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LOCAL MULTIGRID METHOD FOR SOLVING ADAPTIVE FINITE ELEMENT EQUATIONS OF THREE-DIMENSIONAL ELASTICITY PROBLEMS

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Abstract. The adaptive finite element method is an effective numerical approach for solving three-dimensional elasticity problems. This paper presents an adaptive p order finite element method which does not require marking oscillatory terms and does not satisfy the inner node property when refining meshes. Subsequently, aiming at the p order finite element equations under the adaptive meshes, a multigrid (MG) method based on local relaxation iteration is established by utilizing the characteristic that only local units on each layer of the meshes needs to be refined during the adaptive encryption process. Numerical experiments demonstrate that the adaptive finite element method exhibits uniform convergence and quasi-optimal computational complexity for linear and quadrtic finite element equations, and the corresponding multigrid method demonstrates good computational efficiency and robustness.

Keywords. Adaptive finite elements; Hierarchical basis; Local relaxation; Multigrid method; Three-dimensional elasticity problems.

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1. INTRODUCTION

The finite element method is an important numerical method for solving three-dimensional elasticity problems. Due to the complexity of practical problems, using conventional finite elements or even higher dimensions may not necessarily improve accuracy. An effective method is to use the adaptive finite element method, which automatically distributes points on mesh based on the solution characteristics. This approach can effectively overcome the difficulties caused by excessive growth of degrees of freedom due to uniform refinement, and achieve the goal of obtaining maximum computational accuracy with minimal computational amount. Currently, adaptive finite element methods, which are now one of the most effective numerical methods for solving practical problems, such as non-smooth physical domains, discontinuous

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material properties, and stress concentrations, have been widely applied in modern scientific and engineering computations; see, e.g., [1, 2, 3, 7, 8, 15, 12].

In adaptive finite element methods, the solution module is one of the main factors in influencing the overall efficiency of finite element analysis. It is necessary and essential to investigate efficient fast algorithms for finite element algebraic equations based on adaptive mesh. In adaptive mesh refinement, a series of nested meshes are generated. If these finite element algebraic equations depending on these series of meshes are solved by using direct methods, the information between different meshes will be lost.

The most effective method is to use multigrid methods. In the conventional multigrid method, both the pre-smoothing and post-smoothing operators are applied to the global grid, leading to a significant increase in computational workload with each additional level of adaptive refinement mesh. The locally relaxed iterative multigrid method (referred to as the Local MG Method) developed in recent years has achieved excellent solution efficiency in adaptive finite element analysis; see, e.g., [4, 5, 9, 11, 14]. However, existing methods are primarily designed for scalar elliptic problems, and there are still limited results on the treatment of the systems of equations (such as in elasticity problems). Furthermore, the widely used adaptive methods in computations are based on the Z-Z type posterior error estimation [13]. Although this adaptive method converges, it is not optimal. Recently, an adaptive finite element method was presented in [10], and the uniform convergence and quasi-optimal computational complexity was proved for two-dimensional elasticity problems, meanwhile the LocalMG method was designed to prove improved computational efficiency and robustness. However, for the three-dimensional case, no research or exploration has been conducted yet.

In this paper, we extend the ideas of the algorithm and the analytical methods from [10] to the numerical solution of three-dimensional elasticity problems, and establish the p order adaptive finite element algorithm which does not require the marking of oscillatory terms and does not need the elements to satisfy the inner node property. Subsequently, for the p order finite element equations under adaptive meshes, Local MG method is established by performing multiple Gauss-Seidel smoothening on the p order finite element equations to eliminate high-frequency errors, followed by local relaxation iterations on the subsystems of the corresponding linear element space. Numerical experiment demonstrates that the adaptive finite element method exhibits uniform convergence and quasi-optimal computational complexity for linear and quadrtic finite element equations. The corresponding Local MG method shows the improved computational efficiency and robustness. By using hierarchical bases, linear element matrices can be directly obtained from p order finite element discretization matrices, eliminating the need to generate grid transformation matrices. This significantly saves computational CPU time and enhances computational efficiency.

2. Model Problem and High-order Finite Element Discretization

In this paper, we consider the following three-dimensional elasticity problem

$$\begin{aligned} \mathbf{L}u =: -\mathscr{L}^T(\partial_x,\partial_y,\partial_z)D\mathscr{L}(\partial_x,\partial_y,\partial_z)u &= f,\ (x,y,z)\in\Omega, && (2.1)\\ u &= g_D,\ (x,y,z)\in\Gamma_D, && (2.2)\\ \mathscr{L}^T(n_x,n_y,n_z)D\mathscr{L}(\partial_x,\partial_y,\partial_z)u &= g_S,\ (x,y,z)\in\Gamma_S, && (2.3) \end{aligned}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded and Lipschitz polyhedral domain in $\mathbb{R}^3$, $\Gamma_D \cup \Gamma_S \subseteq \partial\Omega$, $\Gamma_D \cap \Gamma_S = \emptyset$, u is a displacement vector, f is an external force, g_D and g_S are given vector functions, $n = (n_x, n_y, n_z)^T$ is a unit normal vector of Γ_S, the perators $\mathscr{L}(a,b,c)$ and D are composed respectively by

$$\mathscr{L}(a,b,c) = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \\ b & a & 0 \\ 0 & c & b \\ c & 0 & a \end{pmatrix}, \; D = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}.$$

Here λ and μ are the Lamé constants, which can be composed by the elastic modulus E and Poisson's ratio ν, i.e.,

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \; \mu = \frac{E}{2(1+\nu)}.$$

Define the following finite element spaces as follows

$$\begin{aligned} H_0^1 &= \{u \in (H^1(\Omega))^3 : u|_{\Gamma_D} = 0\}, \\ H_{g_D}^1 &= \{u \in (H^1(\Omega))^3 : u|_{\Gamma_D} = g_D\}. \end{aligned}$$

The variational formulation of (2.1)-(2.3) is to find $u \in H_{g_D}$ such that

$$a(u,v) = F(v), \forall v \in H_0^1, \tag{2.4}$$

where

$$a(u,v) = \int_\Omega (\mathscr{L}(\partial_x, \partial_y, \partial_z)v)^T D \mathscr{L}(\partial_x, \partial_y, \partial_z) u d\Omega$$

and

$$F(v) = \int_\Omega f \cdot v d\Omega + \int_{\Gamma_S} g_S \cdot v ds.$$

Assume that $\mathscr{T}$ is a regular tetrahedron mesh on the polyhedral domain Ω. We introduce the following finite element space

$$\mathbb{V}(\mathscr{P}_p, \mathscr{T}) = \{v_{\mathscr{T}} \in C(\overline{\Omega}) \cap H_{g_D}^1 : v_{\mathscr{T}}|_\tau \in (P_p(\tau))^3, \forall \tau \in \mathscr{T}\}, \tag{2.5}$$

$$\mathbb{V}_0(\mathscr{P}_p, \mathscr{T}) = \{v_{\mathscr{T}} \in C(\overline{\Omega}) \cap H_0^1 : v_{\mathscr{T}}|_\tau \in (P_p(\tau))^3, \forall \tau \in \mathscr{T}\}, \tag{2.6}$$

where $P_p(\tau)$ denotes the space of homogeneous polynomials with order p based on τ. Figure 1 shows the distribution of interpolation nodes in the subdivision element for homogeneous polynomials with order $p = 1$ and $p = 2$.

The variational formulation of linear elasticity problem (2.4) is to find $u_{\mathscr{T}} \in \mathbb{V}(\mathscr{P}_p, \mathscr{T})$ such that

$$a(u_{\mathscr{T}}, v_{\mathscr{T}}) = F(v_{\mathscr{T}}), \forall v_{\mathscr{T}} \in \mathbb{V}_0(\mathscr{P}_p, \mathscr{T}). \tag{2.7}$$

In practical calculations, the finite element space is often constructed by using standard nodal basis functions. However, when low-order element finite functions (such as $p = 1$) are upgraded to high-order element finite funcitons (such as $p \geq 2$), the low-order interpolation functions will be changed accordingly, and the stiffness matrix and other characteristic matrices of the corresponding high-order elements need to be regenerated, which will inevitably bring inconvenience to the preparation of general-purpose programs, especially for adaptive finite element analysis.

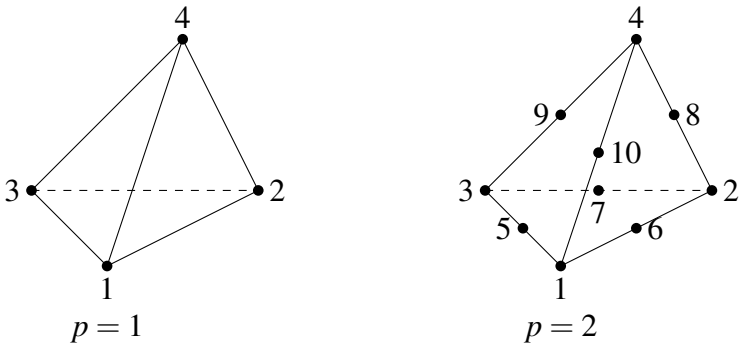

FIGURE 1. The nodes distribution of tetrahedron elements with $p=1,2$

In this paper, we construct the corresponding high-order finite element space by using hierarchical basis functions so that, under the condition of keeping the mesh division unchanged, when the order of the element is increased, the stiffness matrix and other property matrices of the formed low-order elements can be utilized without changing. The p order finite element equation based on hierarchical basis [6] is $A^{(p)}U^{(p)}=F^{(p)}$, that is,

$$\begin{bmatrix} A_{ll} & A_{lq} \\ A_{ql} & A_{qq} \end{bmatrix} \begin{bmatrix} U_l \\ U_q \end{bmatrix} = \begin{bmatrix} F_l \\ F_q \end{bmatrix}, \tag{2.8}$$

where A_{ll} is the global stiffness matrix corresponding to the linear finite element functions, and the corresponding linear element equation is $A_{ll}U_l=F_l$.

In practical calculations, there are many factors that may lead to strong singularities in the problem. Although these singularities can be overcome by uniformly refining the mesh during numerical solution, it will lead to a sharp increase in the physical memory of the computer and the CPU time required for calculation accordingly. However, the adaptive method can automatically generate the required finite element mesh based on the nature of the solution and the required solution accuracy, achieving the maximum computational precision with the minimum computational effort.

In the following section, we will draw on the algorithmic ideas and analytical methods from [10] to establish an adaptive p order finite element method and the corresponding Local MG method that do not require the labeling of oscillatory terms and the refinement of elements does not need to satisfy the inner node property.

3. ADAPTIVE FINITE ELEMENT METHOD AND LOCAL MG METHOD

First, we give a brief introduction about the standad adaptive finite element method (abbreviated as AFEM) based on conforming tetrahedron grid.

For any given initial tetrahedral mesh $\mathscr{T}_0$, a nested sequence of conforming mesh triangulations $\{\mathscr{T}_k\}_{k>0}$ is obtained through local refinement, where $\mathscr{T}_k \to \mathscr{T}_{k+1}$ is mainly composed of the following four modules:

$$\textbf{Solve} \to \textbf{Estimate} \to \textbf{Mark} \to \textbf{Refine}.$$

(1): The module of **Solve**

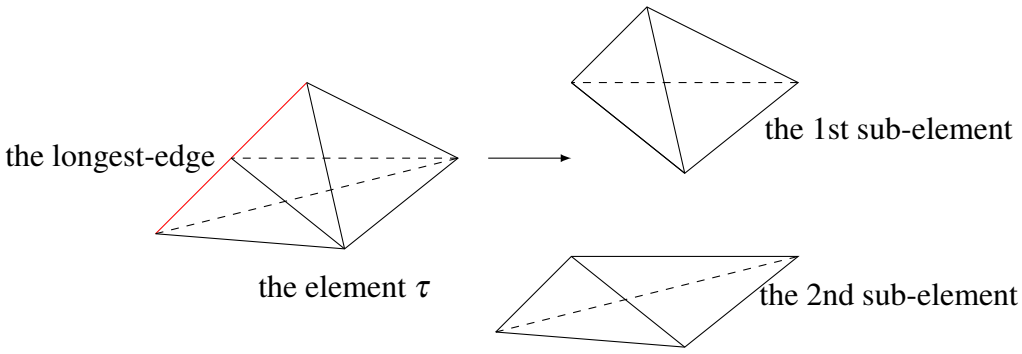

FIGURE 2. The longest edge refinement bisection method

For these given funtions f, g_D, g_S, and a given conforming tetrahedron mesh $\mathscr{T}_k$, we assume that the discrete solution $u_k^{(p)} \in \mathbb{V}(\mathscr{P}_p, \mathscr{T}_k)$ of the discrete variational problem (2.7) can be obtained through this module, where $\mathbb{V}(\mathscr{P}_p, \mathscr{T}_k)$ is the p-order finite element space based on the mesh $\mathscr{T}_k$.

(2): The module of **Estimate**

For the given tetrahedron element $\tau \in \mathscr{T}_k$, we define the posteriori estimator as follow:

$$\eta^2_{\mathscr{T}_k}(u_k^{(p)}, \tau) = h_\tau^2 \|R(u_k^{(p)})\|^2_{0,\tau} + h_\tau \sum_{e \in \partial\tau \setminus \partial\Omega} \|J(u_k^{(p)})\|^2_{0,e},$$

where $R(u_k^{(p)})\big|_\tau = (f - \mathbf{L}u_k^{(p)})\big|_\tau$ and $J(u_k^{(p)})\big|_e = [\mathscr{L}^T(n_x, n_y, n_z)D\mathscr{L}(\partial_x, \partial_y, \partial_z)u_k^{(p)}]\big|_e$ are the residual based on the element and the jump across the interface, $h_\tau = |\tau|^{1/2}$.

For any sets $\mathscr{M}_k \subseteq \mathscr{T}_k$, we define $\eta^2_{\mathscr{T}_k}(u_k^{(p)}, \mathscr{M}_k) = \sum_{\tau \in \mathscr{M}_k} \eta^2_{\mathscr{T}_k}(u_k^{(p)}, \tau)$. In this way, for a given tetrahedron mesh $\mathscr{T}_k$, the error estimator of each element in $\mathscr{T}_k$ can be obtained by using the finite element solution $u_k^{(p)}$ and the error estimator defined above. Similar to the two-dimensional case, the corresponding calculation method can be derived, and a detailed description was presented in [10].

(3): The module of **Mark**

Similar to the two-dimensional case, the Dörfler marking strategy is adopted to select the refinement elements. Given a mesh triangulation $\mathscr{T}_k$, the element error estimator sequence $\{\eta^2_{\mathscr{T}_k}(v_k, \tau)\}_{\tau \in \mathscr{T}_k}$ and the marking parameter $\theta \in (0, 1)$, we can get a marked elements set $\mathscr{M}_k$ which satisfies the following Dörfler marking condition and has the minimum number of elements: $\eta^2_{\mathscr{T}_k}(v_k, \mathscr{M}_k) \geq \theta \eta^2(v_k, \mathscr{T}_k)$.

(4): The module of **Refine**

For the three-dimensional case, we adopt the longest-edge refinement bisection method to generate the mesh $\mathscr{T}_{k+1}$. Let τ be a tetrahedral element that marks the longest edge. We use the longest-edge refinement bisection method to divide the element τ into two new elements, and marks the longest edges for the two sub-elements, as presented in Figure 2. Obviously, this refinement method does not require the elements to satisfy the inner node property.

Using the above four modules, we can obtain an adaptive p order finite element algorithm for solving three-dimensional elasticity problems, which is specifically described as follows.

Algorithm 3.1 (AFEM algorithm)**.** For given functions f, g_D, and g_S, let θ and ε be a given Dörfler marking parameter and an error control constant. The AFEM alogrithm are as follows:

Step 0: Give an initial tetrahedron mesh $\mathscr{T}_0$ and set $k=0$.

Step 1: Use the module of Estimate and get the discrete solution $u_k^{(p)}$ based on mesh $\mathscr{T}_k$.

Step 2: Caculate $\eta^2_{\mathscr{T}_k}(u_k^{(p)},\tau)$. If $\eta^2_{\mathscr{T}_k}(u_k^{(p)},\tau)\le\varepsilon$, the algorithm is over. Otherwise, the algorithm goes on as follow :

Step 3: Execute the module of Mark, and get the marked elements set $\mathscr{M}_k$ of mesh $\mathscr{T}_k$.

Step 4: Execute the module of Refine, and generate the tetrahedron mesh $\mathscr{T}_{k+1}$.

Step 5: let $k=k+1$, and go to **Step 1**.

It is worth mentioning that, compared with the usual AFEM algorithm, the above algorithm is easy to implement in program because it does not need to mark the oscillation term, and the refinement element unit does not need to satisfy the inner node property.

Secondly, we give a detailed introduction about the Local MG method for solving the p order finite element discrete system under adaptive generation mesh.

Let $\mathscr{T}_0$ and $\mathbb{V}_0(=\mathbb{V}(\mathscr{P}_1,\mathscr{T}_0))$ be any given initial tetrahedral mesh and the corresponding linear finite element space, and let J be the number of refinement mesh. Let $\{\mathscr{T}_j\}_{j=1}^J$ be a nested mesh sequence obtained by using AFEM, that is, $\mathscr{T}_j$ is generated by refining the mesh $\mathscr{T}_{j-1}$, $\mathbb{V}_j$ $(=\mathbb{V}(\mathscr{P}_1,\mathscr{T}_j))$ is the corresponding linear finite element space, $\mathscr{N}_j$ is the set which contains all nodes of mesh $\mathscr{T}_j$. For any node $x\in\mathscr{N}_j$, set ϕ_j^x to be the corresponding basis function belongs to the space $\mathbb{V}_j$. In addition, let $\mathbb{V}(\mathscr{P}_p,\mathscr{T}_J)$ be the p order finite element space based on the gird $\mathscr{T}_J$. Then we can obtain the p order finite element equation of three-dimensional elasticity problem (2.1) as follows

$$A_J^{(p)}U_J^{(p)}=F_J^{(p)}. \tag{3.1}$$

Similarly, we can define the linear element equation under adaptive mesh $\mathscr{T}_j$ as follows

$$A_jU_j=F_j,\ 1\le j\le J. \tag{3.2}$$

Due to the use of hierarchical basis, the linear element matrix A_J can be directly obtained from $A_J^{(p)}$ in a blocked form (2.8), without the need to generate the corresponding transformation matrix.

In order to reduce the computation scale and improve computational efficiency, we introduce a local mutigrid method to solve p order finite element equation (3.1), which is based on the adaptive mesh. The essence of this method is by converting high-order finite element calculation into low-order (linear) finite element calculation. For this purpose, we define the index set as follows

$$\bar{\mathscr{N}_j}=\{x\in\mathscr{N}_j:\mathrm{supp}(\phi_j^x)\cap\mathrm{supp}(\phi_j^{x'})\neq\emptyset,\ x'\text{is a new node belong to}\mathscr{N}_j\}.$$

Let $\bar{n}_j=|\bar{\mathscr{N}_j}|$ denote the number of elements in the set $\bar{\mathscr{N}_j}$. For convenience, we abbreviate $\bar{\mathscr{N}_j}=\{x_j^k,k=1,2,\cdots,\bar{n}_j\}$. $\phi_j^k=\phi_j^{x_j^k}$ is the nodal basis function corresponding to the node x_j^k. The usual MG method applies smoothing to all nodes in $\mathscr{N}_j$ (for instance, by using Gauss-Seidel relaxation iterations to smooth all nodes in $\mathscr{N}_j$). However, in the Local MG method, smoothing is only applied to a subset of nodes in $\mathscr{N}_j$ that are related to the refined elements. Then, we give

the V-cycle multigrid iteration format for solving the three-dimensional linear element equation (3.2) based on the adaptive gird as follows

$$U_j^{(k+1)} = U_j^{(k)} + B_j(F_j - A_j U_j^{(k)}), k = 0,1,2,\cdots.$$

For any $g \in \mathbb{V}_j$, the operator $B_j : \mathbb{V}_j \to \mathbb{V}_j$, $0 \le j \le J$ can be defined in a non-recursive manner as follows.

Algorithm 3.2. (V-cycle algorithm based on local relaxation)
$r = g$
For $i = j, j-1, \cdots, 1$
$v_i = 0$
$v_i \leftarrow v_i + R_i(r - A_i v_i)$
$r_i \leftarrow r(ns_i)$
$r \leftarrow (I_{i-1}^i)^T (r - A_i v_i)$
End For
$b = A_0^{-1} r$
For $i = 1, 2, \cdots, j$
$b \leftarrow v_i + I_{i-1}^i b$
$b \leftarrow b + R_i^T (r_i - A_i b)$
End For
$B_j * g = b$.

In the algorithm above, $ns_i = \{k_1, k_2 \cdots, k_{\bar{n}_i}\}$, and $r(ns_i)$ is the vector consisting of the $k_1, k_2, \cdots, k_{\bar{n}_i}$ components of the vector r, I_{i-1}^i is the standard interpolation (promotion) operator, and R_j is the local Gauss-Seidel relaxation operator, defined as $R_j = (I - \prod_{k=1}^{\bar{n}_j} (I - P_j^k)) A_j^{-1}$, where $P_j^k : \mathbb{V}_J \to \mathbb{V}_j^k := span\{\phi_j^k\}$ denotes the projection operator satisfying

$$a(P_j^k w, \phi_j^k) = a(w, \phi_j^k), \ \forall w \in \mathbb{V}_J.$$

Remark 3.3. In Algorithm 3.2, all computations at the i-th level only involve the nodes in $\bar{\mathcal{N}_i}$, and no processing is done for the nodes in $\mathcal{N}_i \backslash \bar{\mathcal{N}_i}$. Therefore, for a given $g \in \mathbb{V}_j$, computing $B_j * g$ in each iteration requires only $O(n_j)$ operations, where $n_j = |\mathcal{N}_j|$.

By utilizing Algorithm 3.2, we can obtain a local relaxation iteration based MG method (abbreviated as Local MG) for solving the p order finite element equation (3.1). The specific description of the algorithm is as follows.

Algorithm 3.4. (Local MG method)

Step 0: Set $k = 0$ and give the controlling accuracy tol. For any initial value of iteration $(U_J^{(p)})^{(0)}$, caculate $(r_J^{(p)})^{(0)} = F_J^{(p)} - A_J^{(p)} (U_J^{(p)})^{(0)}$;

Step 1: Perform the forward Gauss-Seidel iterations with the numbers m in the finite element space $\mathbb{V}(\mathscr{P}_p, \mathscr{T}_J)$ to obtain the solution vector $(U_J^{(p)})^{(k+1)}$;

Step 2: Calculate the residual: $(r_J^{(p)})^{(k+1)} = F_J^{(p)} - A_J^{(p)} (U_J^{(p)})^{(k+1)}$, and calculate the residual corresponding to the linear element equation $r_J^{(k+1)}$;

Step 3: Solve the linear element equation on the adaptive mesh $\mathscr{T}_j$ as follows $A_j e_j^{(k+1)} = r_j^{(k+1)}$, $1 \le j \le J$ and obtain the approximation $\widetilde{e}_J^{(k+1)}$ of the value $e_J^{(k+1)}$;

Step 4: Update the solution vector $(U_J^{(p)})^{(k+1)} = (U_J^{(p)})^{(k+1)} + \begin{pmatrix} \widetilde{e}_J^{(k+1)} \\ 0 \end{pmatrix}$;

Step 5: Perform the backward Gauss-Seidel iterations with the numbers m in the finite element space $\mathbb{V}(\mathscr{P}_p, \mathscr{T}_J)$ to obtain the solution vector, and still name it as $(U_J^{(p)})^{(k+1)}$;

Step 6: Caculate $(r_J^{(p)})^{(k+1)} = F_J^{(p)} - A_J^{(p)}(U_J^{(p)})^{(k+1)}$. If $\frac{\|(r_J^{(p)})^{(k+1)}\|}{\|(r_J^{(p)})^{(0)}\|} \le tol$, the algorithm stops; Otherwise, let $k = k+1$, $(U_J^{(p)})^{(k)} = (U_J^{(p)})^{(k-1)}$ and go to **Step 1**.

4. Numerical Experiment and Result Analysis

In this section, we give a typical example to verify the convergence and quasi-optimal computational complexity of the adaptive p order finite element method, as well as the computational efficiency and robustness of the Local MG method for solving the p order finite element equations on three dimensional adaptive meshes.

In this numerical experiment, the controlling accuracy in Algorithm 3.4 is set to $tol = 10^{-8}$. N_{itr} represents the number of iterations of the Local MG method, and N_{dof} represents the total number of degrees of freedom. Let the elastic modulus $E = 3000$ and Poisson's ratio $\nu = 0.3$.

Let $\Omega = (-1,1)^3 \backslash (0,1) \times (0,1) \times (-1,1)$, the proper vector function f and the boundary function g_S and g_D be chosen to ensure $u = (u_1, u_2, u_3)$ to be the solution of model (2.1), where

$$u_1 = u_2 = u_3 = r^{\frac{2}{3}} \sin \frac{2}{3}\theta.$$

In the experiment, the selected initial mesh is a tetrahedral mesh, as shown in Figure 3.

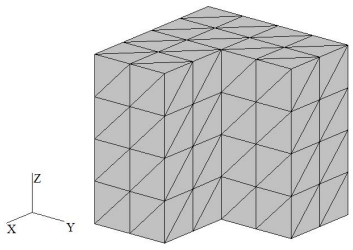

FIGURE 3. The selected initial mesh

In this experiment, we design an adaptive algorithm based on the linear ($p = 1$) and quadratic ($p = 1$) finite element methods. For the case with $p = 1$, Figure 4 shows the 14th refinement mesh in the left with the Dörfler parameter $\theta = 0.5$, and the error curve figures in the right about the errors of the solution u and the linear finite element solution u_k under the energy norm with the Dörfler parameter $\theta = 0.1, 0.3, 0.5$. For the case with $p = 2$, Figure 5 shows the 13th refinement mesh in the left with the Dörfler parameter $\theta = 0.5$, and the error curve figures in the right about the errors of the solution u and the linear finite element solution u_k under the energy norm with the Dörfler parameter $\theta = 0.1, 0.3, 0.5$.

To demonstrate the superiority of the adaptive finite element algorithm designed in this paper, the error reduction under uniform refinement mesh is also listed in the table. Table 1 presents the

number of iterations for solving the linear element equation on adaptive mesh using the Local MG method with the Dörfler parameter $\theta = 0.5$. Table 2 and Table 3 present the number of iterations for solving the layered quadratic element equation on adaptive mesh using the Local MG method with $\theta = 0.3$ and $\theta = 0.5$, respectively.

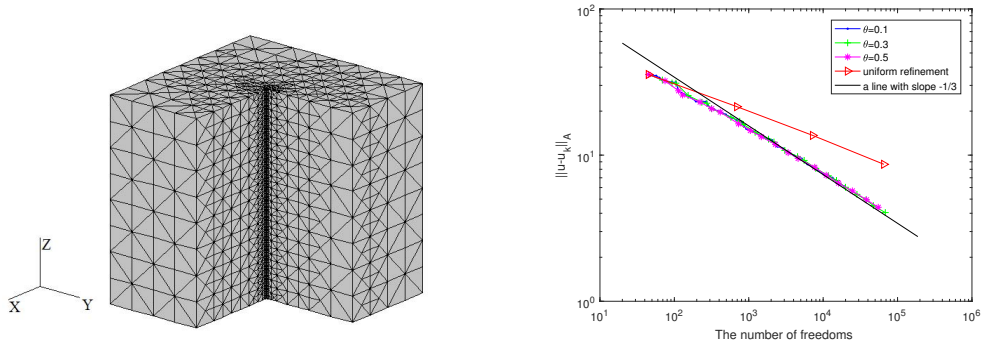

FIGURE 4. The refinement mesh $\mathscr{T}_{14}$ for $\theta = 0.5$ (left) and the error curve of $\|u - u_k\|_A$for $\theta = 0.1, 0.3, 0.5$ (right) by linear finite element method

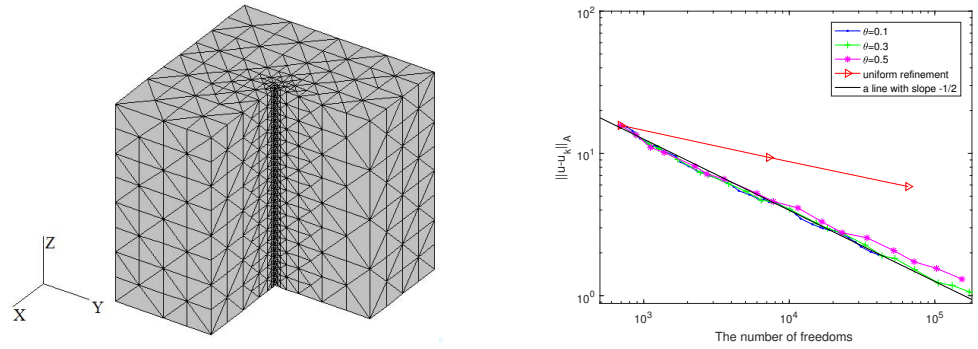

FIGURE 5. The refinement mesh $\mathscr{T}_{13}$ for $\theta = 0.5$ (left) and the error curve of $\|u - u_k\|_A$for $\theta = 0.1, 0.3, 0.5$ (right) by quadratic finite element method

J	2	4	6	8	10	12	14
N_{dof}	114	222	414	1050	2283	4608	10986
N_{itr}	22	20	25	30	33	33	36

TABLE 1. Iterative times for solving linear finite element equations under adaptive mesh by local MG with$\theta = 0.5$

From the above numerical results, it can be seen that the adaptive finite element method designed in this paper is convergent. For the linear element case, the error curve decreases as the number of elements increases and is parallel to the standard line with a slope of $-\frac{1}{3}$, demonstrating quasi-optimal computational complexity. For the quadratic element case, the error curve decreases as the number of elements increases and is parallel to the standard line with a slope of $-\frac{1}{2}$, achieving the goal of obtaining the maximum computational accuracy with the minimum amount of computation. The MG method based on local relaxation exhibits excellent

J	2	6	8	10	14	16	18	20	22
N_{dof}	1083	2775	5037	7740	23805	43380	72633	131547	237771
N_{itr}	35	34	40	43	45	50	55	51	55

TABLE 2. Iterative times for solving quadrtic finite element equations under adaptive mesh by local MG with$\theta = 0.3$

J	2	4	6	8	10	12	14	16
N_{dof}	1113	2247	3561	7785	16782	34185	71892	153765
N_{itr}	35	36	40	43	52	56	60	69

TABLE 3. Iterative times for solving quadrtic finite element equations under adaptive mesh by local MG with$\theta = 0.5$

computational efficiency and robustness in solving linear element equations and layered quadratic element equations on adaptive meshes. The number of iterations is basically independent of the problem size and the number of adaptive refinements J.

5. CONCLUSIONS

Adaptive finite element method is an effective numerical method for solving three-dimensional stress concentration problems. It can automatically refine the mesh in regions with large errors by fully utilizing the solution information of the current mesh, achieving the goal of obtaining the maximum computational accuracy with the minimum amount of computation. For a series of mutually nested meshes formed by the adaptive process, using the MG method to solve the corresponding finite element equations can fully leverage its rapid and efficient characteristics. In this paper, Local MG method with improved computational efficiency and robustness was designed by decomposing the p order finite element space into a "high-frequency" part and a linear element space, combined with the characteristics of mesh refinement. Although we only conducted numerical experiments for the cases of $p = 1$ and $p = 2$, the numerical results obtained are quite encouraging. Of course, we still need to conduct more extensive numerical tests and corresponding theoretical analysis of algorithms for more general practical problems, such as the quasi-optimal computational complexity of adaptive algorithms and the convergence analysis of Local MG method. This is also a topic that the authors will further investigate in the future.

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