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EXISTENCE OF A LOCALLY BOUNDED VARIATION SOLUTION OF A MODEL WITH INFINITE DELAY FOR AN ISOLATED SPECIES

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Abstract. This qualitative study focuses on the existence of solutions of the Volterra integral equation within the space of locally bounded variation functions. In addition, we prove the existence and uniqueness of solutions for a model with infinite delay for an isolated species in the same function space.

Keywords. Infinite delay; Locally bounded variation; Prolongation of solutions; Volterra integral equation.

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1. Introduction and preliminaries

The Volterra integral equation with infinite delay is under the spotlight of academic research due to its wide applications in the mathematical modeling of various problems in ecology, chemistry, and so on. In biological processes, particularly in ecological systems, it is often necessary to consider certain delays in the models. These are conditioned by the growth time of the individuals of the species that enter the systems. These type of models have been studied extensively; see, e.g., [5, 6, 7, 8, 9, 11, 12] and the references therein. Studying an isolation model of a species from the ecological system where it lives is justified since it can, for example, determine or predict the evolution of the number of individuals in a human population or the evolution of bacterial cultures. Some of these models were studied in [1, 2, 3]. One of these models is the generalization of the Hutchinson integro-differential equation using infinite delay given by

$$x'(t) = \alpha \int_{-\infty}^{t} k(t-s)x(s)ds - \rho x^2(t), \tag{1.1}$$

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where α and ρ are positive constants, and $k : [0,+\infty) \to [0,+\infty)$ is a continuous function. The stability of the solutions of this model was studied by Gopalsamy in [4], who further proved that $x^* = \frac{a}{b}$ is globally asymptotically stable.

On the other hand, let us consider the Volterra integral equation

$$x(t) = f(t) + \int_0^t g(t,s,x(s))\,ds, \tag{1.2}$$

where $f \in C(J,\mathbb{R})$ (the continuous functions space), $g \in C(D\times\mathbb{R},\mathbb{R})$, $J = [\beta,\beta+a]$ with $a > 0$, and $D = \{(t,s) \in J\times J : t \geq s\}$. It is known that there exists a number r, with $0 < r \leq a$, such that integral equation (1.2) has at least one solution in $C([\beta,\beta+r],\mathbb{R})$.

Now, questions arise such as: Are these the minimum conditions that f and g must satisfy for a solution to exist? Is it possible to generalize the region D? What conditions can be established for f and g so that solutions exists and belong to a given function space?

In this paper, we study conditions for existence, uniqueness, and prolongation of the solutions of Volterra integral equation (1.2), when function g is a real function defined on $D\times\mathbb{R}$, where

$$D := \left\{(t,s) \in \mathbb{R}^2 : 0 \leq s \leq t < +\infty\right\},$$

in the locally bounded variation function space. The locally bounded variation characteristic of a function does not require restrictions on the interval where it is defined. Thus it is possible to work with the Volterra equation with infinite delay (1.1) and to guarantee the existence and uniqueness of the solutions in the locally bounded variation functions space. It should be noted that many authors investigated the solutions of integral equations in various bounded variation function spaces on the interval $[a,b]$. However, if I is an interval, then every bounded variation function on I is also a locally bounded variation function on I, but the converse is not always true. Therefore, it is relevant to perform qualitative studies of solutions of integral equations, in particular, the solutons of integral equations with infinite delay, in the locally bounded variation space.

In this study, the concepts of bounded variation functions and locally bounded variation functions are required. For this, we include the definitions and some properties that are needed in the rest of this work.

(See [10], page 11). Let I be a real interval of any type, $f : I \to \mathbb{R}$ be a function, and J be a subinterval of I. The **variation** of f on J is the non-negative extended real number

$$V(f;J) = \sup_{S_n} \sum_{i=0}^{n} |f(t_i) - f(t_{i-1})|,$$

where the supremum is taken over any finite subdivision $S_n : t_0 < t_1 < \cdots < t_n$ of J.

The function $f : I \to \mathbb{R}$ is called a **bounded variation function** on I (or a function of bounded variation on I) if and only if $V(f;I) < +\infty$. $f : I \to \mathbb{R}$ is called a **locally bounded variation function** on I if $V(f;J) < +\infty$, for any closed subinterval J of I. In the case that $J = [a,b]$ is a closed subinterval of I, we use the notation $V(f;a,b)$ instead of $V(f;[a,b])$, and we also say that $V(f;a,a) = 0$. It is known that

(1) If I is a interval, then every bounded variation function on I is also a locally bounded variation function on I. The converse is not always true.

(2) Let $f:[a,b]\to\mathbb{R}$ a Lipschitz function on $[a,b]$, that is, there exists a constant $L>0$ such that $|f(x)-f(y)|\leq L|x-y|$ for all $x,y\in[a,b]$. Then f is a function of bounded variation on $[a,b]$.

(3) The space formed by all bounded variation functions over a closed interval $[a,b]$ is denoted by $BV([a,b])$ and is a Banach space with the norm $\|f\|:=\|f\|_\infty+V(f;a,b)$, since every function in this space is also bounded.

A function in $BV([a,b])$ can be expressed as the difference of two monotone functions, which transfers properties, differentiability a.e. (almost eve-rywhere), and integrability, from such monotone functions to the functions of bounded variation on $[a,b]$.

(4) The space formed by all locally bounded variation functions over a interval I is denoted by $lBV(I)$.

2. A Solution of Bounded Variation

In this section, we introduce a qualitative study of the solutions of integral equation of Volterra (1.2). For this, the following conditions are given to the involved functions

(H_1) $f:[0,+\infty)\to\mathbb{R}$ is a locally bounded variation function, this is, $f\in lBV([0,+\infty))$.

(H_2) $f:[0,+\infty)\to\mathbb{R}$ is a continuous function.

(H_3) $g:D\times\mathbb{R}\to\mathbb{R}$, where $D=\left\{(t,s)\in\mathbb{R}^2:0\leq s\leq t<+\infty\right\}$ is a locally Lipschitz function.

Under these conditions, we will establish some preliminary results leading to the proof of the theorem of existence and uniqueness of the solution of equation (1.2). From now on, a and b denote non-negative constants.

Lemma 2.1. *If f satisfies* (H_2), *then* $R(a,b):=\left\{(t,s,x)\in\mathbb{R}^3:0\leq s\leq t\leq a,\ |x-f(s)|\leq b\right\}$ *is a compact subset of* $\mathbb{R}^3$.

Proof. Since f is continuous, then f is bounded on $[0,a]$. Consider $f_a:=\max\{|f(s)|:s\in[0,a]\}$. Then, for any arbitrary point $(t,s,x)\in R(a,b)$, $|x|\leq|x-f(s)|+|f(s)|\leq b+f_a$. Therefore, $\|(t,s,x)\|=|t|+|s|+|x|\leq 2a+b+f_a$. Consequently, $R(a,b)$ is a bounded subset of $\mathbb{R}^3$.

On the other hand, let $\{(t_n,s_n,x_n)\}_{n=1}^{+\infty}$ be a sequence in $R(a,b)$ that converges to (t_0,s_0,x_0). Since $(t_n,s_n,x_n)\in R(a,b)$, it follows that

$$0\leq s_n\leq t_n\leq a\quad\text{and}\quad|x_n-f(s_n)|\leq b.\tag{2.1}$$

For the continuity of f, taking limit as $n\to+\infty$ in (2.1), it follows that $0\leq s_0\leq t_0\leq a$ and $|x_0-f(s_0)|\leq b$. In consequence, one has $(t_0,s_0,x_0)\in R(a,b)$, which implies that $R(a,b)$ is a compact subset of $\mathbb{R}^3$. □

Lemma 2.2. *If f satisfies* (H_2) *and g satisfies* (H_3), *respectively, then g is a Lipschitz function over $R(a,b)$.*

Proof. Since g is a locally Lipschitz function on $D\times\mathbb{R}$, we see by Lemma 2.1 that $R(a,b)$ is compact. Then g is a Lipschitz function on $R(a,b)$. □

We denote the Lipchitz constant of g on $R(a,b)$ by L_a^b.

Remark 2.3. A desirable characteristic of Lipschitz functions on $R(a,b)$ is used frequently in this work. It is uniformly continuous (Thus bounded). That is, given $\varepsilon>0$, there exists

$\delta > 0$ such that, for all (t,s,x), $(\bar{t},\bar{s},\bar{x}) \in R(a,b)$, if $\|(t,s,x)-(\bar{t},\bar{s},\bar{x})\| < \delta$, then $\|g(t,s,x) - g(\bar{t},\bar{s},\bar{x})\| < \varepsilon$.

Lemma 2.4. *Suppose that g satisfies* (H_3)*, $x \in lBV([0,+\infty))$, and $\beta > 0$. For each $t \in [0,\beta]$, the function $g_t : [0,t] \to \mathbb{R}$ defined by $g_t(s) := g(t,s,x(s))$ belongs to $BV([0,t])$.*

Proof. Let $s_0 < s_1 < \ldots < s_n$ be a finite subdivision of $[0,t]$ and consider the set

$$A_\beta := \{(t,s) : 0 \le s \le t \le \beta\}.$$

Since $x \in lBV([0,+\infty))$, then x is bounded on the interval $[0,\beta]$, that is, there exists a constant $k > 0$ such that $|x(s)| \le k$, for all $s \in [0,\beta]$. Now, $A_\beta \times [-k,k] \subseteq D \times \mathbb{R}$ is compact, so g is a Lipschitz function on this set. Let L_β^k be the Lipschitz constant of g on $A_\beta \times [-k,k]$. Then

$$\begin{aligned}
\sum_{i=1}^{n} |g_t(s_i) - g_t(s_{i-1})| &= \sum_{i=1}^{n} |g(t,s_i,x(s_i)) - g(t,s_{i-1},x(s_{i-1}))| \\
&\le L_\beta^k \left(\sum_{i=1}^{n} |s_i - s_{i-1}| + \sum_{i=1}^{n} |x(s_i) - x(s_{i-1})| \right) \\
&\le L_\beta^k \left(\beta + V(x;0,\beta)\right).
\end{aligned}$$

Since this is true for any finite subdivision of $[0,t]$, $V(g_t;0,t) \le L_\beta^k(\beta + V(x;0,\beta)) < \infty$ $\square$

Remark 2.5. An important consequence of Lemma 2.4 is that, under the same assumptions, $g_t(\cdot) = g(t,\cdot,x(\cdot))$ is differentiable a.e. and it is integrable on $[0,\beta]$.

Lemma 2.6. *Suppose that f satisfies* (H_1) *and* (H_2)*, and g satisfies* (H_3)*. Let $0 < \beta \le a$ and $x \in lBV([0,+\infty))$ such that $|x(s) - f(s)| \le b$, for all $0 \le s \le \beta$. Then the function $F : [0,\beta] \to \mathbb{R}$ given by $F(t) = f(t) + \int_0^t g(t,s,x(s))\,ds$ is continuous on $[0,\beta]$ and belongs to $BV([0,\beta])$.*

Proof. Note that if $s \in [0,t]$, then $(t,s,x(s)) \in R(\beta,b)$, so F is well defined, by Remark 2.5. To verify that F is a continuous function and that it belongs to $BV([0,\beta])$, it is enough to verify that the function $\varphi(t) = \int_0^t g(t,s,x(s))\,ds$ also satisfies these conditions in the interval $[0,\beta]$, since f satisfies them. By Lemma 2.1, $R(a,b)$ is compact. By (H_3), g is uniformly continuous on $R(a,b)$. Thus it is bounded on $R(a,b)$, so there exists $M > 0$ such that

$$|g(t,s,x)| \le M, \quad \forall (t,s,x) \in R(a,b). \tag{2.2}$$

Also, given $\varepsilon > 0$, there exists $\delta' > 0$ such that

$$|g(t,s,x) - g(\bar{t},\bar{s},\bar{x})| < \frac{\varepsilon}{2\beta} \tag{2.3}$$

for all $(t,s,x),(\bar{t},\bar{s},\bar{x}) \in R(a,b)$, with $\|(t,s,x)-(\bar{t},\bar{s},\bar{x})\| < \delta'$. Note that $\|(t,s,x)-(\bar{t},\bar{s},\bar{x})\| < \delta$, where $\delta = \min\left\{\delta', \frac{\varepsilon}{2M}\right\}$ implies that (2.3) is satisfied. Thus, for $|t-\bar{t}| < \delta$, without loss of generality, we may assume that $t \le \bar{t}$,

$$\begin{aligned}
|\varphi(t) - \varphi(\bar{t})| &\le \int_0^t |g(t,s,x(s)) - g(\bar{t},s,x(s))|\,ds + \int_t^{\bar{t}} |g(\bar{t},s,x(s))|\,ds \\
&\le \frac{\varepsilon}{2\beta} t + M|\bar{t} - t| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Consequently, φ is continuous on $[0,\beta]$.

Finally, let us prove that $\varphi \in BV([0,\beta])$. Consider $t_0 < t_1 < \cdots < t_n$ a finite subdivision of $[0,\beta]$

$$\begin{aligned}
\sum_{i=1}^{n} |\varphi(t_i) - \varphi(t_{i-1})| =& \sum_{i=1}^{n} \left| \int_0^{t_i} g(t_i,s,x(s))\,ds - \int_0^{t_{i-1}} g(t_{i-1},s,x(s))\,ds \right| \\
=& \sum_{i=1}^{n} \left| \int_0^{t_{i-1}} g(t_i,s,x(s))\,ds - \int_0^{t_{i-1}} g(t_{i-1},s,x(s))\,ds + \int_{t_{i-1}}^{t_i} g(t_i,s,x(s))\,ds \right| \\
\leq& \sum_{i=1}^{n} \int_0^{t_{i-1}} |g(t_i,s,x(s)) - g(t_{i-1},s,x(s))|\,ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |g(t_i,s,x(s))|\,ds \\
\leq& \sum_{i=1}^{n} \int_0^{t_{i-1}} L_a^b \|(t_i,s,x(s)) - (t_{i-1},s,x(s))\| ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} M\,ds \\
=& \sum_{i=1}^{n} \int_0^{t_{i-1}} L_a^b |t_i - t_{i-1}| ds + \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} M\,ds \\
\leq& L_a^b \sum_{i=1}^{n} |t_i - t_{i-1}| t_{i-1} + \sum_{i=1}^{n} M\,|t_i - t_{i-1}| \\
\leq& (L_a^b \beta + M)\beta.
\end{aligned}$$

Since this is true for any finite subdivision of $[0,\beta]$, $V(\varphi;0,\beta) \leq (L_a^b\beta + M)\beta < \infty$, that is, $\varphi \in BV([0,\beta])$. $\square$

The successive Picard approximations technique to be considered for Volterra integral equation (1.2) is defined by the following recurrence relation:

$$\begin{cases} x_0(t) &= f(t) \\ x_{n+1}(t) &= f(t) + \displaystyle\int_0^t g(t,s,x_n(s))ds, \qquad n \geq 0. \end{cases}$$

Theorem 2.7. *(Existence and Uniqueness) Let a and b be positive constants. Let f satisfy* (H_1) *and* (H_2)*, and let g satisfy* (H_3)*. Then, there exists a unique solution of equation* (1.2) *which is continuous and belongs to $BV([0,\beta])$, where $\beta \leq \min\left\{a, \frac{b}{M}\right\}$, where M is as in* (2.2)*.*

Proof. First, we verify (by induction) that each Picard iteration $x_n\,(n \geq 0)$ is a continuous function on $[0,\beta]$, belongs to $BV([0,\beta])$, and satisfies the inequality

$$|x_n(t) - f(t)| \leq Mt, \quad \forall t \in [0,\beta], \tag{2.4}$$

where M is as in (2.2). In effect, given that $x_0(t) = f(t)$, the hypotheses (H_1) and (H_2) imply that x_0 is a continuous function, $x_0 \in BV([0,\beta])$ and $(t,s,f(s)) \in R(a,b)$. By Lemma 2.6, we have that x_1 is continuous on $[0,\beta]$ and belongs to $BV([0,\beta])$. It also satisfies

$$|x_1(t) - f(t)| \;\leq\; \int_0^t |g(t,s,f(s))|\,ds \leq \int_0^t M\,ds = Mt.$$

Suppose that x_n is a continuous function on $[0,\beta]$, belongs to $BV([0,\beta])$, and inequality (2.4) is satisfied. Then, for all $s \in [0,\beta]$, $|x_n(s) - f(s)| \leq Ms \leq M\beta \leq b$. Again, Lemma 2.6 implies that $x_{n+1}(t)$ is continuous on $[0,\beta]$ and belongs to $BV([0,\beta])$. Furthermore,

$$|x_{n+1}(t)-f(t)| = \left|\int_0^t g(t,s,x_n(s))\,ds\right| \leq \int_0^t |g(t,s,x_n(s))|\,ds \leq Mt.$$

Consequently, the Picard iterates $\{x_n\}_{n=0}^{+\infty}$ are continuous on $[0,\beta]$. They belong to $BV([0,\beta])$ and satisfy inequality (2.4). Define $y_n(t)=x_n(t)-x_{n-1}(t)$, for all $n\geq 1$ and $y_0(t)=x_0(t)$. It follows that

$$|y_1(t)|=|x_1(t)-x_0(t)|=|x_1(t)-f(t)|\leq Mt\leq M\beta,\ t\in[0,\beta].$$

On the other hand, by hypothesis (H$_3$), Lemma 2.2, and inequality (2.4), it follows that, for all $t\in[0,\beta]$,

$$\begin{aligned}
|y_2(t)|=|x_2(t)-x_1(t)| &\leq \int_0^t |g(t,s,x_1(s))-g(t,s,x_0(s))|\,ds\\
&\leq L_a^b\int_0^t |x_1(s)-x_0(s)|\,ds\\
&\leq L_a^b\int_0^t Ms\,ds = L_a^b M\frac{t^2}{2!}\\
&\leq L_a^b M\frac{\beta^2}{2!}.
\end{aligned}$$

Recursively, we obtain that

$$|y_n(t)|\leq \frac{M}{L_a^b}\frac{\left(L_a^b t\right)^n}{n!}\leq \frac{M}{L_a^b}\frac{\left(L_a^b \beta\right)^n}{n!},\quad \text{for } n=1,2,\ldots.$$

Since $\sum_{n=1}^{+\infty}\dfrac{\left(L_a^b\beta\right)^n}{n!}$ is a convergent series, we have that $\sum_{n=1}^{+\infty} y_n(t)$ converges absolutely. Since $y_n(t)$ belongs to $BV([0,\beta])$ for each n, this series converges in $BV([0,\beta])$. Thus there exists a function $x\in BV([0,\beta])$ such that $\lim\limits_{k\to\infty} x_k=\lim\limits_{k\to\infty}\sum_{n=0}^{k} y_n = x$. Let us verify that function x satisfies inequality (2.4). If $t\in[0,\beta]$ and $0\leq s\leq\beta\leq a$, then

$$|x(t)-f(t)|=\lim_{k\to\infty}|x_k(t)-f(t)|\leq Mt\leq M\beta\leq b.$$

Thus $(t,s,x(s))\in R(a,b)$ if $0\leq s\leq t\leq\beta$. By Lemma 2.6, we have that $F(t)=f(t)+\int_0^t g(t,s,x(s))\,ds$ is a continuous function on $[0,\beta]$ and belongs to $BV[0,\beta]$. Furthermore, by the definitions of x_{n+1} and F, and by hypothesis (H$_3$), we have

$$\begin{aligned}
|x_{n+1}(t)-F(t)|\leq \int_0^t |g(t,s,x_n(s))-g(t,s,x(s))|\,ds &\leq \int_0^t L_a^b|x_n(s)-x(s)|\,ds\\
&\leq L_a^b\int_0^t \max_{s\in[0,\beta]}|x_n(s)-x(s)|\,ds\\
&\leq L_a^b\max_{s\in[0,\beta]}|x_n(s)-x(s)|\,\beta.
\end{aligned}$$

Making $n\to+\infty$ in the previous inequality, we have that $\lim_{n\to+\infty}x_{n+1}(t)=F(t)$. Hence

$$x(t)=f(t)+\int_0^t g(t,s,x(s))ds,\ \forall t\in[0,\beta]. \tag{2.5}$$

Let us now prove that the solution is unique. Suppose that $y(t)$ is another solution of (2.5) which are continuous and belong to $BV([0,\beta])$. In addition, we define $z(t) = |x(t) - y(t)|$. Then

$$z(t) \leq \int_0^t |g(t,s,x(s)) - g(t,s,y(s))|\,ds \leq L_a^b \int_0^t |x(s) - y(s)|\,ds = L_a^b \int_0^t z(s)\,ds.$$

Thus, $L_a^b \int_0^t z(s)\,ds - z(t) \geq 0$, implying that $e^{-L_a^b t}\left(L_a^b \int_0^t z(s)\,ds - z(t)\right) \geq 0$. Since z is continuous on $[0,\beta]$, it is integrable. In consequence,

$$\frac{d}{dt}\left[-e^{-L_a^b t}\int_0^t z(s)\,ds\right] = e^{-L_a^b t}\left(L_a^b \int_0^t z(s)\,ds - z(t)\right) \geq 0.$$

Since z is continuous, $\frac{d}{dt}\left[-e^{-L_a^b t}\int_0^t z(s)\,ds\right]$ is integrable and non-negative, we have that, for each $w \in [0,\beta]$, $-e^{-L_a^b w}\int_0^w z(s)\,ds \geq 0$, implying that $\int_0^w z(s)ds \leq 0$, but $z(w) \geq 0$. Hence $z(w) = 0$, for each $w \in [0,\beta]$. Then $x(t) = y(t)$, for all $t \in [0,\beta]$. □

3. Prolongation of Solutions

As we have seen in the previous section, Theorem 2.7 guarantees that for given a, b, arbitrary nonnegative real numbers, there exists β such that equation (1.2) has a unique solution defined on $[0,\beta]$ which is continuous and belongs to $BV([0,\beta])$. Specifically, $\beta \leq \min\left\{a, \frac{b}{M}\right\}$, where $|g(t,s,x)| \leq M$ for all $(t,s,x) \in R(a,b)$. That is, M depends on g, a, and b.

Now, under the same conditions, we are going to analyze the existence and uniqueness of a solution defined on the largest possible interval. Consider β as in Theorem 2.7. We translate equation (1.2) into $y(t) = f(t+\beta) + \int_0^{t+\beta} g(t+\beta,s,y(s))ds$. This makes sense since f is defined on $[0,+\infty)$, so

$$\begin{aligned}
y(t) &= f(t+\beta) + \int_0^{t+\beta} g(t+\beta,s,y(s))ds \\
&= f(t+\beta) + \int_0^{\beta} g(t+\beta,s,y(s))ds + \int_{\beta}^{t+\beta} g(t+\beta,s,y(s))ds \\
&= f(t+\beta) + \int_0^{\beta} g(t+\beta,s,y(s))ds + \int_0^{t} g(t+\beta,s+\beta,y(s+\beta))ds \\
&= \widehat{f}(t) + \int_0^t \widehat{g}(t,s,y(s))ds,
\end{aligned}$$

where

$$\widehat{f}(t) = f(t+\beta) + \int_0^{\beta} g(t+\beta,s,y(s))ds$$

and

$$\widehat{g}(t,s,y) = g(t+\beta,s+\beta,y).$$

Let us prove that $\widehat{f}$ and $\widehat{g}$ satisfy the hypotheses of the existence and uniqueness theorem.

Lemma 3.1. *Let f satisfy hypotheses* (H_1) *and* (H_2)*, and let g satisfy hypothesis* (H_3)*. If $|y(s) - f(s)| \leq b$ for all $0 \leq s \leq \beta$, then $\widehat{f} \in lBV([0,+\infty))$.*

Proof. Let $[c,d] \subset [0,+\infty)$, and let $t_0 < t_1 < \cdots < t_n$ be a finite subdivision of $[c,d]$. If $\tau_i = t_i + \beta$, then $\tau_0 < \tau_1 < \cdots < \tau_n$ is a finite subdivision of $[c+\beta, d+\beta]$, so

$$\begin{aligned}
&\sum_{i=1}^{n} \left| \widehat{f}(t_i) - \widehat{f}(t_{i-1}) \right| \\
&= \sum_{i=1}^{n} \left| f(t_i+\beta) + \int_0^\beta g(t_i+\beta,s,y(s))\,ds - f(t_{i-1}+\beta) - \int_0^\beta g(t_{i-1}+\beta,s,y(s))\,ds \right| \\
&\leq \sum_{i=1}^{n} |(f(\tau_i) - f(\tau_{i-1}))| + \sum_{i=1}^{n} \int_0^\beta |g(\tau_i,s,y(s)) - g(\tau_{i-1},s,y(s))|\,ds. \\
&\leq V(f;c+\beta,d+\beta) + \sum_{i=1}^{n} \int_0^\beta |g(\tau_i,s,y(s)) - g(\tau_{i-1},s,y(s))|\,ds.
\end{aligned}$$

For each $s \in [0,\beta]$, we have $s \leq \beta \leq \tau_i \leq d+\beta$ and $|y(s) - f(s)| \leq b$, which implies that $(\tau_i, s, y(s)) \in R(d+\beta, b)$. From (H_2), (H_3), and Lemma 2.1, we have

$$\begin{aligned}
\sum_{i=1}^{n} \int_0^\beta |g(\tau_i,s,y(s)) - g(\tau_{i-1},s,y(s))|\,ds &\leq \int_0^\beta \sum_{i=1}^{n} L_{d+\beta}^b \|(\tau_i,s,y(s)) - (\tau_{i-1},s,y(s))\| ds \\
&\leq \int_0^\beta L_{d+\beta}^b \sum_{i=1}^{n} |\tau_i - \tau_{i-1}| ds = L_{d+\beta}^b (d-c)\beta.
\end{aligned}$$

Then,

$$\sum_{i=1}^{n} \left| \widehat{f}(t_i) - \widehat{f}(t_{i-1}) \right| \leq V(f;c+\beta,d+\beta) + L_{d+\beta}^b (d-c)\beta.$$

Since finite subdivision of $[c,d]$ is arbitrary and (H_1) is satisfied, we have

$$V(\widehat{f};c,d) \leq V(f;c+\beta,d+\beta) + L_{d+\beta}^b (d-c)\beta < \infty,$$

which mean that $\widehat{f} \in lBV([0,\infty),\mathbb{R})$. $\square$

Lemma 3.2. *If f and g satisfy (H_2) and (H_3), respectively, then $\widehat{f}$ is continuous on $[0,+\infty)$.*

Proof. Let $x_0 \in [0,+\infty)$, and let $\varepsilon > 0$. Since f is continuous, there exists $\delta_1 > 0$ such that

$$|x_0 - t| < \delta_1 \Rightarrow |f(x_0) - f(t)| < \frac{\varepsilon}{2}.$$

On the other hand, g is locally Lipschitz. Thus g is continuous, that is, there exists $\delta_2 > 0$ such that

$$|x_0 - t| < \delta_2 \Rightarrow |g(x_0+\beta,s,x(s)) - g(t+\beta,s,x(s))| < \frac{\varepsilon}{2\beta}.$$

Taking $\delta = \min\{\delta_1,\delta_2\}$ and assuming that $|x_0 - t| < \delta$, we have that

$$\begin{aligned}
\left| \widehat{f}(x_0) - \widehat{f}(t) \right| &= \left| (f(x_0+\beta) - f(t+\beta)) + \int_0^\beta (g(x_0+\beta,s,y(s)) - g(t+\beta,s,y(s)))\,ds \right| \\
&\leq |f(x_0+\beta) - f(t+\beta)| + \int_0^\beta |g(x_0+\beta,s,y(s)) - g(t+\beta,s,y(s))|\,ds \\
&\leq \frac{\varepsilon}{2} + \int_0^\beta \frac{\varepsilon}{2\beta}\,ds \\
&= \varepsilon.
\end{aligned}$$

□

Lemma 3.3. *If g satisfies* (H$_3$)*, then $\widehat{g}$ also satisfies it.*

Proof. Let $P = (t_0, s_0, y_0) \in (D\times\mathbb{R})^\circ$, where $(D\times\mathbb{R})^\circ$ denotes the interior of $D\times\mathbb{R}$. Then $P_\beta = (t_0+\beta, s_0+\beta, y_0) \in (D\times\mathbb{R})^\circ$. Since g is locally Lipschitz on $D\times\mathbb{R}$, there exist a constant $L_{P_\beta}(g) > 0$ and an open ball $B(P_\beta, r) \subset (D\times\mathbb{R})^\circ$ such that

$$|g(t,s,y) - g(\bar{t},\bar{s},\bar{y})| \le L_{P_\beta}(g)\|(t,s,y) - (\bar{t},\bar{s},\bar{y})\|,$$

for all $(t,s,y) \in B(P_\beta, r)$. If $(t,s,y) \in B(P,r)$, then $\|(t+\beta, s+\beta, y) - P_\beta\| = \|(t,s,y) - P\| < r$, i.e., $(t+\beta, s+\beta, y) \in B(P_\beta, r)$. Let $(t,s,y), (\bar{t},\bar{s},\bar{y}) \in B(P,r)$ so,

$$\begin{aligned} |\widehat{g}(t,s,y) - \widehat{g}(\bar{t},\bar{s},\bar{y})| &= |g(t+\beta, s+\beta, y) - g(\bar{t}+\beta, \bar{s}+\beta, \bar{y})| \\ &\le L_{P_\beta}(g)\|(t,s,y) - (\bar{t},\bar{s},\bar{y})\|. \end{aligned}$$

That is, $\widehat{g}$ is locally Lipschitz on $(D\times\mathbb{R})^\circ$. □

Lemmas 3.2 and 3.3 together with Theorem 2.7 guarantee the existence of the uniqueness of solutions of equation

$$y(t) = \widehat{f}(t) + \int_0^t \widehat{g}(t,s,y(s))ds, \tag{3.1}$$

which is continuous and belongs to $BV[0,\beta_1]$, where $\beta_1 \le \min\left\{a, \frac{b}{M_1}\right\}$ and M_1 is such that $|\widehat{g}(t,s,y)| \le M_1$ for all $(t,s,y) \in R(a,b)$.

We have found a solution x_0 defined on $[0,\beta]$ and a solution x_1 defined on $[0,\beta_1]$. We may define

$$x(t) := \begin{cases} x_0(t) & \text{if } \ t \in [0,\beta] \\ x_1(t-\beta) & \text{if } \ t \in [\beta, \beta+\beta_1]. \end{cases}$$

This function is a solution of equation (1.2), which is continuous and belongs to $BV[0,\beta+\beta_1]$.

A repeated application of this process gives us a prolongation of the original solution to an interval of $[0,\alpha)$, where $\alpha = \beta + \sum_{j=1}^{\infty}\beta_i$. If $\alpha = \infty$, the process finishes and the solution is prolongable up to the interval $[0,+\infty)$. This could occur, for example, if $|g(t,s,x)|$ is bounded by M on its domain $D\times\mathbb{R}$. Then we may take $M = M_1 = M_2 = \cdots$ and obtain $\beta = \beta_1 = \beta_2 = \cdots$. Otherwise, the interval $[0,\alpha]$ is the maximum interval over which is defined the unique solution, which is of bounded variation.

4. An Ecological Model of an Isolated Community of a Single Species

In this section, we describe a differential model of an isolated population living in an ecologically closed system (see system (4.1)). That is, there is no other species that consumes the products that it consumes or that lives in the places where it lives, In addition, the species does not serve as food for any other species living in the ecological system in consideration. This model is linked with the reproduction or extinction of the species. Let us start by defining a normalized kernel.

A continuous function $K : [0,+\infty) \to [0,+\infty)$ with the property $\int_0^{+\infty} K(s)ds = 1$ is called **normalized kernel**.

Let $\alpha, \rho : [0,+\infty) \to [0,+\infty)$ be locally Lipschitz functions, and K a normalized kernel. Let us consider the system

$$\begin{cases} x'(t) = \alpha(t)\int_{-\infty}^{t} K(t-s)x(s)ds - \rho(t)x^2(t), & \text{if} \quad t \geq 0 \\ x(t) = \phi(t), & \text{if} \quad t \leq 0 \end{cases} \tag{4.1}$$

with $\phi \in FA$, where $FA := \{\phi \in C((-\infty,0],\mathbb{R}) : \phi(t) \geq 0, \phi(0) > 0, \sup_{t\in(-\infty,0]}(\phi(t)) < \infty\}$. We will express equation (4.1) as a Volterra-type equation for $t \geq 0$ in the following way. If $r \geq 0$, then

$$x'(r) = \alpha(r)\int_{-\infty}^{0} K(r-s)x(s)\,ds + \alpha(r)\int_{0}^{r} K(r-s)x(s)\,ds - \rho(r)x^2(r).$$

If

$$F(r) = \alpha(r)\int_{-\infty}^{0} K(r-s)x(s)ds = \alpha(r)\int_{0}^{+\infty} K(r+s)\phi(-s)ds,$$

then equation (4.1) can be written as follows

$$\begin{cases} x'(r) = F(r) + \alpha(r)\int_{0}^{r} K(r-s)x(s)ds - \rho(r)x^2(r), & \text{if} \quad r \geq 0, \\ x(r) = \phi(r), & \text{if} \quad r \leq 0. \end{cases}$$

Integrating from 0 to t on the equation

$$x'(r) = F(r) + \alpha(r)\int_{0}^{r} K(r-s)x(s)ds - \rho(r)x^2(r),$$

we obtain that

$$\begin{aligned} x(t) &= x(0) + \int_0^t F(r)dr + \int_0^t \alpha(r)\left(\int_0^r K(r-s)x(s)ds\right)dr - \int_0^t \rho(r)x^2(r)dr \\ &= x(0) + \int_0^t F(r)dr + \int_0^t \left(\int_s^t \alpha(r)K(r-s)x(s)dr\right)ds - \int_0^t \rho(r)x^2(r)dr \\ &= x(0) + \int_0^t F(r)dr + \int_0^t \left[x(s)\left(\int_s^t \alpha(r)K(r-s)dr\right) - \rho(s)x^2(s)\right]ds. \end{aligned} \tag{4.2}$$

If we denote

$$f(t) = \phi(0) + \int_{0}^{t} F(u)du, \tag{4.3}$$

where $F(u) = \alpha(u)\int_0^{+\infty} K(u+s)\phi(-s)ds$ and

$$g(t,s,y) = y\left(\int_s^t \alpha(u)K(u-s)du\right) - \rho(s)y^2. \tag{4.4}$$

Then, we can express equality (4.2) as

$$x(t) = f(t) + \int_0^t g(t,s,x(s))ds, \quad t \geq 0. \tag{4.5}$$

Now, our objective is to show that the functions defined by (4.3) and (4.4) satisfy the hypotheses required by the existence and uniqueness theorem (Theorem 2.7) to guarantee the existence of a locally bounded variation solution of equation (4.5).

Lemma 4.1. *Let $\phi \in FA$, α a locally Lipschitz function on $[0,+\infty)$, and K a normalized kernel. Then the function $f : [0,+\infty) \to \mathbb{R}$ defined by $f(t) = \phi(0) + \int_0^t F(u)du$, where $F(u) = \alpha(u)\int_0^{+\infty} K(u+s)\phi(-s)ds$; is continuous and belongs to $lBV]([0,+\infty))$.*

Proof. To prove that f is well defined, it is enough to show that F is well defined. Let $\phi_M = \sup_{s\in[0,+\infty)} \phi(-s)$. Then

$$
\begin{aligned}
0 \le F(u) &\le \alpha(u)\phi_M \int_0^{+\infty} K(u+s)ds = \alpha(u)\phi_M \int_u^{+\infty} K(\sigma)d\sigma \\
&\le \alpha(u)\phi_M \int_0^{+\infty} K(\sigma)d\sigma = \alpha(u)\phi_M < \infty.
\end{aligned}
$$

Thus F is not negative and is well defined on $[0,+\infty)$.

Now, let $[r_1,r_2]$ be a closed subinterval of $[0,+\infty)$, and let $t_0 < t_1 < \ldots < t_n$ be a subdivision of $[r_1,r_2]$. Since α is locally Lipschitz, there exists $M_{r_1r_2} > 0$ such that $|\alpha(u)| \le M_{r_1r_2}$, for all $u \in [r_1,r_2]$. Then

$$
\begin{aligned}
\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} \left| \int_{t_{i-1}}^{t_i} F(u)du \right| &\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \left| \alpha(u) \int_0^{+\infty} K(u+s)\phi(-s)ds \right| du \\
&\le \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} M_{r_1r_2}\phi_M \left| \int_0^{+\infty} K(u+s)ds \right| du \\
&\le \sum_{i=1}^{n} M_{r_1r_2}\phi_M \left| \int_{t_{i-1}}^{t_i} \left(\int_0^{+\infty} K(w)dw \right) du \right| \\
&\le M_{r_1r_2}\phi_M |r_2 - r_1|.
\end{aligned}
$$

That is, $f \in lBV([0,+\infty))$.

To verify the continuity of f, it suffices to prove that $\int_0^{+\infty} K(u+s)\phi(-s)ds$ is continuous, since α is continuous. Let $u_0 \in [0,+\infty)$. By the fact that $\int_0^{+\infty} K(\sigma)d\sigma = 1$, given $\varepsilon > 0$, we see that there exists $T > 0$ such that $\left|\int_u^{+\infty} K(\sigma)d\sigma\right| < \frac{\varepsilon}{3\phi_M}$ for all $u > T$.

On the other hand, K is uniformly continuous on $[0,2T+u_0]$. Consequently, there exists $\delta > 0$ such that if $s_1,s_2 \in [0,2T+u_0]$ and $|s_1 - s_2| < \delta$, then $|K(s_1) - K(s_2)| < \frac{\varepsilon}{3\phi_M T}$. If $|u-u_0| < \delta$, then, for all $s \in [0,T]$, $|K(u+s) - K(u_0+s)| < \frac{\varepsilon}{3\phi_M T}$. Thus

$$
\begin{aligned}
&\left| \int_0^{+\infty} K(u+s)\phi(-s)ds - \int_0^{+\infty} K(u_0+s)\phi(-s)ds \right| \\
&\le \phi_M \int_0^T |K(u+s) - K(u_0+s)|\, ds + \phi_M \int_T^{+\infty} |K(u+s) - K(u_0+s)|\, ds \\
&\le \phi_M \int_0^T |K(u+s) - K(u_0+s)|\, ds + \phi_M \left[\int_{T+u}^{+\infty} K(\sigma)d\sigma + \int_{T+u_0}^{+\infty} K(\sigma)d\sigma \right] \\
&< \phi_M \left[\int_0^T \frac{\varepsilon}{3\phi_M T} ds + \frac{\varepsilon}{3\phi_M} + \frac{\varepsilon}{3\phi_M} \right] = \varepsilon.
\end{aligned}
$$

Therefore, f is continuous at u_0. □

Lemma 4.2. *Let $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t < \infty\}$. Let α and ρ be locally Lipschitz functions on $[0,+\infty)$, and let K be a normalized kernel. Then the function $g : D \times \mathbb{R} \to \mathbb{R}$ given by*

$$g(t,s,y) = y\left(\int_s^t \alpha(u)K(u-s)du\right) - \rho(s)y^2$$

is locally Lipschitz on $D \times \mathbb{R}$.

Proof. Let $M > 0$ and $C_M = \{(t,s,y) \in \mathbb{R}^3 : 0 < s < M, t < M, |y| < M\} \cap (D \times \mathbb{R})$. Since α and ρ are locally Lipschitz real functions, the are continuous and, therefore, bounded on $[0,M]$. So, there exist constants $M_1, M_2 > 0$ such that $|\alpha(t)| \le M_1$ and $|\rho(t)| \le M_2$, for all $t \in [0,M]$. Let L_α and L_ρ the Lipschitz constants of α and ρ on $[0,M]$, respectively. If (t_1,s_1,x_1), $(t_2,s_2,x_2) \in C_M$, then

$$\begin{aligned}
&|g(t_1,s_1,x_1) - g(t_2,s_2,x_2)| \\
&= \left| x_1 \int_0^{t_1-s_1} \alpha(\sigma+s_1)K(\sigma)d\sigma - x_2 \int_0^{t_2-s_2} \alpha(\sigma+s_2)K(\sigma)d\sigma - \rho(s_1)x_1^2 + \rho(s_2)x_2^2 \right|.
\end{aligned}$$

Without loss of generality, we assume that $t_1 - s_1 < t_2 - s_2$, then

$$\begin{aligned}
&|g(t_1,s_1,x_1) - g(t_2,s_2,x_2)| \\
&\le \left| x_1 \int_0^{t_2-s_2} \alpha(\sigma+s_1)K(\sigma)d\sigma - x_2 \int_0^{t_2-s_2} \alpha(\sigma+s_2)K(\sigma)d\sigma \right| \\
&\quad + \left|(\rho(s_2)-\rho(s_1))x_2^2 + \rho(s_1)(x_2^2 - x_1^2)\right| \\
&\le \left| \int_0^{t_2-s_2} [x_1\alpha(\sigma+s_1) - x_2\alpha(\sigma+s_2)]\, K(\sigma)d\sigma \right| \\
&\quad + |\rho(s_2)-\rho(s_1)|\,|x_2^2| + |\rho(s_1)|\,|x_2^2 - x_1^2| \\
&\le \left| \int_0^{t_2-s_2} [x_1\alpha(\sigma+s_1) - x_2\alpha(\sigma+s_1) + x_2\alpha(\sigma+s_1) \right. \\
&\quad \left. - x_2\alpha(\sigma+s_2)]\, K(\sigma)d\sigma \right| + M^2\,|\rho(s_2)-\rho(s_1)| + M_2\,|x_1+x_2|\,|x_1-x_2| \\
&\le \int_0^{t_2-s_2} |x_1-x_2|\,|\alpha(\sigma+s_1)||K(\sigma)|d\sigma \\
&\quad + \int_0^{t_2-s_2} |x_2|\,|\alpha(\sigma+s_1)-\alpha(\sigma+s_2)|\,|K(\sigma)|d\sigma + M^2L_\rho\,|s_2-s_1| + 2MM_2\,|x_1-x_2| \\
&\le (M_1+2MM_2)\,|x_1-x_2| + \left(ML_\alpha + M^2L_\rho\right)|s_1-s_2| \\
&\le \max\left\{M_1+2M_2M, ML_\alpha+M^2L_\rho\right\}(|x_1-x_2| + |s_1-s_2|) \\
&\le L_M\,\|(t_1,s_1,x_1) - (t_2,x_2,s_2)\|,
\end{aligned}$$

with $L_M = \max\left\{M_1+2M_2M, ML_\alpha+M^2L_\rho\right\}$. □

Lemma 4.1 guarantees that the function f in (4.3) satisfies (H_1) and (H_2). Lemma 4.2 guarantees that the function g defined in (4.4) satisfies (H_3). Thus, by Theorem 2.7, for each $\phi \in FA$, integral equation ((4.1)) has a unique solution, which is a continuous and locally bounded variation function.

Note that the case

$$\begin{cases} x'(t) = \alpha \int_{-\infty}^{t} K(t-s)x(s)ds - \rho x^2(t), & \text{if} \quad t \geq 0 \\ x(t) = \phi(t), & \text{if} \quad t \leq 0 \end{cases}$$

is a particular case of equation 4.1 by taking $\alpha(t) = \alpha \in \mathbb{R}$ and $\rho(t) = \rho \in \mathbb{R}$.

5. Conclusion

In this work, we established the conditions to ensure the existence and uniqueness of continuous solutions for the Volterra equation within the space of locally bounded variation functions by using Picard's successive approximations. Subsequently, we demonstrated that the prolongation of the solutions also belongs to this space. Finally, we proved the existence and uniqueness of solutions for an infinite delay model with variable coefficients for an isolated species within the same function space. These techniques may be useful for the qualitative study of solutions of this and other models in different function spaces.

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