_t \boldsymbol{Z}_{(n)}^i, \boldsymbol{\upsilon}\right) + \left(\mathrm{div}\boldsymbol{Z}_{(n)}^{i+\frac{1}{2}}, \mathrm{div}\boldsymbol{\upsilon}\right) = -\left(\boldsymbol{Y}_{(n)}^{i+\frac{1}{2}} - \boldsymbol{Y}_d^{i+\frac{1}{2}}, \boldsymbol{\upsilon}\right) - \left(z_{(n)}^{i+\frac{1}{2}}, \mathrm{div}\boldsymbol{\upsilon}\right), \quad \boldsymbol{Z}_{(n)}^N = 0.$$

V. Compute $u_{(n+1)}^i, i = 0,1,\cdots,N-1$ by

$$\begin{cases} b\left(u_{(n+\frac{1}{2})}^i, \mu\right) = b\left(u_{(n)}^i, \mu\right) - \rho_{(n)}\left(J'(u_{(n)}^i, \mu\right), \quad \forall \mu \in U, \\ u_{(n+1)}^i = P_K\left(u_{(n+\frac{1}{2})}^i\right). \end{cases}$$

3. Calculate the iterative error: $E_{n+1} = |||u_{(n)} - u_{(n+1)}|||_{2,0}$.
4. If $E_{n+1} \leq Tol$, stop; Else set $n := n+1$ go to 2.

Let $\Omega = (0,1)\times(0,1)$, $T = 1$ and $\nu = 1$. The following examples of POCPs were solved by Algorithm 5.1 based on AFEPack [35].

Example 5.1. The test exact solutions are as follows:

$$\begin{aligned}
&y = t^2 \sin(2\pi x_1)\sin(2\pi x_2),\\
&\boldsymbol{Y} = -\left(2\pi t^2 \cos(2\pi x_1)\sin(2\pi x_2), 2\pi t^2 \sin(2\pi x_1)\cos(2\pi x_2)\right)^{\mathrm{T}},\\
&\boldsymbol{Z} = \left(2\pi(1-t)^2 \cos(2\pi x_1)\sin(2\pi x_2), 2\pi(1-t)^2 \sin(2\pi x_1)\cos(2\pi x_2)\right)^{\mathrm{T}},\\
&\boldsymbol{Y}_d = \boldsymbol{Y}(x,t) - \boldsymbol{Z}_t(x,t) + \begin{pmatrix} 2\pi(1-t)\cos(2\pi x_1)\sin(2\pi x_2) \\ 2\pi(1-t)\sin(2\pi x_1)\cos(2\pi x_2) \end{pmatrix},\\
&z = -\mathrm{div}\boldsymbol{Z}(x,t) + 2(1-t)\sin(2\pi x_1)\sin(2\pi x_2),\\
&u = \min\{b, \max\{a, z + \mathrm{div}\boldsymbol{Z}\}\}, a = -0.5, b = 0.5,\\
&f = y_t + \mathrm{div}\boldsymbol{Y} - u, y_d = y - z_t.
\end{aligned}$$

We show some numerical results in Table 1 and the relationship between $\log_{10}(node)$ and $\log_{10}(error)$ in Figure 1.

TABLE 1. Numerical results of Example 5.1.

$h=\tau$	$1/10$	$1/20$	$1/40$	$1/80$
$\|\|\|u_h-u\|\|\|_{2,0}$	3.6583e-02	9.1546e-03	2.2864e-03	5.7101e-04
$\|\|\|y_h-y\|\|\|_{\infty,0}$	2.4658e-02	6.1857e-03	1.5403e-03	3.8498e-04
$\|\|\|z_h-z\|\|\|_{\infty,0}$	3.3465e-02	8.3765e-03	2.0926e-03	5.2283e-04
$\|\|\|\boldsymbol{Y}_h-\boldsymbol{Y}\|\|\|_{\infty,0}$	4.8792e-02	1.2206e-02	3.0498e-03	7.6235e-04
$\|\|\|\boldsymbol{Z}_h-\boldsymbol{Z}\|\|\|_{\infty,0}$	5.1254e-02	1.2834e-02	3.2107e-03	8.0084e-04
$\|\|\|\mathrm{div}\,(\boldsymbol{Y}_h-\boldsymbol{Y})\,\|\|\|_{2,0}$	5.4839e-02	1.3712e-02	3.4282e-03	8.5701e-04
$\|\|\|\mathrm{div}\,(\boldsymbol{Z}_h-\boldsymbol{Z})\,\|\|\|_{2,0}$	5.8735e-02	1.4690e-02	3.6711e-03	9.1773e-04

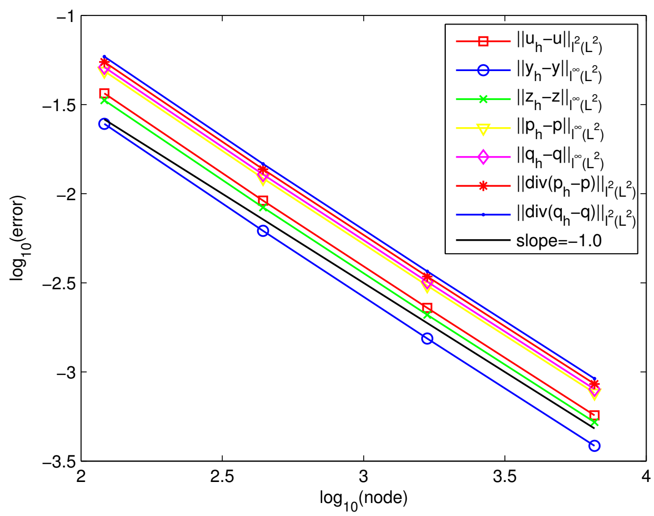

FIGURE 1. Convergence rates of Example 5.1.

Example 5.2. The test exact solutions are as follows:

$$
\begin{aligned}
&y=t^2x_1(1-x_1)x_2(1-x_2),\\
&\boldsymbol{Y}=-\left(t^2(1-2x_1)x_2(1-x_2),t^2x_1(1-x_1)(1-2x_2)\right)^{\mathrm{T}},\\
&\boldsymbol{Z}=\left((1-t)^2(1-2x_1)x_2(1-x_2),(1-t)^2x_1(1-x_1)(1-2x_2)\right)^{\mathrm{T}},\\
&\boldsymbol{Y}_d=\boldsymbol{Y}(x,t)-\boldsymbol{Z}_t(x,t)+\begin{pmatrix}(1-t)(1-2x_1)x_2(1-x_2)\\(1-t)x_1(1-x_1)(1-2x_2)\end{pmatrix},\\
&z=-\mathrm{div}\boldsymbol{Z}(x,t)+2(1-t)x_1(1-x_1)x_2(1-x_2),\\
&u=\min\{b,\max\{a,z+\mathrm{div}\boldsymbol{Z}\}\},a=-0.25,b=0.25,\\
&f=y_t+\mathrm{div}\boldsymbol{Y}-u,y_d=y-z_t.
\end{aligned}
$$

In Table 2 and Figure 2, we can see the optimal convergence rate $O(h^2+k^2)$. They are consistent with our theoretical results.

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TABLE 2. Numerical results of Example 5.2.

$h=\tau$	$1/10$	$1/20$	$1/40$	$1/80$
$\|\|\|u_h-u\|\|\|_{2,0}$	2.4865e-02	6.2162e-03	1.5541e-03	3.8852e-04
$\|\|\|y_h-y\|\|\|_{\infty,0}$	1.4579e-02	3.6448e-03	9.1121e-04	2.2778e-04
$\|\|\|z-z_h\|\|\|_{\infty,0}$	2.5691e-02	6.4228e-03	1.6957e-03	4.0142e-04
$\|\|\|\boldsymbol{Y}_h-\boldsymbol{Y}\|\|\|_{\infty,0}$	3.6957e-02	9.2393e-03	2.3098e-03	5.7745e-04
$\|\|\|\boldsymbol{Z}_h-\boldsymbol{Z}\|\|\|_{\infty,0}$	4.1026e-02	1.0257e-02	2.5641e-03	6.4103e-04
$\|\|\|\mathrm{div}\,(\boldsymbol{Y}_h-\boldsymbol{Y})\,\|\|\|_{2,0}$	4.4052e-02	1.1013e-02	2.7533e-03	6.8831e-04
$\|\|\|\mathrm{div}\,(\boldsymbol{Z}_h-\boldsymbol{Z})\,\|\|\|_{2,0}$	4.6538e-02	1.1635e-02	2.9086e-03	7.2716e-04

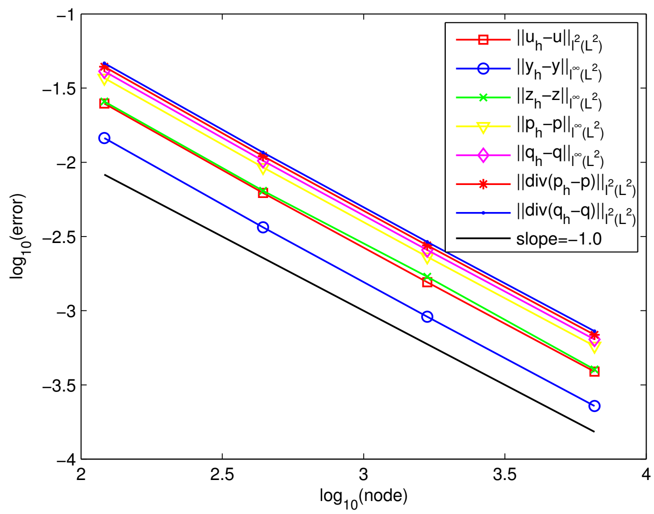

FIGURE 2. Convergence rates of Example 5.2.

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